A Probabilistic description of the bed load sediment flux: 4. Fickian diffusion at low transport rates

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Abstract. High-speed imaging of coarse sand particles transported as bed load reveals how particle motions possess intrinsic periodicities associated with their start-and-stop behavior. The dominant harmonics in these motions have a primary influence on the rate at which the mean squared particle displacement $R(\tau)$ — a measure conventionally used to assess the possibility of anomalous diffusion — increases with the time interval $\tau$. Over a timescale corresponding to the typical travel time of particles, Einstein-Smoluchowsky-like calculations of $R(\tau)$ may ostensibly indicate non-Fickian behavior while actually reflecting the effects of periodicities in particle motions, not anomalous diffusion. We provide the theoretical basis for this observed behavior, and we illustrate how the effective (Fickian) particle diffusivity obtains from G. I. Taylor’s classic definition involving the particle velocity autocovariance, including its relation to the ensemble-averaged particle velocity as articulated by O. M. Phillips. Cross-stream diffusivities are an order of magnitude smaller than streamwise diffusivities.

1. Bed Load Particle Diffusion

The idea of diffusion (or “dispersion”) of bed load particles is a central element of two compelling problems in the study of sediment transport. The first involves understanding the kinematics and mechanics of downstream and cross-stream diffusion (dispersion) of tracer particles in flume experiments or in natural channels at flood and longer timescales [e.g. Sayre and Hubbell, 1965; Drake et al., 1988; Hassan and Church, 1991; Ferguson and Wathen, 1998; Nikora et al., 2002; Ganti et al., 2010; Martin et al., 2012]. The second involves understanding how bed load particle diffusion contributes to the local sediment flux under conditions of unsteady and nonuniform transport [Lisle et al., 1998; Schmeeckle and Furbish, 2007; Furbish et al., 2012a, 2012b; Ball, 2012]. Both problems begin with the recognition that bed load particle motions, although deterministically governed in detail by coupled fluid-particle physics, nonetheless possess a distinct probabilistic nature due to the stochastic (quasi-random) qualities of particle entrainment and disentrainment, and the inherent variability in particle velocities and displacements during transport [Einstein, 1937, 1950].

The possibility that sediment particles exhibit anomalous rather than Fickian (normal) diffusion
during transport [Nikora et al., 2002; Bradley et al., 2010; Ganti et al., 2010; Hill et al., 2010; Martin et al., 2012] has far-reaching implications for how we conceptualize and calculate rates of transport and dispersal of particles and particle-borne substances [Schumer et al., 2009; Furbish et al., 2009a, 2009b; Furbish and Haff, 2010; Furbish et al., 2012a], and begs a fundamental question. What is the physical basis for the appearance of anomalous diffusion in bed load particle motions? Inasmuch as these particle motions involve rolling, sliding and low hops that involve frequent interactions (collisions) with the bed [Drake et al., 1988; Lajeunesse et al., 2010; Roseberry et al., 2012], thereby producing stochastic variations in particle velocities and displacements [Einstein, 1937, 1950; Roseberry et al., 2012; Furbish et al., 2012b], then these motions mimic random-walk behavior attributed to “conventional” diffusive systems, abiotic and biotic [e.g. Einstein, 1905; Taylor, 1921; Viswanathan et al., 1996; Metzler and Klafter, 2000; Okubo and Levin, 2001; Cantrell and Cosner, 2003; Trigger, 2010]. But individual bed load particle motions are brief, involving only a few to tens of collisions with the bed between start and stop [Roseberry et al., 2012], far fewer than what particles in conventional systems experience on similar timescales or over comparably scaled excursion distances. An understanding of bed load particle diffusion therefore requires a sharper description of the kinematics of particle motions than might be represented by simple random-walk models.

Evidence for anomalous diffusion comes from measured displacements of tracer particles seeded in natural channels and flume experiments [Nikora et al., 2002; Bradley et al., 2010; Hill et al., 2010; Martin et al., 2012]. Specifically, letting \( x_p = (x_p, y_p) \) [L] denote the particle position with streamwise and cross-stream coordinates \( x_p \) [L] and \( y_p \) [L], then for Brownian-like (that is, normal or Fickian) diffusion the streamwise particle diffusivity \( \kappa_x \) [L\(^2\) t\(^{-1}\)] can be calculated from measurements using the Einstein-Smoluchovsky equation,

\[
2\kappa_x \tau = \langle [x_p(t + \tau) - x_p(t) - \bar{x}_p(\tau)]^2 \rangle,
\]

where \( \tau \) [t] is a time (lag) interval and \( \bar{x}_p(\tau) = \langle x_p(t + \tau) - x_p(t) \rangle \) [L] is the expected (average) displacement associated with the time interval \( \tau \). The angle brackets in (1) denote an average over many starting times for a single particle, or an average over a specified group of particles, where in practice these two types of averaging can be combined. Assuming the expected cross-stream displacement \( \bar{x}_p(\tau) = \langle y_p(t + \tau) - y_p(t) \rangle \) [L] is zero, then for cross-stream motions, \( 2\kappa_x \tau = \langle [y_p(t + \tau) - y_p(t)]^2 \rangle \), with diffusivity \( \kappa_y \) [L\(^2\) t\(^{-1}\)]. More generally, letting \( R_p \) [L\(^2\)] denote the right side of (1), namely \( R_p(\tau) = \langle [x_p(t + \tau) - x_p(t) - \bar{x}_p(\tau)]^2 \rangle \), then the idea of anomalous diffusion considers the scaling of the mean squared displacement \( R_p(\tau) \) with the time interval \( \tau \) as \( R_p(\tau) \sim \tau^\sigma \), where for normal (Fickian) diffusion the exponent \( \sigma = 1 \), for subdiffusion \( 0 < \sigma < 1 \), and for superdiffusion \( \sigma > 1 \) [e.g. Metzler and Klafter, 2000; Nikora et al., 2002; Schumer et al., 2009; Trigger, 2010]. Similar comments apply to the mean squared (cross-stream) displacement \( R_y(\tau) = \langle [y_p(t + \tau) - y_p(t)]^2 \rangle \) and the relation \( R_y(\tau) \sim \tau^\sigma \). To calculate \( R_p(\tau) \) or \( R_y(\tau) \) for an individual particle or for a group of particles observed at discrete intervals, the average for the interval \( \tau \) is obtained over all paired observations separated by \( \tau \), where the number of paired observations necessarily diminishes with increasing \( \tau \). By definition, the value of \( R_p(\tau) \) or \( R_y(\tau) \) calculated for an individual particle approaches zero as \( \tau \) approaches the particle travel time \( T_p \) [t] [Roseberry et al., 2012].

Bed load particle motions involve three timescales [Nikora et al., 2002]: a short timescale characteristic of the interval between particle-bed collisions, analogous to the mean free time as defined for molecular systems; an intermediate timescale corresponding to the typical particle travel time (start to stop) involving multiple particle-bed collisions; and a long timescale spanning multiple
particle hops and intervening rest periods. Plots of $R(\tau)$ for particle motions at the short timescale reflect a ballistic-like behavior, namely $\sigma = 2$ [Roseberry et al., 2012], as in conventional diffusive systems. Anomalous diffusion has been tentatively identified [Nikora et al., 2002; Martin et al., 2012] at intermediate and long timescales from plots of $R_j(\tau)$ and $R_y(\tau)$ for particle tracer motions from flume experiments and from a re-analysis of tracer motions in a field experiment [see Nikora et al., 2002 with reference to Drake et al., 1988]. The idea of subdiffusive behavior ($\sigma < 1$) of tracer particles involving multiple hops and rest times at long timescales is compelling [Nikora et al., 2002; Bradley et al., 2010; Hill et al., 2010]. For intermediate timescales coinciding with particle travel times, however, the Einstein-Smoluchowsky and related equations may ostensibly indicate non-Fickian behavior while actually reflecting effects of correlated random walks [Viswanathan et al., 2005] associated with intrinsic periodicities in particle motions, not anomalous diffusion [Roseberry et al., 2012]. Herein we provide the theoretical basis for this observed behavior, and we illustrate how the effective (Fickian) particle diffusivities $\kappa_x$ and $\kappa_y$, specifically relevant to calculations of the bed load sediment flux [Furbish et al., 2012a], obtain from G. I. Taylor’s classic definition [Taylor, 1921] involving the particle velocity autocovariance, including its relation to the ensemble-averaged particle velocity as articulated by O. M. Phillips [Phillips, 1991].

Using results of high-speed imaging of sand particles transported as bed load over a planar bed [Schmeeckle and Furbish, 2007; Roseberry et al., 2012], our analysis reveals the ballistic-like behavior of particles at short timescales, behavior that Nikora et al. [2002] correctly anticipated but could not demonstrate with the data available to them, and it clarifies why superdiffusion is not possible at the intermediate timescale corresponding to the typical particle travel time. We present a proof-of-concept that Taylor’s formulation yields a proper description of diffusion of bed load particles at low transport rates, consistent with Fickian diffusion. The analysis also points to the design of experimental measurements required to obtain precise estimates of the particle diffusivity and related quantities. [Note: Although not yet accepted for publication, the companion papers Roseberry et al. [2012] and Furbish et al. [2012a, 2012b] are cited with a 2012 date for simplicity of reference.]

2. Experimental Measurements

Our analysis highlights results of high-speed imaging of sand particles transported as bed load over a planar bed. As described in Roseberry et al. [2012], the experiments were conducted with an 8.5 m × 0.3 m recirculating flume in the River Dynamics Laboratory at Arizona State University. For several flow conditions, fluid velocities were measured with an acoustic doppler velocimeter at a position one cm above the bed surface, from which bed shear stresses were calculated using the logarithmic law of the wall and a value of the roughness length $z_0$ [L] equal to $D_{50}/30$. Bed material consisted of relatively uniform coarse sand with an average diameter $D_{50}$ of 0.05 cm. The bed was smoothed before each experiment. High-speed imaging at 250 frames per second over a 7.57 cm (streamwise) by 6.05 cm (cross-stream) bed-surface domain with 1,280 × 1,024 pixel resolution provided the basis for tracking particle motions. A small plexiglass “sled” window was placed on the water surface so that the camera had a clear view of the bed surface through the water column without effects of image distortion by light refraction with water-surface undulations. Flow depths were sufficiently large that the window did not interfere with the flow at the bed surface in the area filmed.

The image series involved a duration of 19.65 seconds (4,912 frames) for each of four stress conditions. We then performed two sets of measurements. In the first set, designated as A, we used runs R1, R2, R3 and R5 to track all active particles within a specified window at one of two
sampling intervals over varying time durations (Table 1). In the second set of measurements, designated as B, we used runs R2 and R3 to track virtually all active particles over the full 1,280 × 1,024 pixel domain using a frame interval of 0.004 sec over a shorter duration (0.4 sec). The four series in set A provide a description of particle activities and velocities, and fluctuations in these quantities, over durations much longer than the average particle hop time. The two series in set B, although of shorter duration, provide a detailed description of particle motions over the full image domain at a finer resolution than that provided in set A.

For set A, we used ImageJ (an open source code available from the National Institutes of Health) to mark the centroid of each active particle as it moved within successive frames, recording the centroid pixel coordinates, $x_p, [L]$ and $y_p, [L]$. For set B, images were imported into ArcGIS 9.3 and spatial coordinates were edited as a point shapefile. All particles that visibly moved over the entire duration of video R2B were tracked, giving 870 unique spatial coordinates from 20 particles. In video R3B, the spatial coordinates of approximately 95% of all particles in motion were tracked, giving over 13,000 spatial positions from 311 particles. The particles not tracked in R3B were those whose identities were too difficult to maintain through the video or that exited the field of view early, or entered the field of view late.

For both sets of measurements (A and B), we calculated the streamwise and cross-stream particle displacements $\Delta x_p = x_p(t + \Delta t) - x_p(t), [L]$ and $\Delta y_p = y_p(t + \Delta t) - y_p(t), [L]$ between frames, and from these we estimated the “instantaneous” particle velocity components $u_p = \Delta x_p/\Delta t, [L/ T]$ and $v_p = \Delta y_p/\Delta t, [L/ T]$, where $\Delta t [T]$ is the selected sampling interval (0.012 sec for R1, R2; 0.004 sec for R3, R5, R2B, R3B). These paired velocity components involved numerous instants with $v_p = 0$ and finite $u_p$, and fewer instants with $u_p = 0$ and finite $v_p$. Although particles mostly moved downstream, some particles occasionally moved upstream ($u_p < 0$). We considered a particle with $u_p = v_p = 0$ to be at rest, even if for only one frame interval. Conversely, a particle is considered to be active if either $u_p$ or $v_p$ is finite.

These experiments indicate that particle motions consist of rolling, sliding and low hops that involve frequent interactions with particles on the bed [Drake et al., 1988; Lajeunesse et al., 2010; Roseberry et al., 2012]. Particles are accelerated to their highest velocities by sweeping fluid motions rather than being carried upward into high momentum flow, and most of the total hop distance of a particle (start to stop) occurs during periods of high velocity rather than during prolonged periods of low velocity [Roseberry et al., 2012]. The particle activity, the solid volume of particles in motion per unit streambed area, fluctuates as particles respond to near-bed fluid turbulence while simultaneously interacting with the bed, where the magnitude of the fluctuations in activity relative to the overall level of activity depends on the size of the sampling area. The activity increases with increasing bed stress faster than does the average particle velocity. Moreover, the probability density functions, $f_{u_p}(u_p), [L/ T]$, and $f_{v_p}(v_p), [L/ T]$, of the streamwise and cross-stream particle velocities, $u_p, [L/ T]$ and $v_p, [L/ T]$, are exponential-like, consistent with the experimental results of Lajeunesse et al. [2010], whereas the probability density functions of the streamwise particle hop distance $L_s, [L]$ and the associated travel time $T_p, [T]$ are gamma-like. In turn, the hop distance varies with travel time as $L_s \sim T_p^{-5/3}$.

3. Particle Motions and the Mean Squared Displacement

3.1. The Effect of Periodicities in Particle Motions

The motion of a particle, start to stop, over a hop distance $L_s$ during the travel time $T_p$ by definition starts and ends with zero velocity ($u_p(0) = u_p(T_p) = 0$) with finite peak velocity in between.
So regardless of how the particle velocity \( u_p(t) \) varies in detail during the interval \( 0 \leq t \leq T_p \), the velocity signal \( u_p(t) \) must possess at its most basic level a fundamental harmonic with period \( T = 2T_p \) \([t]\) (although variations on this assertion, elaborated below, are possible). Treating this harmonic as a sinusoid, it is straightforward to show that integration of the velocity signal \( u_p(t) \) yields a displacement signal \( x_p(t) \) composed of the sum of two parts, a mean motion equal to \((L_x/T_p)t\), and a fluctuating motion possessing a fundamental harmonic with period \( T = T_p \), normally the dominant harmonic [Roseberry *et al.*, 2012; Ball, 2012; but see section 3.2 below] (Figure 1). Moreover, the travel time \( T_p \) of a particle influences the amplitude of its fundamental velocity harmonic and, in turn, the amplitude of the harmonic of the fluctuating part of the displacement signal \( x_p(t) \). Namely, particles with long travel times on average are more likely to be accelerated to large peak velocities than are particles with short travel times [Roseberry *et al.*, 2012]. The corollary is that particles with short travel times on average are limited to relatively small peak velocities. For illustration let \( x_p(t) = (L_x/T_p)t + \sin(2\pi t/T_p) \), where \( a \) [L] is the amplitude of the dominant (and in this case, the fundamental) harmonic with period \( T = T_p \). If \( U_p \) [L t^{-1}] is the amplitude of the underlying harmonic of the velocity signal \( u_p(t) \) with period \( T = 2T_p \), then \( U_p \sim L_x/T_p \) and the amplitude \( a \) is directly proportional to the product \( U_p T_p \sim L_x \).

Consider the contribution to the mean squared displacement \( R_x(\tau) \) due to the periodic part of the motion of a particle. We start by assuming that the fundamental, or dominant, harmonic of the displacement signal \( x_p(t) \) of a particle is given by \( x_p(t) = (L_x/T_p)t + \sin(\omega t) \), where \( \omega = 2\pi/T \) is the angular frequency and the period \( T \) is not necessarily equal to the travel time \( T_p \). The expected displacement \( \bar{x}_p(\tau) \) associated with an interval \( \tau \) is

\[
\bar{x}_p(\tau) = \frac{1}{T_p - \tau} \int_0^{T_p - \tau} [x_p(t + \tau) - x_p(t)] \, dt ,
\]

where it becomes clear that \( \bar{x}_p(\tau) \) is not the same as the average displacement of the particle given by \((L_x/T_p)t \) (with \( t = \tau \)). Using the right side of the Einstein-Smoluchowsky equation (1) the mean squared displacement \( R_x(\tau) \) is

\[
R_x(\tau) = \frac{1}{T_p - \tau} \int_0^{T_p - \tau} [x_p(t + \tau) - x_p(t) - \bar{x}_p(\tau)]^2 \, dt ,
\]

and

\[
R_x(\tau) = a^2 [1 + F^2 - \cos(\omega \tau)]
\]

\[
- \frac{2Fa^2}{\omega (T_p - \tau)} [1 - \cos(\omega T_p - \omega \tau) + \cos(\omega T_p) - \cos(\omega \tau)]
\]

\[
- \frac{a^2}{4\omega (T_p - \tau)} [\sin(2\omega T_p - 2\omega \tau) + \sin(2\omega T_p) - \sin(2\omega \tau) - 2\sin(2\omega T_p - \omega \tau) + 2\sin(\omega \tau)] ,
\]

in which the function \( F \) is
where it is clear that \( R_s(\tau) \) is not the same as the average of the squared deviations given by \( \langle [x_p(t) - (L/T)_p]^2 \rangle \) (with \( t = \tau \)) [Ball, 2012]. Thus, as the amplitude \( a \) goes to zero, the expected displacement \( \overline{x_p(\tau)} \) becomes linear in \( \tau \) and the mean squared displacement \( R_s(\tau) \) vanishes, so any contribution to the mean squared displacement of a particle is entirely due to its Brownian-like (non-periodic) part. And, whereas the expected displacement \( \overline{x_p(\tau)} \) depends on the hop distance \( L_p \), the mean squared displacement \( R_s(\tau) \) of an individual particle does not. Moreover, expanding terms in (5) in powers of \( \tau/T \), then at lowest order for small times \( \tau \),

\[
R_s(\tau) = 2\pi^2 a^2 \frac{\tau^2}{T^2},
\]

which has the appearance of ballistic behavior, namely \( R_s(\tau) \sim \tau^2 \).

In turn, when calculated for a group of particles the expected displacement \( \overline{x_p(\tau)} \) is an average over all particles. Thus, whereas the sinusoidal motion of an individual particle gives positive and negative deviations about the (individual) expected displacement, this motion may involve mostly positive or mostly negative deviations about the expected displacement of the group, depending on the average particle velocity and hop distance relative to the group averaged velocity and hop distance. This is a fundamentally distinguishing feature in physical interpretations of the expected displacement \( \overline{x_p(\tau)} \) and the mean squared displacement \( R_s(\tau) \) for bed load particle motions versus particle motions that continue indefinitely, as in molecular systems or as envisioned by Taylor [1921] for particles suspended in a turbulent flow. Because bed load particles start and stop, the expected values \( \overline{x_p(\tau)} \) and \( R_s(\tau) \) of individual particles do not in any ergodic sense possess an asymptotic long-time equality with the expected values of the group of particles. For example, each particle suspended in a homogeneous turbulent flow eventually experiences (in a probabilistic sense) the full suite of possible turbulent motions, so after an interval longer than the decorrelation timescale (see below), plots of the expected values \( \overline{x_p(\tau)} \) and \( R_s(\tau) \) versus \( \tau \) for individual particles converge and equal the expected values calculated for the group. In contrast, each bed load particle does not remain in motion long enough to experience the full suite of possible interactions with the fluid and bed, so the expected values \( \overline{x_p(\tau)} \) and \( R_s(\tau) \) versus \( \tau \) for each particle is unlike any other, and unlike values calculated for the group. Similar comments pertain to the mean squared cross-stream displacement \( R_y(\tau) \).

### 3.2. Experimental Results

Imaging reveals that streamwise particle motions, start to stop, typically involve one of three types of net displacement. Most typically, a particle gradually accelerates from rest to its peak velocity then gradually decelerates before returning to rest. In this case the displacement about the average motion, \( x_p - (L/T_p)t \), is approximately sinusoidal with a principal (dominant) harmonic whose period is equal to the travel time, namely \( T = T_p \) (Figure 1). Some particles undergo a brief interval of rapid acceleration from rest to a peak velocity followed by gradual deceleration before deposition, or conversely, a gradual acceleration followed by a brief interval of rapid deceleration in returning to rest. In this case the displacement about the average motion appears as a dominant
harmonic whose period $T = 2T_p$ (Figure 2). Finally, a few particles accelerate then decelerate more than once during a full hop such that the displacement about the average motion involves a (dominant) harmonic whose period is a fraction $m$ of the travel time, namely $T = mT_p$ (Figure 3).

Cross-stream motions similarly involve net displacements that possess periodic structure, but are less systematic than streamwise displacements (Figure 4). That is, dominant harmonics are not systematically related to the travel time $T_p$. Cross-stream motions are more erratic than streamwise motions.

The effect of brief, rapid accelerations due to particle-fluid and particle-bed interactions, including collisions with particles on the bed, is to add high-frequency “noise” to the periodic part of the motion of a particle, that is, to randomize this periodic motion. Thus one may consider the motion of a particle as consisting of a correlated random walk — albeit a brief walk — involving a few to tens of collisions during an individual hop.

The close fits between plots of (5) and (7) and empirically calculated values of $R_X(\tau)$ for individual particles from R2B and R3B (Figure 5) reveal the primary influence of the dominant harmonic in the streamwise motion of a particle on the mean squared displacement $R_X(\tau)$. (Similar fits were obtained for virtually all recorded particle motions completing full hops, start to stop.) Namely, with reference to the scaling relation $R_X(\tau) \sim \tau^\sigma$, calculated values of $R_X(\tau)$ typically exhibit a ballistic-like behavior with $\sigma \approx 2$ for $\tau \leq 0.01$ sec [Roseberry et al., 2012; Ball, 2012]. This represents for the specific conditions of our experiments the characteristic interval between particle-bed collisions, analogous to the mean-free path, a behavior that Nikora et al. [2002] correctly anticipated (but could not demonstrate with the data available to them). Moreover, the effect of the intrinsically periodic motion of a particle is to give $R_X(\tau)$ the appearance of non-Fickian (superdiffusive) behavior with $\sigma > 1$ for $0.01 \leq \tau \leq 0.1$. (We note that the small window size of R3 sampled only short (censored) particle motions, so the slope of $R_X(\tau)$ declines at $\tau$ less than 0.1 sec. The window size of R5 is smaller than that of R3, so we have not plotted the R5 data.) However, this apparent superdiffusive behavior ($\sigma > 1$) merely represents the collective effect of the correlated (sinusoidal) random walks of particles that are increasingly (but not completely) randomized by particle-fluid and particle-bed interactions over a timescale corresponding to the typical travel time of particles. Moreover, short motions are more akin to Brownian-like motions than are long duration motions, so short motions contribute more to the random part of $R_X(\tau)$, just as do higher harmonics within longer duration motions. With increasing $\tau$, fewer particle motions are involved in the calculation of $R_X(\tau)$, specifically, only those whose travel time $T_p \geq \tau$. Values of $R_X(\tau)$ at the largest values of $\tau$ are based on particles with the largest travel times $T_p$. Thus, $R_X(\tau)$ tends to decline at large $\tau$ (but does not necessarily return to zero as with individual particles).

For particles completing full hops in R2B and R3B, estimates of the amplitude $a$ systematically increase with the hop distance $L_x$ and, the hop distance $L_x$ systematically increases with the (individual) mean velocity calculated as $L_x/T_p$ (Figure 7). This reinforces the point made above, that particles with long travel times (or hop distances) on average are more likely to be accelerated to large peak velocities $U_p$ than are particles with short travel times. In turn, with $L_x \sim T_p^{5/3}$ [Roseberry...
et al., 2012], then \( U_p \sim L_p / T_p \sim T_p^{2/3} \) [Ball, 2012]. Thus, whereas the hop distance on average increases at a growing rate with increasing travel time, the mean velocity increases less rapidly with increasing travel time.

When \( R_*(\tau) \) is calculated for all cross-stream motions in each experiment, no ballistic-like behavior is apparent at small \( \tau \), and the slope \( \sigma \) (i.e. the exponent in \( R_*(\tau) \sim \tau^\sigma \)) over the domain \( 0.01 \leq \tau \leq 0.1 \) varies from about 1 to 1.8 (Figure 8). The magnitudes of cross-stream particle velocities typically are much smaller than streamwise velocities [Roseberry et al., 2012], and our measurements of small cross-stream displacements are less precise.

4. Particle Diffusivity
4.1. Definition of the Diffusivity

The motion of a bed load particle over its travel time \( T_p \) is continuous, albeit involving quasi-random (high-frequency) fluctuations in the velocity associated with fluid accelerations and particle-bed collisions [Lajeunesse et al., 2010; Roseberry et al., 2012; Furbish et al., 2012b]. An appropriate description of the particle diffusivities \( \kappa_x \) and \( \kappa_y \) therefore can be obtained from the classic definition provided by Taylor [1921]. It is important, however, to be explicit about the averaging involved in this definition, inasmuch as G. I. Taylor envisioned particle motions that continue indefinitely, as opposed to the start-and-stop motions of bed load particles.

As described in Furbish et al. [2012a] and Roseberry et al. [2012], consider a planar streambed area \( B \) [L^2] large enough to fully sample steady, homogeneous near-bed conditions of turbulence and transport. At any instant the number \( N \) of active particles is approximately constant. That is, the rate of disentrainment within \( B \) equals the rate of entrainment, and the rate at which particles leave \( B \) across its boundaries equals the rate at which particles enter \( B \) across its boundaries. Imagine recording particle motions within \( B \) for an interval of time \( T \) [e.g. Lajeunesse et al., 2010; Roseberry et al., 2012]. For \( T \) much longer than the mean particle travel time (also see below), particle motions during \( T \) adequately represent the joint probability density \( f_{L_x, L_y, T_p}(L_x, L_y, T_p) \) [L^{-2} t^{-1}] of hop distances \( L_x \) and \( L_y \), and travel times \( T_p \) [Furbish et al., 2012a] without bias due to censorship of motions at times \( t = 0 \) and \( t = T \) [Furbish et al., 1990]. The marginal probability densities \( f_{L_x}(L_x) \) [L^{-1}], \( f_{L_y}(L_y) \) [L^{-1}] and \( f_{T_p}(T_p) \) [t^{-1}] possess the means \( \overline{L_x} \) [L], \( \overline{L_y} \) [L] and \( \overline{T_p} \) [t]. Moreover, at any instant the velocities of active particles within \( B \) possess the probability densities \( f_{u_x}(u_x) \) [L^{-1} t] and \( f_{u_y}(u_y) \) [L^{-1} t] with means \( \overline{u_x} \) [L t^{-1}] and \( \overline{u_y} \) [L t^{-1}] and variances \( \sigma_{u_x}^2 \) [L^2 t^{-2}] and \( \sigma_{u_y}^2 \) [L^2 t^{-2}], where it may be assumed that these represent ensemble averaged quantities [Furbish et al., 2012a, 2012b; Roseberry et al., 2012].

The average streamwise velocity of the \( i \)th active particle with travel time \( T_p \), is

\[
\overline{u_{pi}} = \frac{1}{T_p} \int_0^{T_p} u_{pi}(t) \, dt = \frac{L_{pi}}{T_{pi}}.
\]  

In turn, letting \( N_i \) denote the number of particle motions during \( T_s \) (note that \( N_i \gg N \)), and assuming that \( N_i \) is large, the ensemble average velocity
Thus, contrary to the assertion of Lajeunesse et al. [2010], the ensemble average hop distance $\bar{L}_x$ is equal to the product of the ensemble averaged velocity $\bar{u}_p$ and the mean travel time $\bar{T}_x$. (But note that $\bar{L}_x/\bar{T}_x \neq \langle L_{x,i}/T_{p,i} \rangle$.)

As a point of reference, when particles continue their motions indefinitely (that is, they do not start and stop), then experimentally $T_{p,i} = T_s$ (the sample time) and (9) becomes

$$\bar{u}_p = \frac{\sum_{i=1}^{N_s} \int_0^{T_{p,i}} u_{p,i}(t) \, dt}{N_s T_s} = \frac{N_s \bar{T}_{x,i}}{N_s T_s} = \frac{\bar{T}_{x,i}}{T_s},$$

where now $\bar{T}_{x,i}$ is the average displacement during $T_s$, and the average in (10) is the same as the average of an individual particle over long time.

Letting $u_{p,i}' = u_{p,i} - \bar{u}_p$, then the autocovariance $C_x(\tau)$ [L$^2$ t$^2$] of the streamwise particle velocities is

$$C_x(\tau) = \langle u_{p,i}'(t+\tau)u_{p,i}'(t) \rangle = \frac{\sum_{i=1}^{N_s} \int_0^{T_{p,i}} u_{p,i}'(t+\tau)u_{p,i}'(t) \, dt}{N_s \int_\tau^{T_p} f_{T_p}(T_p) \, dT_p},$$

where

$$N_s = N_s \int_0^{T_p} f_{T_p}(T_p) \, dT_p$$

is the number of particle motions with travel time $T_p \geq \tau$. When $\tau = 0$, $N_s = N_s$, and $C_x(0) = \sigma^2_{u_x}$. And, when particles continue their motions indefinitely,

$$C_x(\tau) = \frac{\sum_{i=1}^{N_s} \int_0^{T_{p,i}} u_{p,i}'(t+\tau)u_{p,i}'(t) \, dt}{N_s (T_p - \tau)}.$$

Taylor [1921] demonstrated that for continuous particle motions,

$$\frac{dR_x(\tau)}{d\tau} = 2 \int_0^\tau C_x(\tau') \, d\tau'.$$

Inasmuch as the integral in (14) converges as $\tau \to \infty$, then the streamwise particle diffusivity is

$$\kappa_x = \int_0^{\infty} C_x(\tau') \, d\tau' = \sigma^2_{u_x} \tau_x,$$
in which \( \tau_L \) [t] is the Lagrangian integral timescale defined by

\[
\tau_L = \frac{1}{\sigma_{u_p}^2} \int_0^\infty C_s(\tau') \, d\tau' = \int_0^\infty A_s(\tau') \, d\tau',
\]

(16)

where \( A_s(\tau) = C_s(\tau)/\sigma_{u_p}^2 \) is the autocorrelation of the streamwise particle velocities. Note that in this development we are envisioning \( T_s \gg \tau_L \). By a similar development the cross-stream diffusivity is

\[
\kappa_y = \int_0^\infty C_y(\tau') \, d\tau' = \sigma_{u_p}^2 \tau_L,
\]

(17)

where \( \sigma_{u_p}^2 \) is the variance of the cross-stream velocities \( v_p \).

4.2. Experimental Results

The autocovariance \( C_s(\tau) \) decays to zero by about \( \tau \approx 0.1 \) sec for our experiments (Figure 9). In turn, the integral in (15) converges, where numerically computed values level off at \( \tau \approx 0.1 \) to 0.15 sec (Figure 10), giving estimates of \( \kappa_x \) from about 0.3 cm\(^2\) s\(^{-1}\) to 0.8 cm\(^2\) s\(^{-1}\) over the range of experimental conditions. Inasmuch as variations in particle velocity are of the same order as the mean velocity \( \bar{u}_p \), [Roseberry et al., 2012; Furbish et al., 2012b], then as suggested by Phillips [1991],

\[
\kappa_x = \sigma_{u_p}^2 \tau_L \sim \bar{u}_p \delta,
\]

(18)

where \( \delta \sim \bar{u}_p \tau_L \) [L] is a characteristic distance of motion over which the autocovariance \( C_s(\tau) \) is significant, and is similar in magnitude to the mean hop distance. Moreover, there is clear evidence that particles velocities \( u_p \) possess an exponential density \( f_{u_p}(u_p) \) with ensemble average \( \bar{u}_p \) [Lajeunesse et al., 2010; Roseberry et al., 2012; Furbish et al., 2012b], in which case \( \sigma_{u_p}^2 = \bar{u}_p^2 \), which reinforces the point in (18), that \( \kappa_x \sim \bar{u}_p \delta \). In the language of transport in porous media flows, \( \delta \) is the so-called “dispersivity.” In addition, estimates of the Lagrangian integral timescale suggest that \( \tau_L \approx 0.1 \) sec, similar to the mean travel time estimated for particles in R2B and R3B [Roseberry et al., 2012]. We may therefore assume that \( \kappa_x = k\bar{u}_p^2 \bar{T}_p = k\bar{u}_p \bar{I}_x \), where \( k \) is a dimensionless factor of order unity.

For cross-stream motions the autocovariance \( C_y(\tau) \) decays to zero by about \( \tau \approx 0.1 \) sec (Figure 11). The integral in (17) converges, where numerically computed values level off at \( \tau \approx 0.1 \) sec (Figure 12), giving estimates of \( \kappa_y \) from about 0.05 cm\(^2\) s\(^{-1}\) to 0.1 cm\(^2\) s\(^{-1}\) over the range of experimental conditions, approximately an order of magnitude smaller than values of \( \kappa_x \).

We emphasize that calculated values of \( C_s(\tau) \) and \( C_s(\tau) \) at a given interval \( \tau \) are based on all velocity signals for which \( T_p \geq \tau \). Thus, for small \( \tau \) these values are based on the velocity signals of most particles in each experiment, and for large \( \tau \) these values are based on fewer signals with longer travel times. Uncertainty in the estimates of \( C_s(\tau) \) and \( C_s(\tau) \) (or \( A_s(\tau) \) and \( A_s(\tau) \)) therefore increases with increasing \( \tau \). Estimates at \( \tau \approx \tau_L \) cannot be interpreted as being significantly different from zero.
5. Discussion and Conclusions

As pointed out in Roseberry et al. [2012], problems of diffusion in molecular systems typically involve processes in which an individual particle experiences, say, \(10^6 - 10^{10}\) collisions per second. Motions continue indefinitely, and any scale invariant superdiffusive behavior (as characterized, for example, by hard sphere theory) emerges rapidly and persists. In ecological systems, hundreds to thousands of “collisions” (meaning changes in direction) of a “particle” — such as an albatross or a honey bee — can occur during an individual Lévy flight [Viswanathan et al., 1996; Reynolds et al., 2007]. In contrast, the sediment particle motions described herein involve a few to tens of collisions with the bed during one particle hop. The (apparent) superdiffusive behavior manifest in plots of the mean squared displacement \(R(\tau)\) over a timescale \(0.01 \leq \tau \leq 0.1\) sec actually reflects the effects of periodicities that are inherent in streamwise particle motions, not (scale invariant) superdiffusion — a behavior that cannot in any case persist at longer timescales.

The idea that bed load particle motions exhibit a ballistic-like behavior at small time intervals \(\tau\) is not the same as ballistic behavior as (conventionally) defined for molecular systems in which particles travel unimpeded within a vacuum between collisions. For example, because the mean-free time and associated mean-free path of air molecules are small at Earth-surface pressure and temperature conditions, molecular motions can be approximated as straight lines with constant velocity between collisions. (In detail these motions are parabolic in Earth’s gravitational field, decidedly so at the rarified conditions of the outer atmosphere.) In contrast, bed load particle motions are strongly coupled with fluid motions, and particles rarely move faster than the fluid. Insofar as bed load particles travel at approximately constant velocity between collisions with the bed — leading to \(R(\tau) \sim \tau^2\) — then this is a result of the particle-fluid coupling, not true ballistic behavior. The magnitudes of cross-stream particle velocities typically are much smaller than streamwise velocities [Roseberry et al., 2012], and the absence of ballistic-like behavior in plots of \(R(\tau)\) at small \(\tau\) likely reflects imprecision in our measurements of small (cross-stream) particle displacements.

At any instant, and from one instant to the next, the \(N\) active particles within the streambed area \(B\) represent all possible stages of motions over each possible particle hop represented by the underlying distribution of hop distances, \(f_{L_x}(L_x)\) and \(f_{L_y}(L_y)\). It is therefore appropriate to calculate \(R_x(\tau), R_y(\tau), C_x(\tau)\) and \(C_y(\tau)\) based on all paired observations of \(x_p, y_p, u_p\) and \(v_p\) separated by the interval \(\tau\) (as opposed to setting the initial time \(t\) of each motion to zero and calculating averages only across the number of motions for each time \(\tau\)) in order to ensure that these quantities represent ensemble averages. For example, a value of \(R_x(\tau)\) or \(C_x(\tau)\) calculated for small \(\tau\) includes those particles near the end of long hops as well as those near the beginning of short hops and long hops, in proportion to the likely occurrence of the various stages of motion of the \(N\) active particles sampled from all hops at any instant.

That the autocovariances \(C_x(\tau)\) and \(C_y(\tau)\) decay to zero over a short interval \(\tau\) indicates a Fickian-like diffusive behavior [Taylor, 1921; Garrett, 2006; Ferrari, 2007] in both streamwise and cross-stream particle motions. In contrast, if \(C(\tau)\), for example, possesses a long tail, then (15) does not converge rapidly and the diffusivity increases, albeit slowly, at times larger than \(\tau_x\) [Garrett, 2006]. But here it is important to reemphasize that bed load particle motions do not continue indefinitely, so in fact a finite \(C_x(\tau)\) for \(\tau \gg \tau_x\) is not physically meaningful in characterizing diffusive behavior. More basically, a Fickian-like behavior is anticipated from the exponential-like particle velocity distributions of \(u_p\) and \(v_p\), possibly involving “light” tails [Roseberry et al., 2012; Furbish et al., 2012b], where the diffusivity arises from the time derivative of the second moment of the underlying probability density function of particle displacements occurring during a small
The rapid decay of $C_\tau(\tau)$ and $C_y(\tau)$ over $\tau$ provides a clearer measure of diffusive behavior than does the form of $R_\tau(\tau)$ or $R_y(\tau)$, which is intrinsically sensitive to effects of periodicities in particle motions that start and stop with an average travel time $\tau_t = \tau$. As described in companion papers [Furbish et al., 2012a, 2012b], the volumetric bed load sediment flux involves an advective part equal to the product of the average particle velocity and the particle activity (the solid volume of particles in motion per unit streambed area), and a diffusive part involving the gradient of the product of the particle activity and the diffusivity. This diffusive contribution to the flux may be important under conditions of nonuniform transport, and Taylor’s formulation of the diffusivity, as described above, yields a proper description of the diffusion of bed load particles at low transport rates, consistent with Fickian diffusion. The problem of diffusion (dispersion) of tracer particles involving the effects of multiple hops and rest times [Sayre and Hubbell, 1965; Drake et al., 1988; Hassan and Church, 1991; Ferguson and Wathen, 1998; Nikora et al., 2002; Martin et al., 2012] requires a different formalism [Bradley et al., 2010; Ganti et al., 2010; Hill et al., 2010].

The analysis also points to the design of experimental measurements required to obtain precise estimates of the particle diffusivity and related quantities. Here are lessons we have learned. To confidently estimate the displacement and velocity statistics described above, including the mean hop distance and travel time (and their distributions), the sampling window size and time interval are critical. The window must be large enough, and the sampling time must be long enough ($T_o > \tau_t$), to obtain a sufficient count of hops representing the full range of hop distances occurring in the near-bed conditions of turbulence. Our runs R2B and R3B, like R1 and R2, had sufficiently large windows, but suffered from short sampling times. Runs R3 and R5, like R1 and R2, involved sufficient sampling times, but suffered from small windows. Moreover, to see ballistic-like behavior requires a sampling interval shorter than the typical interval between particle-bed collisions. Experiments involving a wider range of flow conditions and particle sizes are required to clarify the relation between particle velocities and diffusivities as suggested in section 4.2.

**Notation**

- $a$ amplitude of sinusoidal particle displacement [L].
- $A_x, A_y$ autocorrelation of streamwise and cross-stream particle velocities.
- $B$ streambed area [L$^2$].
- $C_x, C_y$ autocovariance of streamwise and cross-stream particle velocities [L$^2$ t$^{-2}$].
- $D$ particle diameter [L].
- $f_{L_x}, f_{L_y}$ probability density functions of streamwise and cross-stream particle hop distances [L$^{-1}$].
- $f_{\tau}$ probability density function of particle travel times [t$^{-1}$].
- $f_{L_x, L_y, \tau}$ joint probability density function of particle hop distances and travel times [L$^{-1}$ t$^{-1}$].
- $f_{u_x, u_y}$ probability density functions of streamwise and cross-stream particle velocities [L$^{-1}$ t$^{-1}$].
- $F$ function defined by (6).
- $k$ dimensionless factor of order unity.
- $L_x, L_y$ streamwise and cross-stream particle hop distances [L].
- $m$ fraction of travel time.
- $N$ number of active particles within the streambed area $B$. 

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References


Figure Captions

Figure 1. Plot of examples of (a) streamwise particle displacement $x_p$ versus time $t$ and (b) deviation in particle displacement about the average motion, $x_p - (L/T_p)t$, versus time $t$ for particles that gradually accelerate from rest to their peak velocities then gradually decelerate before returning to rest; in this case the displacement about the average motion is approximately sinusoidal with a principal (dominant) harmonic whose period $T$ is equal to the travel time, namely $T = T_p$.

Figure 2. Plot of examples of (a) streamwise particle displacement $x_p$ versus time $t$ and (b) deviation in particle displacement about the average motion, $x_p - (L/T_p)t$, versus time $t$ for particles that undergo a brief interval of rapid acceleration from rest to a peak velocity followed by gradual deceleration before deposition, or conversely, a gradual acceleration followed by a brief interval of rapid deceleration in returning to rest; in this case the displacement about the average motion appears as a dominant harmonic whose period $T_{2T_p}$.

Figure 3. Plot of examples of (a) streamwise particle displacement $x_p$ versus time $t$ and (b) deviation in particle displacement about the average motion, $x_p - (L/T_p)t$, versus time $t$ for particles that accelerate then decelerate more than once during a full hop such that the displacement about the average motion involves a (dominant) harmonic whose period $T = mT_p$.

Figure 4. Plot of cross-stream particle displacements $y_p$ versus time $t$ for particles motions illustrated in (a) Figure 1 and (b) Figure 2.

Figure 5. Plots of calculated (circles) and theoretical (solid lines) values of the mean squared displacement $R_p(\tau)$ versus time interval $\tau$ for particle motions illustrated in (a) Figure 1 with $T = T_p$, (b) Figure 2 with $T = 2T_p$, and (c) Figure 3 with $T = 0.5T_p$ (black), $T = 0.4T_p$ (light gray), $T = 0.7T_p$ (dark gray) and $T = 0.5T_p$ (white); dashed lines are given by (7) and possess slopes of $\sigma = 2$.

Figure 6. Plot of mean squared displacement $R_p(\tau)$ versus time interval $\tau$ for all particle motions within experiment R1 (black), R2 (light gray circles), R3 (white circles), R2B (light gray triangles) and R3B (white triangles; eye-fit estimates of the slope of $R_p(\tau)$ over the domain $0.01 \leq \tau \leq 0.1$ sec vary from about 1.5 to 1.8, and dashed line possesses slope of $\sigma = 2$.

Figure 7. Plot of (a) amplitude $a$ versus hop distance $L$ and (b) hop distance $L$ versus mean velocity $L/T_p$ for particles completing full hops in R2B (white) and R3B (black); not shown is one datum in (a) with coordinates $(L, a) = (2.31, 0.74)$.

Figure 8. Plot of mean squared displacement $R_p(\tau)$ versus time interval $\tau$ for all particle motions within each experiment; eye-fit estimates of the slope of $R_p(\tau)$ over the domain $0.01 \leq \tau \leq 0.1$ sec vary from about 1 to 1.8.

Figure 9. Plot of autocorrelation $A_\tau(\tau) = C_\tau(\tau)/\sigma_{\tau p}^2$ of streamwise particle velocities $u_p$ calculated for R1 (black), R2 (light gray circles), R3 (dark gray circles), R5 (white), R2B (light gray triangles) and R3B (dark gray triangles).

Figure 10. Plot of integral of the autocovariance $C_\tau(\tau)$ of streamwise particle velocities $u_p$ calculated for R1 (black), R2 (light gray circles), R3 (dark gray circles), R5 (white), R2B (light gray triangles) and R3B (dark gray triangles).

Figure 11. Plot of autocorrelation $A_\tau(\tau) = C_\tau(\tau)/\sigma_{\tau p}^2$ of cross-stream particle velocities $v_p$ calculated for R1 (black), R2 (light gray circles), R3 (dark gray circles), R5 (white) and R2B (light gray triangles).

Figure 12. Plot of integral of the autocovariance $C_\tau(\tau)$ of cross-stream particle velocities $v_p$
calculated for R1 (black), R2 (light gray circles), R3 (dark gray circles), R5 (white) and R2B (light gray triangles).

**Table 1. Experimental Conditions.**

<table>
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<tr>
<th>Run</th>
<th>Sampling window size (pixels)</th>
<th>Run time (sec)</th>
<th>Sampling interval (sec)</th>
<th>Mean activity (number cm⁻²)</th>
<th>Mean particle velocity (cm s⁻¹)</th>
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<td>2.83</td>
<td>4.57</td>
</tr>
</tbody>
</table>
Figure 1

(a) Displacement $x_p$ (cm) vs. Time $t$ (sec)

(b) Deviation $x_p - (L/T_p)$ vs. Time $t$ (sec)
Figure 2

(a) Displacement $x_p$ (cm) vs. Time $t$ (sec)

(b) Deviation $x_p - (L/T_p)$ (cm) vs. Time $t$ (sec)
Figure 3

(a) Displacement $x_p$ (cm) vs. Time $t$ (sec)

(b) Deviation $x_p - (L/T_p)t$ (cm) vs. Time $t$ (sec)
Figure 4
Figure 6

Mean squared displacement $R_x(\tau)$ (cm$^2$) vs. time interval $\tau$ (sec).

- Time interval $\tau$ (sec): 0.001, 0.01, 0.1, 1
- Mean squared displacement $R_x(\tau)$ (cm$^2$): 0.0001, 0.001, 0.01, 0.1, 1
Figure 8
Figure 9

Autocorrelation $A_x(\tau)$ vs. Lag interval $\tau$ (sec)
Figure 10

Integral of covariance $C_x(\tau)$ (cm$^2$ s$^{-1}$)

Lag interval $\tau$ (sec)
Figure 12

Integral of covariance $C_y(\tau)$ (cm$^2$ s$^{-1}$) vs. Lag interval $\tau$ (sec)