

Partisan Strength and Legislative Bargaining

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1. Introduction

Casual observers of American politics are constantly bombarded with headlines about how contemporary legislative politics have become increasingly polarized and partisan. Whether due to the natural geographic sorting of voters into more ideologically homogeneous constituencies (e.g., Dodd and Oppenheimer, 2017; Chen and Rodden, 2013), gerrymandering (e.g., La Raja, 2009), or the general empowerment of party leaders (e.g., Aldrich and Rohde, 2017), many scholars of American politics have suggested that parties in the United States Congress are more ideologically divergent than they have been for over a century (e.g., Hare and Poole, 2014). Likewise, scholars of political psychology (e.g., Hetherington and Rudolph, 2015) have argued that this increase in party polarization has corresponded to a decrease in trust across the parties, which makes the policymaking process increasingly contentious in Washington.

These scholarly and journalistic accounts suggest that the rise of partisan conflicts in Congress has led to a rise in legislative gridlock or, at least, an increase in the incidence of one-sided policy outcomes being obtained, with the minority party being effectively shut out of the policymaking process. As legislators have come to place greater value on partisanship, majority party members have had less of an incentive to reach out to members of the minority party for their support even on relatively minor legislative issues (Lee, 2016); and the majority party leadership has had more of an incentive to make concessions to dissonant voices in their own party. In addition, members of the minority party are increasingly expected to do whatever they can to prevent a policy victory for the majority party, even if that victory might benefit the minority party's interests as well.

While such arguments seem sensible on their face, and comport nicely with recent journalistic accounts of contemporary legislative politics, other factors may limit their applicability. Of particular importance is the way that partisanship interacts with the value that legislators place on future outcomes, such as a party obtaining (and/or retaining) majority party status or successfully implementing its agenda in later sessions of the legislature. It is plausible to conjecture that in *some* circumstances, in order to secure policy agreements the leadership of a majority party would be either willing to reach out to members of the minority party or willing not to cater to extreme interests in its own party. It is also plausible to believe that there are circumstances in which members of a minority party are able to offer some members of the majority party *something* so as to secure policy outcomes that they find mutually advantageous.

To engage with these issues, we develop a model of legislative bargaining over a fixed divisible resource, a dollar, which builds on the models of Baron and Ferejohn (1989) and Calvert and Dietz (2005). Our model accounts for both the strength of a legislator's party affiliation and the extent to which future outcomes

matter for current negotiations. Specifically, we consider three legislators, two of whom (the partisans) belong to the same political party, who bargain over how to divide a dollar. In each period until majority agreement is reached, a legislator is recognized with equal probability to make a proposal. Future shares are discounted by a common discount factor. The nonpartisan's payoff is simply the present value of his share of the dollar. A partisan's payoff, however, consists of a fraction of what his copartisan obtains added to the amount that he receives (discounted to the present). The weight that a partisan places on his copartisan's share of the dollar is our measure of the strength of partisanship (i.e., of party affiliation). We restrict attention to partisan symmetric stationary subgame perfect equilibria in this bargaining game. As in Baron and Ferejohn (1989) and Calvert and Dietz (2005), agreement is reached without delay.¹

We allow both the party affiliation parameter and the discount factor to vary. Hence, our model allows us to assess how the outcome of legislative bargaining depends on both the strength of party affiliation and on the value that legislators place on future payoffs. When there are three legislators, the models of Baron and Ferejohn (1989) and Calvert and Dietz (2005) are special cases of our own. Baron and Ferejohn permit the discount factor to vary, but assume that legislators only care about their own shares. Calvert and Dietz allow for any degree of partisan affiliation, but only consider the case in which the legislators are infinitely patient (i.e., the discount factor is 1).

In our main proposition, we provide a complete description of the equilibrium of our model as a function of the strength of party affiliation and the discount factor. We show that the range of possible values for the discount factor can be partitioned into three intervals whose boundaries depend on the value of the party affiliation parameter, with the qualitative features of the equilibrium proposals differing between these intervals. When legislators place a high value on potential future interactions, a majority party proposer is willing to seek the support of either his copartisan or the nonpartisan by offering enough of the dollar to obtain a vote in favor of his proposal. Thus, bipartisan coalitions can be obtained in equilibrium with positive probability in this case. For smaller values of the discount factor, however, the prospects for bipartisan agreements break down. For intermediate values, a majority party proposer is only willing to split the dollar between himself and his copartisan. When legislators are sufficiently impatient, a majority party proposer keeps the whole dollar for himself, relying on the strength of his copartisan's party affiliation to obtain his support. (Only the first of these three possibilities is identified by Calvert and Dietz (2005) because they assume

¹Payoff externalities also feature in Montero (2007). Montero extends the n -person Baron-Ferejohn legislative bargaining model by having each legislator care about the equitableness of the division of the dollar, not just his own share.

that the discount factor takes on its highest possible value of 1.)

In addition to characterizing the equilibria of our model, we also perform a number of comparative static exercises. In particular, we investigate how the features of the equilibrium proposals and how the shares of the dollar that each legislator can expect to receive, *ex ante*, respond to changes in the strength of party affiliation and in the discount factor.

We are not the first to model legislative partisanship. The models of negative agenda setting analyzed by Cox and McCubbins (2005), for example, account for the role of partisanship by identifying the majority party median as the agent that has particular procedural rights; specifically, the power to keep policy proposals off the agenda. Likewise, Krehbiel et al. (2015) account for legislative partisanship by identifying party agenda setters as the agents that have particular parliamentary rights and/or the ability to engage in vote buying. Krehbiel and Meirowitz (2002) assume that party affiliations determine which agents move first in advancing a sequence of policy proposals and amendments in the legislative process, with a majority party member moving first, followed by the minority party legislator. However, none of these models (nor many others), parsimoniously capture the value of partisanship to rank-and-file members of the majority or minority parties independent of the spatial location of their ideal points (which are often assumed to be positively correlated with party affiliation) or their procedural rights. Hence, analysis of our model reveals numerous insights that speak to ongoing debates about the impact of partisanship on legislative politics; and the model is sufficiently tractable so as to facilitate further analytical extensions in future scholarship.

2. The Partisan Legislative Bargaining Game

There are three legislators (indexed by $i = 1, 2, 3$) whose objective is to determine how to distribute a unit of benefits, which we hereafter refer to as a *dollar*, among themselves. A *distribution* is a nonnegative vector $x = (x_1, x_2, x_3)$ for which $\sum_i x_i = 1$. The set of possible distributions is X .

As in the baseline model analyzed by Calvert and Dietz (2005), legislators 1 and 2 are members of the same party (hereafter referred to as the *partisans*), each of whom cares not only for his own share of the dollar, but also for the share received by his *copartisan*. Each of the partisan legislators assigns the same weight to his copartisan's share, with this weight being strictly less than the weight of 1 that he assigns to his own share. Legislator 3, the *nonpartisan*, only cares about his own share of the dollar. Formally, the three legislators' utilities obtained with the distribution x in the period in which x is adopted are:

$$U^1(x) = x_1 + \alpha x_2, \tag{1}$$

$$U^2(x) = x_2 + \alpha x_1, \tag{2}$$

and

$$U^3(x) = x_3, \tag{3}$$

where $\alpha \in [0, 1)$. The parameter α measures the extent to which either of the partisans takes account of his copartisan's share of the dollar. It can be interpreted as being the value of party affiliation. If $\alpha = 0$, each partisan legislator only cares about his own share, which is the case considered by Baron and Ferejohn (1989).

Legislators discount future payoffs using a common *discount factor* of $\delta \in [0, 1]$. Thus, if t periods in the future legislator i receives a utility payoff of u^i , this has a present value of $\delta^t u^i$. If $\delta = 1$, there is no discounting, which is the case considered by Calvert and Dietz (2005). When $\delta < 1$, the legislators exhibit impatience, preferring to receive benefits sooner rather than later.

As in Baron and Ferejohn (1989), in the initial period, a legislator is recognized with equal probability to make a proposal for dividing the dollar. We assume that the legislature uses a *closed rule* in which the distribution x that has been proposed is voted on without amendment against the status quo. If a simple majority votes in favor of x , the bargaining ends and the dollar is distributed according to the agreed upon allocation. If x fails to secure a majority, however, this procedure is repeated with a one period delay. Bargaining continues until there is majoritarian agreement.

A *strategy* for a legislator consists of a proposal rule and a voting rule. A legislator's *proposal rule* specifies for each period what proposal he would make should he be recognized. His *voting rule* specifies for each period which distributions he would vote for. In general, the decisions in each period could depend on the past history of the bargaining. Following Baron and Ferejohn (1989) and Calvert and Dietz (2005), however, we restrict attention to stationary strategies in which the decisions made in any period are not contingent on past history. With a *stationary strategy*, a legislator makes the same proposal every time he is recognized and only considers the current distribution being voted on when deciding whether to support it. We exploit the symmetry of legislators 1 and 2 in the model by supposing that legislator 2's proposal is the same as that of legislator 1 *mutatis mutandis*.² We call such a strategy *partisan symmetric*. Our equilibrium concept is partisan symmetric stationary subgame perfect equilibrium. In a *partisan symmetric stationary subgame perfect equilibrium*, (i) each legislator uses a partisan symmetric stationary strategy and (ii) the profile of the three legislators' strategies is a Nash equilibrium when restricted to any subgame.

Because the proposer only needs the support of one of the other legislators to obtain a majority, he will only offer just enough of the dollar to one other legislator

²This assumption merely facilitates equilibrium selection rather than preventing any legislator from playing a best response.

so as to secure his support. For legislator 1, he proposes the distribution

$$x^{12} = (1 - x_P, x_P, 0) \quad (4)$$

if he wants to secure the support of his copartisan and the distribution

$$x^{13} = (1 - x_N, 0, x_N) \quad (5)$$

if he wants to secure the support of the nonpartisan. He chooses to seek the support of his copartisan with probability p and of the nonpartisan with probability $1 - p$, where $p \in [0, 1]$. In summary, a *proposal for legislator 1* consists of (i) the two distributions x^{12} and x^{13} and (ii) the probability p with which the first distribution is offered.³

A *proposal for legislator 2* consists of the distributions

$$x^{21} = (x_P, 1 - x_P, 0) \quad (6)$$

and

$$x^{23} = (0, 1 - x_N, x_N) \quad (7)$$

together with the probability p that the first distribution is offered.

Because legislator 3 wants to secure the support of one other legislator at the least cost to himself, he will offer the same amount x_S to secure the support of either of them. Because the other two legislators stand in exactly the same relationship to legislator 3, we suppose that he seeks their support with equal probability. Thus, a *proposal for legislator 3* consists of the distributions

$$x^{31} = (x_S, 0, 1 - x_S) \quad (8)$$

and

$$x^{32} = (0, x_S, 1 - x_S), \quad (9)$$

with each offered with probability $1/2$.⁴

A legislator's *continuation value* is his expected utility at the beginning of the next period should agreement not be reached in the current period. With our stationarity assumption, this value is time invariant. The continuation value

³Because of the utility externality due to party affiliation, the utility that a copartisan obtains when a partisan makes an offer to the nonpartisan may be sufficient to obtain his support even if he is not offered any of the dollar himself. Nevertheless, it is never in the interest of a partisan to make an unacceptable offer to the nonpartisan so as to obtain his copartisan's support because if he simply kept the amount offered to the nonpartisan, the utilities of both partisans would be increased. Thus, a partisan only makes an offer to the nonpartisan if he wants his support.

⁴The symmetry assumptions we have made in defining the three proposals are the same as in Calvert and Dietz (2005) and are analogous to those made implicitly in Proposition 3 of Baron and Ferejohn (1989).

discounted by one period is the utility that a legislator must be provided in order to gain his support for the distribution being voted on.

The continuation value of legislator i is:

$$V^i = \frac{1}{3} [pU^i(x^{12}) + (1-p)U^i(x^{13})] + \frac{1}{3} [pU^i(x^{21}) + (1-p)U^i(x^{23})] + \frac{1}{3} \left[\frac{1}{2}U^i(x^{31}) + \frac{1}{2}U^i(x^{32}) \right]. \quad (10)$$

The first term in square brackets in this expression is the expected utility obtained with legislator 1's proposal. Similarly, the second and third terms in square brackets are his expected utilities with the proposals of legislators 2 and 3, respectively. All three terms receive a weight of $1/3$ because each legislator is recognized with equal probability. Using (1)–(9), the continuation values in (10) can be rewritten as

$$V^1 = V^2 = \frac{1}{3} (1 + \alpha) \left[1 - (1-p)x_N + \frac{1}{2}x_S \right], \quad (11)$$

and

$$V^3 = \frac{1}{3} [2(1-p)x_N + 1 - x_S]. \quad (12)$$

Henceforth, we refer to the game introduced in this section as the *Partisan Legislative Bargaining Game*.

3. Benchmark Cases

Before analyzing our general model of partisan bargaining, we begin by presenting the benchmark cases that are considered by Baron and Ferejohn (1989) and Calvert and Dietz (2005). The former is the special case of our model in which legislators are purely self-interested, but discounting of future payoffs is permitted. The latter is the special case in which partisans may care about each other's welfare, but there is no discounting.

3.1. Baron–Ferejohn

Baron and Ferejohn (1989, pp. 1191–1193) consider the special case of our model in which $\alpha = 0$.⁵ Because each legislator is only concerned with his own share of the benefit, party affiliation has no influence over the equilibrium allocations of the dollar.

For the case where $n = 3$, in equilibrium, any proposer offers with equal probability the fraction $\delta/3$ of the dollar to one of the other two legislators, with

⁵Baron and Ferejohn (1989) allow for an arbitrary odd number of legislators. Our description of their equilibrium is for the three legislator case illustrated in their Figure 3.

the remaining $1 - \delta/3$ allocated to himself. Each legislator has a continuation value of $1/3$, as can be confirmed by setting $\alpha = 0$, $p = 1/2$, and $x_N = x_S = \delta/3$ in (11) and (12). Thus, $\delta/3$ is what needs to be offered to gain a legislator's support; so each legislator will vote for any offer of at least this amount. Thus, the distribution voted on in the first period is adopted, and each pair of legislators forms a winning coalition with equal probability.

The proposer's share is decreasing in δ . If $\delta = 1$, the legislators are infinitely patient, with the consequence that in equilibrium, the proposer receives $2/3$, whereas any coalition partner receives $1/3$. If $\delta = 0$, however, nobody cares about the future, thereby removing any threat value of withholding support for a proposal. Hence, the proposer can allocate all of the dollar to himself.

3.2. Calvert–Dietz

Calvert and Dietz (2005, sec. 4.1) consider the special case of our model in which there is no discounting ($\delta = 1$). As we do here, they allow for any amount of symmetric altruism between the two partisans provided that a partisan assigns less weight to his copartisan than to himself.

The equilibrium values of the choice variables are functions of the value α that partisans place on each other's share. A partisan proposer (i) seeks the support of his copartisan with probability $p = (1 + \alpha)/2$ and (ii) he offers $x_P = 1/(\alpha + 3)$ to his copartisan and $x_N = (1 - \alpha)/(\alpha + 3)$ to the nonpartisan. The nonpartisan offers $x_S = \alpha + x_N = (\alpha^2 + 2\alpha + 1)/(\alpha + 3)$ with equal probability to the other two legislators, if recognized. The continuation values are x_S for each of the partisans, and x_N for the nonpartisan, with each legislator voting to accept any distribution that provides him with his continuation value. Thus, as in the Baron–Ferejohn equilibrium, agreement is reached without delay.

For the special case in which $\alpha = 0$, the equilibrium is identical to the Baron–Ferejohn equilibrium for the case in which there is no discounting. The probability that a partisan seeks the support of his copartisan is increasing in the parameter α , attaining its minimum value of $1/2$ when $\alpha = 0$ and approaching a limiting maximum value of 1 as α goes to 1 . A partisan proposer exploits his copartisan's party affiliation by decreasing his offer x_P to him as the value of α is increased. The offer attains its maximum value of $1/3$ when $\alpha = 0$ and it is bounded below by the limiting value of $1/4$ as α goes to 1 . As α increases, there is less need for a partisan to solicit the support of the nonpartisan. Consequently, the offer x_N made to the nonpartisan is decreasing in α , attaining its maximum value of $1/3$ when $\alpha = 0$ and having a value of 0 in the limit when α goes to 1 . The difference $x_P - x_N$ in the offers made to a copartisan and to the nonpartisan is increasing in α , ranging from 0 when $\alpha = 0$ to a limiting value of $1/4$ as α approaches 1 . As α increases, the nonpartisan must make an ever more generous offer x_S to gain

the support of a partisan, ranging from an offer of $1/3$ when $\alpha = 0$ to a limiting value of 1 as α approaches 1. The probability that the partisans form a winning coalition is equal to $2p/3 = (1 + \alpha)/3$. This value is increasing in α , attaining its minimum value of $1/3$ when $\alpha = 0$ and approaching a limiting maximum value of $2/3$ as α goes to 1.

3.3. Comparison of the Benchmarks

In comparing across these benchmark results, we see that regardless of the strength of party affiliation, with positive probability, a partisan proposer in the Calvert–Dietz model offers the nonpartisan a share of the dollar. In other words, when $\delta = 1$, there is always the prospect for a bipartisan outcome. Moreover, should a partisan offer a positive share to his copartisan, he retains a larger share of the dollar $((2 + \alpha)/(3 + \alpha))$ than the share retained ($2/3$) in the Baron–Ferejohn model when there is no discounting. Nevertheless, the copartisan supports receiving a smaller share himself because of the benefit he receives from the partisan proposer’s share. Because x_N and $1 - x_S$ are decreasing in α , if the nonpartisan legislator is part of the winning coalition, he receives less of the dollar in the Calvert–Dietz equilibrium than he would be in the Baron–Ferejohn equilibrium regardless of who makes the proposal. Of course, if he is not part of the winning coalition, he receives nothing in either case.

4. Equilibria of the Partisan Legislative Bargaining Game

We begin our analysis of the more general model by noting that the equilibrium of the Partisan Legislative Bargaining Game depends on the values of the preference parameter α and the discount factor δ . For each value of α , let

$$\underline{\delta}(\alpha) = \frac{6\alpha}{(1 + \alpha)(2 + \alpha)} \quad (13)$$

and

$$\bar{\delta}(\alpha) = \frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1 + \alpha)}. \quad (14)$$

The graphs of these two functions are shown in Figure 1.⁶ Both of these functions are increasing in α , with $\underline{\delta}(0) = \bar{\delta}(0)$ and both $\underline{\delta}(\alpha)$ and $\bar{\delta}(\alpha)$ having limiting values of 1 as α goes to 1. Furthermore, $0 < \underline{\delta}(\alpha) < \bar{\delta}(\alpha) < 1$ when $0 < \alpha < 1$. For a given value of α , the critical values $\underline{\delta}(\alpha)$ and $\bar{\delta}(\alpha)$ partition the interval $[0, 1]$ into three regions: $[\bar{\delta}(\alpha), 1]$, $[\underline{\delta}(\alpha), \bar{\delta}(\alpha))$, and $[0, \underline{\delta}(\alpha))$, which we refer to as Regions 1, 2, and 3, respectively. As we shall see, the qualitative features of the

⁶The *Mathematica* code for plotting the graphs in Figures 1 and 3 is provided in the Appendix.

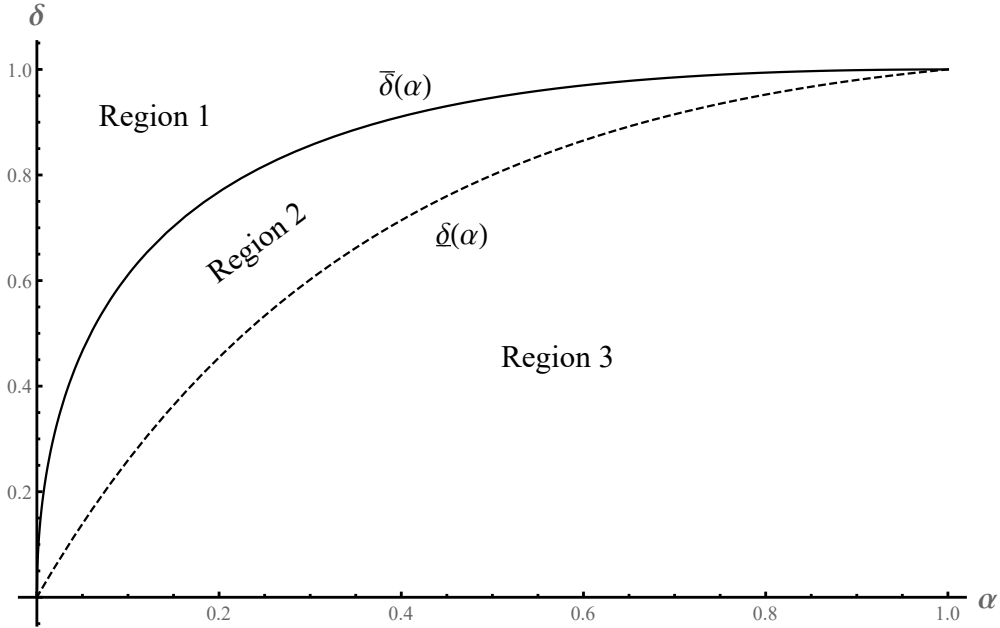


Figure 1: Partisan Legislative Bargaining Game Equilibrium Regions

equilibrium depend on which of these three regions δ is in.⁷ Note that if $\alpha = 0$, only the first region is nonempty, whereas for any $\alpha \in (0, 1)$, all three regions are nonempty.

For each $\alpha \in [0, 1)$ and $\delta \in [0, 1]$, the equilibrium of the Partisan Legislative Bargaining Game is characterized in Proposition 1.⁸

Proposition 1. *A set of strategies is a partisan symmetric stationary subgame perfect equilibrium if and only if*

- (a) For any $\delta \in [\bar{\delta}(\alpha), 1]$,
 - (i) when a partisan is the proposer,
 - i. with probability $p = \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{-2\delta(-2\alpha\delta + \delta + \alpha\delta)}$, he offers $x_P = \frac{-2\alpha + \alpha\delta + \delta}{(3+\alpha)(1-\alpha)}$ to his copartisan, 0 to the nonpartisan, and proposes for himself to receive $\frac{3-\alpha^2-\delta-\alpha\delta}{(3+\alpha)(1-\alpha)}$;
 - ii. with probability $1 - p$, he offers $x_N = \frac{-2\alpha + \alpha\delta + \delta}{3+\alpha}$ to the nonpartisan, 0 to his copartisan, and proposes for himself to receive $\frac{(1+\alpha)(3-\delta)}{3+\alpha}$;

⁷The equilibrium values of p , x_P , x_N , and x_S are continuous in the parameter values. In our discussion, it is sometimes more convenient to include $\bar{\delta}(\alpha)$ in Region 2 and $\delta(\alpha)$ in Region 3.

⁸The proof of Proposition 1 and all subsequent results are in the Appendix.

- (ii) when the nonpartisan is the proposer, with equal probability, he offers $x_S = \frac{(1+\alpha)(\alpha+\delta)}{3+\alpha}$ to one of the partisans and 0 to the other and proposes for himself to receive $\frac{3-\alpha^2-\delta-\alpha\delta}{3+\alpha}$.

Regardless of who is the proposer, the nonpartisan votes for any distribution in which he receives at least $\frac{-2\alpha+\alpha\delta+\delta}{3+\alpha}$ and either partisan votes for any distribution that offers him utility at least $\frac{(1+\alpha)(\alpha+\delta)}{3+\alpha}$.

- (b) For any $\delta \in [\underline{\delta}(\alpha), \bar{\delta}(\alpha))$, when a partisan is the proposer, with probability 1, he offers $x_P = \frac{2\delta+3\alpha\delta+\alpha^2\delta-6\alpha}{(6-\delta-\alpha\delta)(1-\alpha)}$ to his copartisan, 0 to the nonpartisan, and proposes for himself to receive $\frac{6-3\delta-3\alpha\delta}{(6-\delta-\alpha\delta)(1-\alpha)}$.
- (c) For any $\delta \in [0, \underline{\delta}(\alpha))$, when a partisan is the proposer, with probability 1, he offers 0 to both of the other legislators and proposes for himself to receive 1.
- (d) For any $\delta \in [0, \bar{\delta}(\alpha))$, when the nonpartisan is the proposer, with equal probability, he offers $x_S = \frac{2\delta(1+\alpha)}{6-\delta-\alpha\delta}$ to one of the partisans and 0 to the other, and proposes for himself to receive $\frac{3(2-\delta-\alpha\delta)}{6-\delta-\alpha\delta}$.
- (e) In (b), (c), and (d), regardless of who is the proposer, the nonpartisan votes for any distribution in which he receives at least $\frac{\delta(2-\delta-\alpha\delta)}{6-\delta-\alpha\delta}$ and either partisan votes for any distribution that offers him utility at least $\frac{2\delta(1+\alpha)}{6-\delta-\alpha\delta}$.

Each of the distributions proposed receives the support of a majority.

The equilibrium proposals characterized in Proposition 1 are summarized in Figure 2. For both a partisan and nonpartisan proposer, what is offered if recognized depends on the values of the party affiliation parameter α and the discount factor δ . Specifically, the proposal depends on the value of δ relative to the thresholds $\underline{\delta}(\alpha)$ and $\bar{\delta}(\alpha)$; that is, on which of the three regions in Figure 1 δ lies in.

If a partisan proposer is recognized, working from right to left in Figure 2, for $\delta \in (\bar{\delta}(\alpha), 1]$ (Region 1), he offers a positive share of the dollar to one of the other legislators, with the choice of who is to receive the offer determined probabilistically. For $\delta \in (\underline{\delta}(\alpha), \bar{\delta}(\alpha)]$ (Region 2), a partisan proposer offers a positive share to his copartisan and excludes the nonpartisan from the bargaining process. Finally, for $\delta \in [0, \underline{\delta}(\alpha)]$ (Region 3), a partisan proposer proposes to keep the whole dollar for himself. It is noteworthy that the finding by Calvert and Dietz (2005) that a partisan proposer offers a nonzero share of the dollar to a nonpartisan with positive probability when there is no discounting does not hold for all values of the discount factor. Rather, the bipartisanship that Calvert and Dietz identify *only* holds for values of δ in Region 1; that is, only if the legislators have sufficient concern about the future.

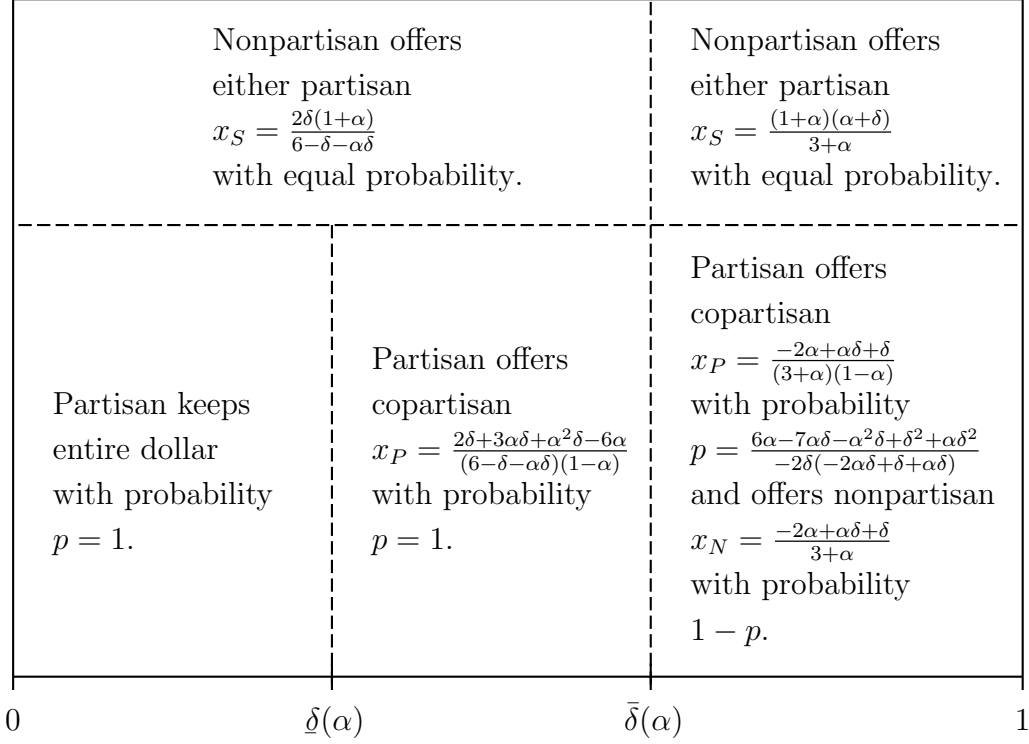


Figure 2: Equilibrium Proposals in the Partisan Legislative Bargaining Game

In all three regions, the nonpartisan proposer makes a positive offer to one of the other legislators if recognized, provided that $\delta \neq 0$. Because of our symmetry assumption, each of them has an equal chance of receiving the offer, but the size of the offer does not depend on who it is made to. The formula used to determine the magnitude of the nonpartisan's offer depends on whether δ is in Regions 1 or 2, or in Region 3. When the strength of party affiliation nears 1, for almost all values of δ , a partisan proposer keeps the whole dollar.

When $\delta = 0$, no legislator cares about the future. In this case, the proposer keeps the whole dollar, regardless of the value of α .

In interpreting these results, it is important to bear in mind that the boundaries between the three regions for δ depend on the value of the party affiliation parameter α , as illustrated in Figure 1. As α increases, Region 1 shrinks, with the consequence that the threshold value $\bar{\delta}(\alpha)$ above which bipartisanship can be sustained increases. Likewise, as α increases, Region 3 expands, with the consequence that the threshold value $\underline{\delta}(\alpha)$ below which a partisan proposer exploits his copartisan's strength of party affiliation in order to keep the whole dollar increases. That is, as party affiliation is strengthened, the likelihood that the nonpartisan

is cut out of the process and that a partisan proposer keeps the whole dollar increases. When the strength of party affiliation nears 1, for almost all values of δ , a partisan proposer keeps the whole dollar and bipartisanship cannot be sustained. In other words, when party strength is very strong, a partisan takes advantage of his proposal power to appropriate all of the current resources that are available unless the legislators exhibit substantial patience about when they receive any benefit. When the strength of party affiliation is close to 0, the possibility of bipartisanship exists for almost all values of δ . In the limit, when no value is placed on party affiliation ($\alpha = 0$), a partisan proposer is willing to offer a positive share of the dollar to the nonpartisan for any value of $\delta \neq 0$. The size of the interval $\delta \in (\underline{\delta}(\alpha), \bar{\delta}(\alpha)]$ in which a partisan proposer only makes a positive offer to his copartisan first increases and then decreases as α increases.

Each of the proposals receives the support of a majority, with agreement reached without delay. The utility of a proposer, whoever he is, always exceeds his discounted continuation value, so there is a benefit to being recognized. In all three regions, the offer made by the nonpartisan to one of the partisans makes the recipient indifferent between accepting the offer or rejecting it and moving to the next round of bargaining. In others words, the utility value of the share offered is equal to a partisan's continuation value, discounted by one period. In Region 1, any offer that a partisan makes to the nonpartisan is one that he is just willing to accept. In Regions 1 and 2, the same is true about any offer made by a partisan to his copartisan. However, in Region 3 (i.e., when the legislators are sufficiently impatient), the copartisan's utility exceeds his discounted continuation value even though he is offered none of the dollar. In this case, the positive value that the copartisan places on the partisan proposer keeping all of the dollar is more than enough to win his support. The lower bound of 0 on an offer prevents a partisan proposer from extracting all of the surplus from his copartisan.⁹

5. Comparative Statics of the Equilibrium Proposals

In many cases, it is not apparent from inspection of the formulas in Proposition 1 how the equilibrium values of the proposal variables respond to changes in the strength of party affiliation and the discount factor. Identifying the signs of these comparative static responses adds further insight into the nature of the legislative bargaining problem that we are considering. There are four endogenous variables in Proposition 1: (i) the probability p that a partisan proposer makes a positive offer to his copartisan, (ii) the share x_P that a partisan proposer offers to his copartisan, (iii) the share x_N that a partisan proposer offers to the nonpartisan, and (iv) the share x_S that a nonpartisan proposer offers to a partisan. We deter-

⁹The statements made in this paragraph are established formally in the proof of Proposition 1.

mine how the equilibrium values of each of these variables changes in response to changes in α and δ .

5.1. Comparative Statics for the Value of Party Affiliation

Proposition 2 presents our comparative static analysis for the strength of party affiliation parameter α . In this proposition, for each of the endogenous variables, we restrict attention to the region of the parameter space in which its value varies with α . The value of the discount factor δ is held fixed when determining these comparative static results.

Proposition 2. *In any partisan symmetric stationary subgame perfect equilibrium, an increase in the value of party affiliation α results in*

- (a) *an increase in the probability p that a partisan proposer makes a positive offer to his copartisan for $\delta \in (\bar{\delta}(\alpha), 1]$;*
- (b) *a decrease in the share x_P that a partisan proposer offers to his copartisan for $\delta \in (\underline{\delta}(\alpha), 1]$;*
- (c) *a decrease in the share x_N that a partisan proposer offers to the nonpartisan for $\delta \in (\bar{\delta}(\alpha), 1]$;*
- (d) *an increase in the share x_S that a nonpartisan proposer offers to a partisan for $\delta \in (0, 1]$.*

A partisan proposer only makes an offer to the nonpartisan with positive probability if $\delta \in (\bar{\delta}(\alpha), 1]$; that is, if δ is in the interior of Region 1. In this region, we see from Part (a) of Proposition 2 that the probability p with which the copartisan is made an offer is increasing in the strength of party affiliation and, hence, the probability with which the nonpartisan is made an offer is decreasing in this parameter. The intuition for this result is straightforward. As the value of party affiliation increases, a partisan proposer is able to offer less of the dollar to his copartisan and still obtain his support because the utility value to the copartisan of whatever the partisan keeps for himself is increasing in this parameter. As a consequence, a partisan proposer benefits from proposing an increased share of the dollar for himself; and he likewise benefits from any additional portion of the dollar that he allocates to his copartisan rather than to the nonpartisan (which would yield no positive copartisanship externalities).

Any offer x_P that a partisan proposer makes to his copartisan is only positive if $\delta \in (\underline{\delta}(\alpha), 1]$; that is, if δ is in Region 1 or in the interior of Region 2. When this is the case, from Part (b) of Proposition 2, we see that the size of this offer is

decreasing in the strength of party affiliation. Because an increase in the strength of party affiliation increases the utility value to the copartisan of the share of the dollar that goes to the partisan proposer, the latter can offer a smaller portion of the dollar to his copartisan as α increases and still have his offer be accepted.

As we have noted, a positive offer x_N is made by a partisan proposer to the nonpartisan only if $\delta \in (\bar{\delta}(\alpha), 1]$. By Part (c) of Proposition 2, the size of this offer is also decreasing in the strength of party affiliation. As α increases, it becomes more desirable for the partisans to form a winning coalition because of the increased value of the utility externalities, which reduces the bargaining power of the nonpartisan. As a consequence, the nonpartisan is willing to reduce the amount needed in order to obtain his support.¹⁰

No matter what the value of δ , the nonpartisan needs to obtain the support of one of the partisans in order to have his proposal accepted. By Part (d) of Proposition 2, the nonpartisan needs to offer a larger share of the dollar x_S to one of the partisans in order to induce a partisan to accept as the strength of party affiliation increases whenever $\delta \neq 0$. This conclusion is an implication of his reduced bargaining power discussed above. The nonpartisan prefers to have a minimally acceptable offer accepted in the current period than to having the offer turned down and moving to the next period. In the next period, either (i) the nonpartisan is recognized as the proposer, in which case he would not only have to make a better offer in order to get it accepted, he would also incur the cost of delaying agreement or (ii) one of the partisans is recognized as the proposer, in which case the nonpartisan would lose the added benefit that accrues to a proposer.

In the benchmark case considered by Baron and Ferejohn (1989), there is no party affiliation ($\alpha = 0$). In their equilibrium, any proposer offers $\delta/3$ with equal probability to either of the other legislators, keeping $2\delta/3$ for himself. When there is party affiliation ($\alpha > 0$), it is readily confirmed from Part (a) of Proposition 1 that in the case in which bipartisanship is possible ($\delta > \bar{\delta}(\alpha)$), a partisan proposer favors his copartisan by making an offer to him instead of to the nonpartisan with a probability that exceeds $1/2$. Moreover, as is shown in Proposition 2, regardless of whom he makes an offer to, the amount that a partisan proposer keeps is increasing in α , provided that a positive offer is made. Hence, for any $\delta > 0$ and any $\alpha > 0$, a partisan proposer keeps more for himself than in the Baron–Ferejohn benchmark case. We also know from Proposition 2 that a nonpartisan proposer’s

¹⁰The threshold values $\underline{\delta}(\alpha)$ and $\bar{\delta}(\alpha)$ that define the region boundaries are increasing in α . When δ is on the lower boundary of Region 1 ($\delta = \bar{\delta}(\alpha)$), a marginal increase in α results in this value of δ being in Region 2. Before and after this change, a partisan only makes an offer to his copartisan if $\delta \neq 0$. Similarly, when δ is on the lower boundary of Region 2 ($\delta = \underline{\delta}(\alpha)$), a marginal increase in α results in this value of δ being in Region 3. No share of the dollar is offered by a partisan proposer to his copartisan either before or after this change.

own share is decreasing in α for any value of δ . Hence, he secures less of the dollar when the other legislators are affiliated than when they are not.

5.2. Comparative Statics for the Value of the Discount Factor

We now turn in Proposition 3 to our comparative static analysis for the discount factor δ . For each of the endogenous variables, we restrict attention to the region of the parameter space in which its value varies with changes in δ holding α fixed.

Proposition 3. *In any partisan symmetric stationary subgame perfect equilibrium, an increase in the discount factor δ results in*

- (a) *a decrease in the probability p that a partisan proposer makes a positive offer to his copartisan for $\delta \in [\bar{\delta}(\alpha), 1)$ and $\alpha \in (0, 1)$;*
- (b) *an increase in the share x_P that a partisan proposer offers to his copartisan for $\delta \in [\bar{\delta}(\alpha), 1]$;*
- (c) *an increase in the share x_N that a partisan proposer offers to the nonpartisan for $\delta \in [\bar{\delta}(\alpha), 1)$;*
- (d) *an increase in the share x_S that a nonpartisan proposer offers to a partisan for $\delta \in [0, 1)$.*

As δ increases, the legislators become more patient, which implies that each of them is more willing to hold out for a better offer in the future, making them more expensive coalition partners. That is, each legislator's discounted continuation value is increasing in δ . As a consequence, whenever a proposer (either partisan or nonpartisan) is willing to offer a positive share to a legislator (in which case, he offers the discounted continuation value), he must offer a larger share of the dollar to his coalition partner in order to induce acceptance in the current period. Thus, as shown in Parts (b), (c), and (d) of Proposition 3, x_P , x_N , and x_S are all increasing in δ in the relevant regions.

When bipartisanship is viable ($\delta > \bar{\delta}$), it is shown in the proof of Proposition 1 that the continuation values of the two partisans are

$$V^1 = V^2 = \frac{\alpha + \delta + \alpha\delta + \delta^2}{\delta(3 + \alpha)}, \quad (15)$$

and that of the nonpartisan is

$$V^3 = \frac{-2\alpha + \delta + \alpha\delta}{\delta(3 + \alpha)}. \quad (16)$$

By differentiating (15) and (16) with respect to δ , we see that a partisan's continuation value increases at a faster rate than that of the nonpartisan as δ is increased. Intuitively, this is the case because the partisans both benefit from whatever share of the dollar they receive, whereas the nonpartisan only benefits from his own share. Because a coalition partner must be offered his discounted continuation value for his support, a partisan proposer thus finds the nonpartisan a marginally more attractive coalition partner than his copartisan when δ is increased, which makes it more likely that he will include the nonpartisan in his winning coalition, as shown in Part (a) of Proposition 3

Consistent with conventional wisdom, our comparative static results suggest that when parties are highly polarized (i.e., when α is very high), bipartisan outcomes are not likely to be obtained. Holding polarization constant, however, one is more likely to see bipartisanship when legislators are more patient. Given the longer time horizons of U.S. Senators in comparison to members of the House, this result might help to explain why the Senate has traditionally cultivated a reputation for bipartisan cooperation. This result is also consistent with recent empirical findings that point to how majority party membership appears to be less important for facilitating legislative success in the Senate than in the House (Volden and Wiseman, 2017).

6. Expected Shares and Their Comparative Statics

The offers that are characterized in Proposition 1 are the equilibrium offers in the Partisan Legislative Bargaining Game that are made *conditional* on being recognized as the proposer. The shares of the dollar that a legislator expects to receive prior to commencing bargaining is a measure of his relative bargaining strength. It is of interest to determine what these ex ante shares are and how they respond to changes in the strength of party affiliation and the discount factor.

From an ex ante perspective, the two partisans are in completely symmetric situations, and so have the same expected shares. Without loss of generality, we focus on the share of legislator 1. The *ex ante expected share of a partisan* is

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[p(1 - x_P) + (1 - p)(1 - x_N) + px_P + \frac{1}{2}x_S \right] \\ &= \frac{1}{3} \left[p + (1 - p)(1 - x_N) + \frac{1}{2}x_S \right] \end{aligned} \tag{17}$$

for $\delta \in [\bar{\delta}(\alpha), 1]$ and

$$E[x_1] = \frac{1}{3} \left[1 - x_P + x_P + \frac{1}{2}x_S \right] = \frac{1}{3} \left[1 + \frac{1}{2}x_S \right] \tag{18}$$

for $\delta \in [0, \bar{\delta}(\alpha)]$. Because the expected shares sum to 1 and the two partisans have the same expected shares, the *ex ante expected share of a nonpartisan* is

$$E[x_3] = 1 - 2E[x_1]. \quad (19)$$

In Proposition 4, we use the equilibrium values of p , x_P , x_N , and x_S from Proposition 1 to express these expected shares in terms of the parameters of our model.

Proposition 4. *In the Partisan Legislative Bargaining Game:*

- (a) For any $\delta \in [\bar{\delta}(\alpha), 1]$, the *ex ante expected share of a partisan* is $E[x_1] = \frac{(\alpha+\delta)}{(3+\alpha)\delta}$ and the *ex ante expected share of a nonpartisan* is $E[x_3] = \frac{-2\alpha+\delta+\alpha\delta}{(3+\alpha)\delta}$.
- (b) For any $\delta \in [0, \bar{\delta}(\alpha))$, the *ex ante expected share of a partisan* is $E[x_1] = \frac{2}{6-\delta-\alpha\delta}$ and the *ex ante expected share of a nonpartisan* is $E[x_3] = \frac{2-\delta-\alpha\delta}{6-\delta-\alpha\delta}$.

In the Baron and Ferejohn (1989) model, $\alpha = 0$. By Proposition 4, the expected share of any legislator is then $1/3$ for any value of δ . The three legislators are symmetrically situated, and thus have equal expected shares of the dollar. For any $\alpha \neq 0$, we see from Part (b) of Proposition 4 that these expected shares are also $1/3$ if $\delta = 0$. In this case, a legislator is recognized with probability $1/3$ and keeps the whole dollar if recognized. We do not consider these special cases further.

For $\delta \in [\bar{\delta}(\alpha), 1]$, the expected shares may be rewritten as

$$E[x_1] = \frac{1}{3} + \frac{1}{3} \left[\frac{(3-\delta)\alpha}{(3+\alpha)\delta} \right] \quad (20)$$

and

$$E[x_3] = \frac{1}{3} - \frac{1}{3} \left[\frac{2(3-\delta)\alpha}{(3+\alpha)\delta} \right]. \quad (21)$$

Similarly, for $\delta \in [0, \bar{\delta}(\alpha))$, they may be rewritten as

$$E[x_1] = \frac{1}{3} + \frac{1}{3} \left[\frac{(1+\alpha)\delta}{[6-(1+\alpha)\delta]} \right] \quad (22)$$

$$E[x_3] = \frac{1}{3} - \frac{1}{3} \left[\frac{2(1+\alpha)\delta}{[6-(1+\alpha)\delta]} \right].^{11} \quad (23)$$

The expressions in (20)–(23) show how the expected shares deviate from the Baron–Ferejohn benchmark case when $\alpha > 0$. When $\alpha = 0$, as we have seen,

¹¹The algebra used to derive (20)–(23) may be found in the Appendix.

each legislator has an expected share equal to $1/3$. For $\alpha > 0$, the second terms in (20) and (22) show how much is added to a partisan's expected share due to party affiliation, with the magnitude of this term depending on both α and δ . Similarly, the second terms in (21) and (23) show how much must be subtracted from a nonpartisan's expected share because of the partisans' party affiliation.¹²

In Proposition 5, we determine how the expected shares respond to changes in the strength of party affiliation and the discount factor. Note that an immediate implication of (19) is that a partisan's expected share is increasing in response to a change in one of these parameters if and only if the nonpartisan's expected share is decreasing.

Proposition 5. *In any partisan symmetric stationary subgame perfect equilibrium:*

- (a) *If the discount factor δ is positive, an increase in the value of party affiliation α results in an increase in the expected share $E[x_1]$ of a partisan legislator and a decrease in the expected share $E[x_3]$ of the nonpartisan legislator.*
- (b) *If the value of party affiliation α is positive, an increase in the discount factor δ results in*
 - (i) *a decrease in the expected share $E[x_1]$ of a partisan legislator and an increase in the expected share $E[x_3]$ of the nonpartisan legislator for $\delta \in [\bar{\delta}(\alpha), 1]$, and*
 - (ii) *an increase in the expected share $E[x_1]$ of a partisan legislator and a decrease in the expected share $E[x_3]$ of the nonpartisan legislator for $\delta \in [0, \bar{\delta}(\alpha))$.*

The relationship between the ex ante share for legislator 1 and δ is displayed graphically in Figure 3 for several values of α . In conformity with Proposition 5, we see that legislator 1's expected share of the dollar is (i) increasing in α for fixed δ and (ii) increasing and then decreasing in δ for a fixed positive value of α . For each $\alpha > 0$, the kink occurs at $\bar{\delta}(\alpha)$, the threshold beyond which the nonpartisan has a positive probability of being offered a positive share by a partisan. Because $\bar{\delta}(\alpha)$ is increasing in α (see Figure 1), this kink moves to the right as α is increased.¹³ From (18), it can be seen that there is no kink at $\underline{\delta}(\alpha)$, the threshold beyond which a partisan offers his copartisan a positive share. While there is a kink in the graph

¹²Setting $\delta = 1$ in Proposition 4 or in (20)–(23), we obtain formulae for the expected shares in the Calvert and Dietz (2005) model.

¹³Because $\bar{\delta}(\alpha)$ is only smaller than 0.5 for values of α significantly below 0.1, we do not include values of δ smaller than 0.5 in Figure 3.

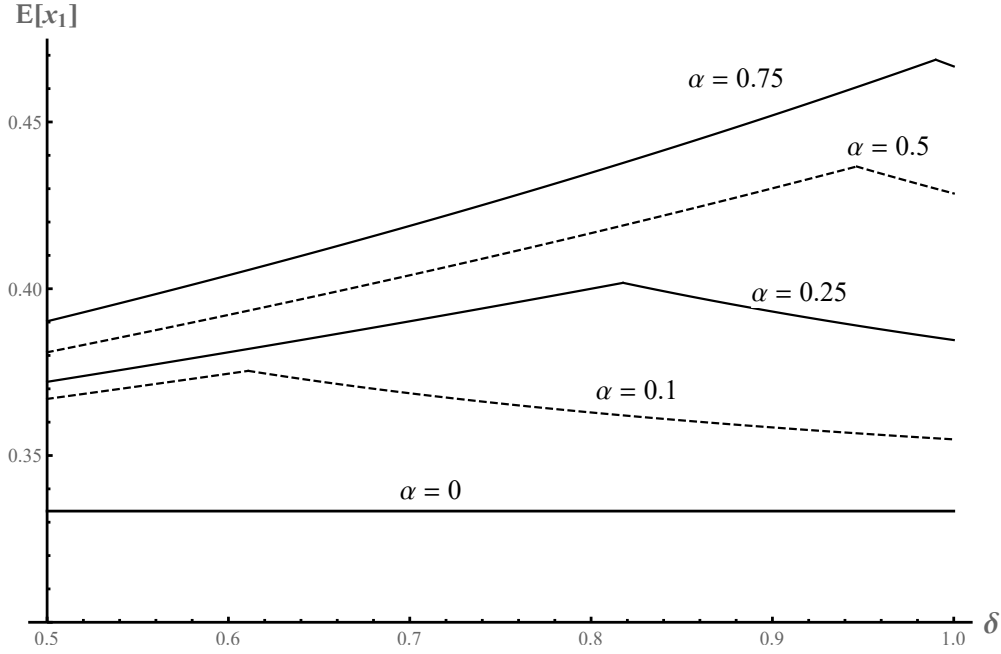


Figure 3: Ex Ante Expected Share for a Partisan Legislator

of x_P at this point, the symmetry of the two partisans' situations smooths this kink out when expected shares are computed.

Propositions 2 and 3 can be used to provide some intuition for Proposition 5. First, consider Part (a). For $\delta \leq \bar{\delta}(\alpha)$, the two partisans share the dollar among themselves if one of them is recognized. Because they are symmetrically situated, conditional on one of them being recognized, they each have an expected share of $1/2$. This happens with probability $2/3$, so the expected share due to a partisan being the proposer is $1/3$, which is independent of α . Thus, as shown in (18), a partisan legislator's expected share only varies with the amount x_S that a non-partisan offers if recognized, which Proposition 2 shows is increasing in α . For $\delta > \bar{\delta}(\alpha)$, by (17), a partisan legislator's expected share due to a partisan being recognized is no longer 1; rather, it is $[p + (1-p)(1-x_N)]/3 = [1 - (1-p)x_N]/3$, which is increasing in α by Proposition 2. This effect reinforces the effect that an increase in x_S has on a partisan's expected share. Turning now to Part (b), for $\delta \leq \bar{\delta}(\alpha)$, the preceding reasoning applies as above, except that appeal is now made to the fact that x_S is increasing in δ , as shown in Proposition 3. Finally, for $\delta > \bar{\delta}(\alpha)$, the preceding argument is modified by observing that, by Proposition 3, $[1 - (1-p)x_N]/3$ is decreasing in δ .¹⁴

¹⁴Our discussion of Propositions 2 and 3 provides further intuition for Proposition 5 because

7. Conclusion

While the contemporary political environment might lend itself to dire predictions about the likelihood of overcoming gridlock and/or the incidence of bipartisan policy outcomes, we know that political compromises do, in fact, occur, and bipartisan policies are, in fact, created. The relevant questions, then, are: When might bipartisan policy outcomes be obtained? What form will these bipartisan policies take, and why? We have advanced a parsimonious model of legislative bargaining with partisan legislators that allows us to engage with these questions in a tractable way. Our analysis reveals that bipartisan outcomes are more likely to be obtained when legislators exhibit substantial concern for the future; but as the strength of party affiliation increases, it is increasingly difficult to forge bipartisan coalitions. Indeed, when the strength of party affiliation is sufficiently large, policy outcomes are quite one sided, with a minority legislator being either entirely shut out of the policy making process or only able to extract a small share of the benefits that are available.

Our results point to many possible extensions that are worthy of further study. From an empirical perspective, as alluded to above, we find that bipartisan coalitions are more likely to be obtained in political environments in which legislators are relatively more patient. Exploring the relationship between time-until-elections and the prevalence of bipartisan coalition formation would be of interest. Our results also suggest that bipartisanship is more likely to be realized when the value of party affiliation is relatively low. To the extent that one equates the value of party affiliation with the value of the party brand in the electoral arena (Kiewiet and McCubbins, 1991) or with the degree to which ideological polarization between parties conveys information about their ideological composition (Snyder and Ting, 2002), one might expect that bipartisanship would be more common when the parties are less ideologically distinct and/or party affiliations carry less weight among voters. Moving beyond exploring the empirical implications of our model with observational data, the tractability of our model and the clarity of its predictions make it ripe for experimental exploration, perhaps in a laboratory setting.

From a theoretical standpoint, our model is obviously quite spartan in its representation of the policymaking process, and there are several ways that one might seek to enrich it. Obvious extensions would be to explore the impact of party affiliation on legislative bargaining outcomes when the chamber is choosing among policies that do not only have particularistic benefits, as is the case here, but also have collective components (Volden and Wiseman, 2007) or (spatial) ideological dimensions (Jackson and Moselle, 2002). Likewise, analyzing the model with

it supplies intuition for the comparative static results that are appealed to here.

more legislators would allow one to obtain more nuanced predictions regarding the likelihood and form of bipartisan policy outcomes, and how those outcomes might vary under different parliamentary arrangements (such as the presence of supermajoritarian voting requirements). Regardless of which directions are taken, we hope that the model advanced here serves as a foundation for providing a greater appreciation of the factors that facilitate and/or inhibit the realization of partisan and bipartisan policy outcomes.

References

- Aldrich, J. H., Rohde, D. W., 2017. Lending and reclaiming power: Majority leadership in the House since the 1950s. In: Dodd, L. C., Oppenheimer, B. I. (Eds.), *Congress Reconsidered, Eleventh Edition*. CQ Press, Thousand Oaks, CA, pp. 29–59.
- Baron, D. P., Ferejohn, J. A., 1989. Bargaining in legislatures. *American Political Science Review* 83, 1181–1206.
- Calvert, R. L., Dietz, N., 2005. Legislative coalitions in a bargaining model with externalities. In: Austen-Smith, D., Duggan, J. (Eds.), *Social Choice and Strategic Decisions: Essays in Honor of Jeffrey S. Banks*. Springer, Berlin, pp. 227–247.
- Chen, J., Rodden, J., 2013. Unintentional gerrymandering: Political geography and electoral bias in legislatures. *Quarterly Journal of Political Science* 8, 239–269.
- Cox, G. W., McCubbins, M. D., 2005. *Setting the Agenda: Responsible Party Government in the U. S. House of Representatives*. Cambridge University Press, Cambridge.
- Dodd, L. C., Oppenheimer, B. I., 2017. Congress in the age of Trump: The 2016 national elections and their aftermath. In: Dodd, L. C., Oppenheimer, B. I. (Eds.), *Congress Reconsidered, Eleventh Edition*. CQ Press, Thousand Oaks, CA, pp. 451–477.
- Hare, C., Poole, K., 2014. The polarization of contemporary American politics. *Polity* 46, 411–429.
- Hetherington, M. J., Rudolph, T. J., 2015. *Why Washington Won't Work: Polarization, Political Trust, and the Governing Crisis*. University of Chicago Press, Chicago.
- Jackson, M. O., Moselle, B., 2002. Coalition and party formation in a legislative voting game. *Journal of Economic Theory* 103, 49–87.
- Kiewiet, D. R., McCubbins, M. D., 1991. *The Logic of Delegation: Congressional Parties and the Appropriations Process*. University of Chicago Press, Chicago.
- Krehbiel, K., Meirowitz, A., 2002. Minority rights and majority power: Theoretical consequences of the motion to recommit. *Legislative Studies Quarterly* 27,

191–217.

- Krehbiel, K., Meirowitz, A., Wiseman, A. E., 2015. A theory of competitive partisan lawmaking. *Political Science Research and Methods* 3, 423–448.
- La Raja, R., 2009. Redistricting: Reading between the lines. *Annual Review of Political Science* 12, 203–223.
- Lee, F., 2016. *Insecure Majorities: Congress and the Perpetual Campaign*. University of Chicago Press, Chicago.
- Montero, M., 2007. Inequity aversion may increase inequity. *Economic Journal* 117, C192–C204.
- Snyder, J. M., Ting, M. M., 2002. An informational rationale for political parties. *American Journal of Political Science* 46, 90–110.
- Volden, C., Wiseman, A. E., 2007. Bargaining in legislatures over particularistic and collective goods. *American Political Science Review* 101, 79–92.
- Volden, C., Wiseman, A. E., 2017. Legislative effectiveness in the United States Senate. Discussion Paper No. 4-2017, Center for the Study of Democratic Institutions, Vanderbilt University.

Appendix A. Proofs¹

Proof of Proposition 1. We first establish the necessity part of the proof.

(a) Suppose that $\delta \in [\bar{\delta}(\alpha), 1]$. We first need to find expressions for the equilibrium values of x_N , x_P , x_S , V^1 , V^2 , V^3 , and p in terms of the model's two exogenous parameters, α and δ . We initially proceed on the assumption that the expressions we obtain for each of the three share variables and for a partisan's probability of making an offer to his copartisan are nonnegative and, in the latter case, do not exceed 1. We then show that this is indeed the case when $\delta \in (\bar{\delta}(\alpha), 1]$. By our symmetry assumption, the nonpartisan legislator makes the same offer to each partisan with equal probability.

For a partisan legislator to accept a proposal of x_S , he must weakly prefer it to his discounted continuation value. Using (11), we thus have

$$x_S \geq \frac{\delta}{3} (1 + \alpha) \left[1 - x_N + px_N + \frac{1}{2}x_S \right]. \quad (\text{A.1})$$

Because a nonpartisan proposer would never offer a partisan legislator more than the minimum necessary to induce him to accept a proposal, this inequality binds in equilibrium. Combining terms involving x_S in the equality version of (A.1), we obtain

$$\left[3 - \delta(1 + \alpha)\frac{1}{2} \right] x_S = \delta(1 + \alpha) [1 - (1 - p)x_N],$$

from which it follows that

$$x_S = \frac{2\delta(1 + \alpha) [1 - (1 - p)x_N]}{6 - \delta(1 + \alpha)}. \quad (\text{A.2})$$

Similarly, the share x_N offered to the nonpartisan legislator is equal to his discounted continuation value in equilibrium. Substituting the value of x_S from (A.2) into the expression in (12) for the nonpartisan's continuation value, we thus have that

$$x_N = \frac{\delta}{3} \left[2(1 - p)x_N + 1 - \frac{2\delta(1 + \alpha) [1 - (1 - p)x_N]}{6 - \delta(1 + \alpha)} \right]$$

or, equivalently,

$$x_N = \frac{\delta}{3} \left[\frac{12(1 - p)x_N + 6 - 3\delta(1 + \alpha)}{6 - \delta(1 + \alpha)} \right].$$

¹Some of the expressions in this appendix were derived using *Mathematica*. The *Mathematica* code for these derivations may be found in Appendix B.

Combining terms involving x_N and expressing the coefficient of x_N in terms of a common denominator, it then follows that

$$\left[\frac{18 - 3\delta(1 + \alpha) - 12\delta(1 - p)}{6 - \delta(1 + \alpha)} \right] x_N = \delta \left[\frac{6 - 3\delta(1 + \alpha)}{6 - \delta(1 + \alpha)} \right].$$

Dividing both sides of this equation by 3 and expanding the numerator on the left hand side, we obtain

$$\left[\frac{6 - 5\delta + 4\delta p - \alpha\delta}{6 - \delta(1 + \alpha)} \right] x_N = \delta \left[\frac{2 - \delta(1 + \alpha)}{6 - \delta(1 + \alpha)} \right].$$

Thus,

$$x_N = \frac{\delta [2 - \delta - \alpha\delta]}{6 - 5\delta + 4\delta p - \alpha\delta}. \quad (\text{A.3})$$

We thus have an expression for x_N in terms of the model's parameters and p (whose value has yet to be determined).

To express x_S in terms of these parameters, we now substitute x_N from (A.3) into (A.2), thereby obtaining

$$x_S = \frac{2\delta(1 + \alpha) \left[1 - (1 - p) \frac{\delta[2 - \delta - \alpha\delta]}{6 - 5\delta + 4\delta p - \alpha\delta} \right]}{6 - \delta(1 + \alpha)}$$

or, equivalently,

$$x_S = \frac{2\delta(1 + \alpha) \left[\frac{6 - 7\delta + 6\delta p - \alpha\delta + \delta^2 - \delta^2 p + \alpha\delta^2 - \alpha\delta^2 p}{6 - 5\delta + 4\delta p - \alpha\delta} \right]}{6 - \delta(1 + \alpha)}.$$

Factoring the numerator in the term in square brackets yields

$$x_S = \frac{2\delta(1 + \alpha) \left[\frac{(6 - \delta(1 + \alpha))(1 - (1 - p)\delta)}{6 - 5\delta + 4\delta p - \alpha\delta} \right]}{6 - \delta(1 + \alpha)},$$

from which it follows that

$$x_S = \frac{2\delta(1 + \alpha) [1 - (1 - p)\delta]}{6 - 5\delta + 4\delta p - \alpha\delta}. \quad (\text{A.4})$$

Substituting the expressions for x_N and x_S in (A.3) and (A.4) into the expression in (11) for the partisan continuation value, after expanding terms we obtain

$$\begin{aligned} V^1 = V^2 &= \frac{1}{3} (1 + \alpha) \\ &\times \left[\frac{6 - 5\delta + 4\delta p - \alpha\delta - \delta(1 - p) [2 - \delta - \alpha\delta] + \delta(1 + \alpha) [1 - (1 - p)\delta]}{6 - 5\delta + 4\delta p - \alpha\delta} \right]. \end{aligned}$$

Simple algebra then shows that the partisan continuation value is

$$V^1 = V^2 = (1 + \alpha) \left[\frac{2[1 - (1 - p)\delta]}{6 - 5\delta + 4\delta p - \alpha\delta} \right]. \quad (\text{A.5})$$

Similarly, substituting the expressions for x_N and x_S in (A.3) and (A.4) into the expression in (12) for the nonpartisan continuation value, after expressing every term using a common denominator, we obtain

$$V^3 = \frac{1}{3} \left[\frac{2(1 - p)\delta(2 - \delta - \alpha\delta) + 6 - 5\delta + 4\delta p - \alpha\delta - 2\delta(1 + \alpha)[1 - (1 - p)\delta]}{6 - 5\delta + 4\delta p - \alpha\delta} \right].$$

Collecting terms and simplifying, the nonpartisan continuation value is

$$V^3 = \frac{2 - \delta - \alpha\delta}{6 - 5\delta + 4\delta p - \alpha\delta}. \quad (\text{A.6})$$

We now determine the share x_P offered by a partisan to his copartisan. Without loss of generality, suppose that legislator 1 is the proposer and he offers the share x_P to legislator 2. This share would be accepted provided that the utility that legislator 2 receives from this proposal is at least as large as his discounted continuation value. That is, he will accept x_P if and only if

$$x_P + \alpha(1 - x_P) \geq \delta V^2$$

or, equivalently,

$$x_P \geq \frac{\delta V^2 - \alpha}{1 - \alpha}. \quad (\text{A.7})$$

Because $\alpha < 1$, $x_P + \alpha(1 - x_P)$ is increasing in x_P . Thus, because x_P cannot be negative, the constraint in (A.7) binds if and only if the right hand side of (A.7) is nonnegative. Rearranging (A.7), we see that it is optimal for legislator 1 to choose x_P so that (A.7) binds if and only if

$$\delta V^2 \geq \alpha. \quad (\text{A.8})$$

Later, we shall show that (A.8) holds when $\delta \in [\bar{\delta}(\alpha), 1]$.

For now, we proceed on the assumption that (A.8) is satisfied. In this case, from (A.5) and (A.7),

$$x_P = \frac{1}{(1 - \alpha)} \left\{ \delta(1 + \alpha) \left[\frac{2[1 - (1 - p)\delta]}{6 - 5\delta + 4\delta p - \alpha\delta} \right] - \alpha \right\}.$$

Expressing the terms on the right hand side of this equality in terms of a common denominator, we obtain

$$x_P = \frac{2\delta(1 + \alpha)[1 - \delta + \delta p] - \alpha(6 - 5\delta + 4\delta p - \alpha\delta)}{(6 - 5\delta + 4\delta p - \alpha\delta)(1 - \alpha)}$$

or, equivalently,

$$x_P = \frac{-6\alpha + 2\delta + 7\alpha\delta - 4\alpha\delta p + \alpha^2\delta - 2\delta^2 + 2\delta^2 p - 2\alpha\delta^2 + 2\alpha\delta^2 p}{(6 - 5\delta + 4\delta p - \alpha\delta)(1 - \alpha)}. \quad (\text{A.9})$$

If a partisan proposer chooses $p \in (0, 1)$, he must be indifferent between offering the value of x_P in (A.9) to his partisan and the value of x_N in (A.3) to the nonpartisan. For now, we proceed on the assumption that the values of α and δ are such that this is the case and consider the possibility that p is either 0 or 1 later. Thus, we need to identify the value of p for which a partisan would be indifferent between offering a share to his copartisan and to the nonpartisan. As above, without loss of generality, suppose that the partisan proposer is legislator 1.

If he offers the value of x_N in (A.3) to the nonpartisan, legislator 1's utility would be

$$U^1(x^{13}) = 1 - x_N = 1 - \frac{\delta(2 - \delta - \alpha\delta)}{6 - 5\delta + 4\delta p - \alpha\delta}. \quad (\text{A.10})$$

If, however, he offers the value of x_P in (A.9) to his copartisan, his utility would be

$$\begin{aligned} U^1(x^{12}) &= 1 - x_P + \alpha x_P \\ &= 1 - \frac{(-6\alpha + 2\delta + 7\alpha\delta - 4\alpha\delta p + \alpha^2\delta - 2\delta^2 + 2\delta^2 p - 2\alpha\delta^2 + 2\alpha\delta^2 p)}{(6 - 5\delta + 4\delta p - \alpha\delta)}. \end{aligned} \quad (\text{A.11})$$

Setting the right hand sides of (A.10) and (A.11) equal to each other and solving for p , we obtain

$$p = \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{2\delta(\delta + \alpha\delta - 2\alpha)}, \quad (\text{A.12})$$

which is the equilibrium value for p .

If $x_P = 0$ in (A.11), the right-hand sides of (A.10) and (A.11) are equal if and only

$$\delta(2 - \delta - \alpha\delta) = 0,$$

which is only satisfied if either (i) $\delta = 0$ or (ii) $2 - \delta - \alpha\delta = 0$. Case (i) does not apply because 0 lies outside the range of values for δ being considered. Case (ii) applies if and only if $\delta = 2/(1 + \alpha)$. Because $\alpha < 1$, this case is impossible because δ cannot exceed 1. Thus, it must be the case that $x_P > 0$ if $p \in (0, 1)$ and, hence, (A.8) holds with a strict inequality.

Substituting the value of p from (A.12) into the expressions for the share offers x_P , x_N , x_S in (A.9), (A.3), and (A.4), respectively, we obtain

$$x_P = \frac{-2\alpha + \delta + \alpha\delta}{(3 + \alpha)(1 - \alpha)}, \quad (\text{A.13})$$

$$x_N = \frac{-2\alpha + \delta + \alpha\delta}{3 + \alpha}, \quad (\text{A.14})$$

and

$$x_S = \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha}. \quad (\text{A.15})$$

Because the nonpartisan offers a partisan a share equal to his discounted continuation value, (A.15) implies that

$$V^1 = V^2 = \frac{(1 + \alpha)(\alpha + \delta)}{\delta(3 + \alpha)}. \quad (\text{A.16})$$

Similarly, because a partisan offers the nonpartisan a share equal to his discounted continuation value, (A.14) implies that

$$V^3 = \frac{-2\alpha + \delta + \alpha\delta}{\delta(3 + \alpha)}. \quad (\text{A.17})$$

We have shown that the expressions in equations (A.12)–(A.17) are the equilibrium values of the endogenous variables expressed in terms of the two parameters α and δ provided that the three share values are nonnegative and that the partisan's offer probability p in (A.12) lies in $(0, 1)$. We proceed on the assumption that the same formulae for the share offers and the partisan's offer probability also apply when $p = 0$ or $p = 1$ and confirm that this is in fact the case below. Moreover, we shall show that these shares are nonnegative and that this probability is in $[0, 1]$ when $\delta \in [\bar{\delta}(\alpha), 1]$.

We now use (A.13)–(A.15) to determine the shares a proposer would keep for himself in equilibrium. A partisan proposer's share when offering x_P is

$$1 - x_P = \frac{3 - \alpha^2 - \delta - \alpha\delta}{(3 + \alpha)(1 - \alpha)}. \quad (\text{A.18})$$

A partisan proposer's share when offering x_N is

$$1 - x_N = \frac{(1 + \alpha)(3 - \delta)}{3 + \alpha}. \quad (\text{A.19})$$

The nonpartisan proposer's share when offering x_S is

$$1 - x_S = \frac{3 - \alpha^2 - \delta - \alpha\delta}{3 + \alpha}. \quad (\text{A.20})$$

A legislator will vote for any proposal in which he receives at least his discounted continuation value. Thus, using (A.16), a partisan will vote for any proposal in which he receives utility u for which

$$u \geq \frac{(1 + \alpha)(\alpha + \delta)}{(3 + \alpha)}. \quad (\text{A.21})$$

Similarly, using (A.17), the nonpartisan will vote for any proposal in which he receives utility u for which

$$u \geq \frac{-2\alpha + \delta + \alpha\delta}{(3 + \alpha)}. \quad (\text{A.22})$$

We have already seen that (A.8) must hold in order for x_P to be nonnegative. By (A.16), this holds if and only if

$$(1 + \alpha)(\alpha + \delta) \geq (3 + \alpha)\alpha$$

or, equivalently, if and only if

$$\delta \geq \frac{2\alpha}{1 + \alpha}. \quad (\text{A.23})$$

We now show that (A.23) holds with a strict inequality when $\delta = \bar{\delta}$, which is the lowest value for δ in the region being considered. First, note that

$$\sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4} > 7\alpha + \alpha^2$$

if and only if

$$24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4 > 49\alpha^2 + 14\alpha^3 + \alpha^4$$

if and only if

$$3 > 2\alpha + \alpha^2.$$

Thus,

$$\bar{\delta} = \frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1 + \alpha)} > \frac{2\alpha}{1 + \alpha} \quad (\text{A.24})$$

because $\alpha < 1$. Hence, the expression for x_P in (A.13) is nonnegative.

We now determine the values for α and δ for which the value of p in (A.12) lies in $[0, 1]$. The upper bound on p is satisfied if and only if

$$p = \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{2\delta(\delta + \alpha\delta - 2\alpha)} \leq 1. \quad (\text{A.25})$$

When $\alpha = 0$, $p = 1/2$ in (A.25), and so this inequality is satisfied for all δ . When δ satisfies (A.23) and $\alpha \in (0, 1)$, (A.25) holds if and only if

$$\delta \geq \frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1 + \alpha)}.^2 \quad (\text{A.26})$$

²The expression on the right hand side of this inequality is obtained by replacing the inequality in (A.25) with an equality and then using the quadratic formula to solve the resulting equation. Only the positive root is relevant. The restriction on δ implied by (A.27) is obtained similarly. See Appendix B for *Mathematica* code to implement these calculations.

This inequality holds with equality if and only if $p = 1$. Thus, $p = 1$ if and only if $\delta = \bar{\delta}$.³

The lower bound on p is satisfied if and only if

$$p = \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{2\delta(\delta + \alpha\delta - 2\alpha)} \geq 0. \quad (\text{A.27})$$

For all $\alpha \in [0, 1)$, (A.27) holds if and only if $\delta > 2\alpha/(1 + \alpha)$. Moreover, there is no value of the parameters for which $p = 0$. Thus, for the range of values for δ that we are considering, a partisan makes an offer to his copartisan with positive probability.

Because p is never 0 for the range of values for δ being considered, we only need to confirm that the expression for p in (A.12) is valid when $p = 1$. That is, we need to show that the value of $U^1(x^{12})$ in (A.11) is at least as large as the value of $U^1(x^{13})$ in (A.10) when $p = 1$. The preceding argument shows that these two values are in fact the same when $p = 1$ (i.e., when $\delta = \bar{\delta}$ and $\alpha \neq 0$).

We have already confirmed that the expression we have found for x_P is non-negative when (A.26) is satisfied. It remains to be shown that the same is true for x_N and x_S .

From (A.14), (A.24), and (A.26), we have

$$\begin{aligned} x_N &= \frac{-2\alpha + \delta + \alpha\delta}{3 + \alpha} \\ &\geq \frac{-2\alpha + (1 + \alpha) \left[\frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1 + \alpha)} \right]}{3 + \alpha} \\ &> \frac{-2\alpha + (1 + \alpha) \frac{(2\alpha)}{(1 + \alpha)}}{3 + \alpha} \\ &= 0. \end{aligned}$$

Similarly, from (A.15), (A.24), and (A.26), we have

$$\begin{aligned} x_S &= \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \\ &\geq \frac{(1 + \alpha) \left(\alpha + \left[\frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1 + \alpha)} \right] \right)}{3 + \alpha} \\ &> \frac{(1 + \alpha) \left(\alpha + \left[\frac{2\alpha}{(1 + \alpha)} \right] \right)}{3 + \alpha} \\ &> 0. \end{aligned}$$

³If $p = 1$, then a.(i).ii is trivially true.

(b) Suppose that $\delta \in [\underline{\delta}(\alpha), \bar{\delta}(\alpha))$ and that a partisan is the proposer. For these values of δ , it is not possible to have $\alpha = 0$, so $\alpha \in (0, 1)$. We proceed on the assumption that it is optimal for a partisan to make his copartisan an offer that he is just willing to accept. That is, we suppose that (A.8) is satisfied. Later, we shall confirm that this is in fact the case for the values of δ being considered.

From (A.26), we know that it is not possible to have $p < 1$ when $\alpha \neq 0$ if $\delta \leq \frac{-3\alpha - \alpha^2 + \sqrt{24\alpha + 33\alpha^2 + 6\alpha^3 + \alpha^4}}{2(1+\alpha)}$. Thus, $p = 1$. Because a partisan proposer strictly prefers to offer his copartisan his continuation value rather than make a positive offer to the nonpartisan, x_N is not relevant. The formula for x_P in (A.9) continues to apply provided that it is nonnegative. Substituting $p = 1$ into this equation, we obtain

$$x_P = \frac{2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha}{(6 - \delta - \alpha\delta)(1 - \alpha)} \quad (\text{A.28})$$

which is the equilibrium value of x_P . The partisan proposer's share when offering x_P is

$$1 - x_P = 1 - \frac{2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha}{(6 - \delta - \alpha\delta)(1 - \alpha)} = \frac{6 - 3\delta - 3\alpha\delta}{(6 - \delta - \alpha\delta)(1 - \alpha)}. \quad (\text{A.29})$$

Because $\alpha < 1$ and $\delta \leq 1$, this share is positive.

From (A.28), we have $x_P \geq 0$ if and only if

$$2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha \geq 0$$

or, equivalently,

$$\delta \geq \frac{6\alpha}{(1 + \alpha)(2 + \alpha)}. \quad (\text{A.30})$$

The right hand side of this inequality is $\underline{\delta}(\alpha)$, which is the smallest value of δ in the interval being considered, so we have confirmed that the share offers are nonnegative.

It remains to confirm that it is optimal for a partisan to offer his copartisan exactly his discounted continuation value. That is we need to confirm that (A.8) holds. Setting $p = 1$ in (A.5), we find that

$$V^1 = V^2 = \frac{2(1 + \alpha)}{6 - \delta - \alpha\delta}. \quad (\text{A.31})$$

Using (A.31), (A.8) holds if and only if

$$2\delta(1 + \alpha) \geq \alpha(6 - \delta - \alpha\delta).$$

Simple algebra shows that this inequality is equivalent to (A.30).

(c) Suppose that $\delta \in [0, \underline{\delta}(\alpha))$ and that a partisan is the proposer. As in Part (b), for values of δ in this interval, it is not possible to have $\alpha = 0$, so $\alpha \in (0, 1)$. The argument in the proof of Part (b) has also established that $p = 1$. Furthermore, because $\delta < \underline{\delta}(\alpha)$, the inequality in (A.30) is violated and, therefore, the nonnegativity constraint on x_P binds (i.e., $x_P = 0$). Thus, with probability 1, a partisan proposes to keep all of the dollar for himself. Because $p = 1$, the continuation values for two partisans are given in (A.31).

(d) Suppose that $\delta \in [0, \bar{\delta}(\alpha))$ and that the nonpartisan is the proposer. From the proofs of Parts (b) and (c), we know that when δ is in this interval, $p = 1$. The formula for x_S in (A.4) continues to apply provided that it is nonnegative. Substituting $p = 1$ into (A.4), we obtain

$$x_S = \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta}, \quad (\text{A.32})$$

which is 0 if $\delta = 0$; otherwise, it is positive. The nonpartisan proposer's share when offering x_S is

$$1 - x_S = 1 - \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} = \frac{3(2 - \delta - \alpha\delta)}{6 - \delta - \alpha\delta}, \quad (\text{A.33})$$

which is positive.

(e) Suppose that $\delta \in [0, \bar{\delta}(\alpha))$. The continuation values for the partisans are given in (A.31). We have shown that $p = 1$ when $\delta < \bar{\delta}(\alpha)$. By setting $p = 1$ in (A.6), we find that the nonpartisan's continuation value is

$$V^3 = \frac{2 - \delta - \alpha\delta}{6 - \delta - \alpha\delta}. \quad (\text{A.34})$$

A legislator will vote for any proposal in which he receives at least his discounted continuation value. Thus, using (A.31), a partisan will vote for any proposal in which he receives utility u for which

$$u \geq \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta}. \quad (\text{A.35})$$

Similarly, using (A.17), the nonpartisan will vote for any proposal in which he receives utility u for which

$$u \geq \frac{\delta(2 - \delta - \alpha\delta)}{6 - \delta - \alpha\delta}. \quad (\text{A.36})$$

It remains to be shown that each of the distributions proposed receives the support of a majority. Someone who is offered his discounted continuation value is indifferent between supporting or opposing the proposal. However, if he does

not support it, the proposal is defeated and we do not have an equilibrium. There are three cases.

Case 1: $\delta \in [\bar{\delta}(\alpha), 1]$. From (A.13), (A.16), and (A.21), it follows that both partisans vote for $x^{12} = (1 - x_P, x_P, 0)$ because

$$\begin{aligned}
U^1(x^{12}) &= 1 - x_P + \alpha x_P \\
&= 1 - (1 - \alpha) \frac{[-2\alpha + \delta + \alpha\delta]}{(3 + \alpha)(1 - \alpha)} \\
&= \frac{(3 - \delta)(1 + \alpha)}{3 + \alpha} \\
&> \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \\
&= \delta V^1
\end{aligned}$$

and

$$\begin{aligned}
U^2(x^{12}) &= \alpha(1 - x_P) + x_P \\
&= \alpha + (1 - \alpha) \frac{[-2\alpha + \delta + \alpha\delta]}{(3 + \alpha)(1 - \alpha)} \\
&= \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \\
&= \delta V^2.
\end{aligned}$$

From (A.14), (A.16), (A.17), (A.21), and (A.22), it follows that the partisan proposer and the nonpartisan vote for $x^{13} = (1 - x_N, 0, x_N)$ because

$$\begin{aligned}
U^1(x^{13}) &= 1 - x_N \\
&= 1 - \frac{[-2\alpha + \delta + \alpha\delta]}{3 + \alpha} \\
&= \frac{(1 + \alpha)(3 - \delta)}{3 + \alpha} \\
&> \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \\
&= \delta V^1
\end{aligned}$$

and

$$U^3(x^{13}) = x_N = \frac{-2\alpha + \delta + \alpha\delta}{3 + \alpha} = \delta V^3.$$

From (A.15), (A.16), (A.17), (A.21), and (A.22), it follows that the nonpartisan proposer and the partisan who is offered a positive share vote for $x^{31} =$

$(x_S, 0, 1 - x_S)$ because

$$U^3(x^{31}) = 1 - x_S = \frac{3 - \alpha^2 - \delta - \alpha\delta}{3 + \alpha} > \frac{-2\alpha + \delta + \alpha\delta}{3 + \alpha} = \delta V^3$$

and

$$U^1(x^{31}) = x_S = \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} = \delta V^1.$$

Case 2: $\delta \in [\underline{\delta}(\alpha), \bar{\delta}(\alpha)]$. From (A.28) and (A.31), it follows that both partisans vote for $x^{12} = (1 - x_P, x_P, 0)$ because

$$\begin{aligned} U^1(x^{12}) &= 1 - x_P + \alpha x_P \\ &= 1 - (1 - \alpha) \frac{[2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha]}{(6 - \delta - \alpha\delta)(1 - \alpha)} \\ &= \frac{6 - 3\delta - 4\alpha\delta - \alpha^2\delta + 6\alpha}{6 - \delta - \alpha\delta} \\ &= \frac{(6 - 3\delta - \alpha\delta)(1 + \alpha)}{6 - \delta - \alpha\delta} \\ &> \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} \\ &= \delta V^1 \end{aligned}$$

and

$$\begin{aligned} U^2(x^{12}) &= \alpha(1 - x_P) + x_P \\ &= \alpha + (1 - \alpha) \frac{[2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha]}{(6 - \delta - \alpha\delta)(1 - \alpha)} \\ &= \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} \\ &= \delta V^2 \end{aligned}$$

From (A.32) and (A.33)–(A.34), it follows that the nonpartisan proposer and the partisan who is offered a positive share vote for $x^{31} = (x_S, 0, 1 - x_S)$ because

$$U^3(x^{31}) = 1 - x_S = \frac{3(2 - \delta - \alpha\delta)}{6 - \delta - \alpha\delta} > \frac{2 - \delta - \alpha\delta}{6 - \delta - \alpha\delta} = \delta V^3$$

and

$$U^1(x^{31}) = x_S = \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} = \delta V^1.$$

Case 3: $\delta \in [0, \underline{\delta}(\alpha)]$. The proof in Case 2 for the nonpartisan's proposal applies in this case as well.

It remains to confirm that when a partisan proposes $x^{12} = (1, 0, 0)$, it will be approved. A partisan votes for $x^{12} = (1, 0, 0)$ because

$$U^1(x^{12}) = 1 > \frac{2\alpha}{2 + \alpha} > \frac{\delta(1 + \alpha)}{3} \geq \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} = \delta V^1,$$

where the first inequality holds because $\alpha < 1$, the second because $\delta < \frac{6\alpha}{(1+\alpha)(2+\alpha)}$, and the third because $\delta + \alpha\delta \geq 0$.

The copartisan also supports $x^{12} = (1, 0, 0)$ because

$$U^2(x^{12}) = \alpha \geq \frac{2\alpha}{2 + \alpha} > \frac{\delta(1 + \alpha)}{3} \geq \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} = \delta V^2,$$

where the first inequality is strict if $\alpha \neq 0$.

This completes the necessity part of the proof.

For the sufficiency part of the proof, we need to show that the strategies described in the statement of the proposition are a partisan symmetric stationary subgame perfect equilibrium. In other words, we need to show that no legislator wants to deviate unilaterally from these strategies. To do this, we must show that: (i) no legislator in his role as a proposer wants to modify the share offered to one of the other legislators in order to receive his support, (ii) no legislator in his role as a proposer wants to modify the probabilities with which he makes offers to the other legislators, and (iii) no legislator wants to deviate from his voting strategy.

We have already shown that the specified shares are the minimal amounts needed to attain the support of the relevant legislator, so (i) holds. The last part of the necessity proof has established (iii). For a partisan proposer, the proofs of Parts (a), (b), and (c) have shown that deviating from the specified probability p would require reducing a partisan proposer's share in favor of one of the other legislators. When the nonpartisan is the proposer, he receives the share $1 - x_S$ regardless of who ends up supporting him. Thus, the nonpartisan has no incentive to deviate from making offers to each partisan with probability $\frac{1}{2}$. Hence, (ii) holds as well. \square

Proof of Proposition 2. Differentiation for all the comparative statics in this and the subsequent two proofs were carried out using *Mathematica*. The code for these derivations may be found in Appendix B. At a boundary of any of the three regions for δ , the relevant one-sided derivative is used.⁴

(a) For $\delta \in [\bar{\delta}(\alpha), 1]$, by Part (a) of Proposition 1,

$$p = \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{-4\alpha\delta + 2\delta^2 + 2\alpha\delta^2}. \quad (\text{A.37})$$

⁴The equilibrium values of the variables are continuous in the parameters at a boundary between two regions, but they may not be differentiable.

Differentiating (A.37) with respect to α for $\delta \in (\bar{\delta}(\alpha), 1]$, we obtain

$$\begin{aligned}
\frac{\partial p}{\partial \alpha} &= \frac{6 + 2\alpha^2 - \alpha^2\delta - 5\delta - 2\alpha\delta}{2(2\alpha - \alpha\delta - \delta)^2} \\
&\geq \frac{6 + 2\alpha^2 - \alpha^2(1) - 5(1) - 2\alpha(1)}{2(2\alpha - \alpha\delta - \delta)^2} \\
&= \frac{1 - 2\alpha + \alpha^2}{2(2\alpha - \alpha\delta - \delta)^2} \\
&= \frac{(1 - \alpha)^2}{2(2\alpha - \alpha\delta - \delta)^2} \\
&> 0,
\end{aligned} \tag{A.38}$$

where the first inequality follows because $\delta \leq 1$ and the last because $\alpha < 1$.

(b) For $\delta \in [\bar{\delta}(\alpha), 1]$, by Part (a) of Proposition 1,

$$x_P = \frac{-2\alpha + \alpha\delta + \delta}{(3 + \alpha)(1 - \alpha)}. \tag{A.39}$$

Differentiating (A.39) with respect to α for $\delta \in (\bar{\delta}(\alpha), 1]$, we obtain

$$\begin{aligned}
\frac{\partial x_P}{\partial \alpha} &= \frac{-6 - 2\alpha^2 + \alpha^2\delta + 5\delta + 2\alpha\delta}{(3 + \alpha)^2(1 - \alpha)^2} \\
&\leq \frac{-6 - 2\alpha^2 + \alpha^2(1) + 5(1) + 2\alpha(1)}{(3 + \alpha)^2(1 - \alpha)^2} \\
&= \frac{-1 + 2\alpha - \alpha^2}{(3 + \alpha)^2(1 - \alpha)^2} \\
&= -\frac{(1 - \alpha)^2}{(3 + \alpha)^2(1 - \alpha)^2} \\
&= -\frac{1}{(3 + \alpha)^2} \\
&< 0,
\end{aligned} \tag{A.40}$$

where the first inequality follows because $\delta \leq 1$.

For $\delta \in [\underline{\delta}(\alpha), \bar{\delta}(\alpha))$, by Part (b) of Proposition 1,

$$x_P = \frac{2\delta + 3\alpha\delta + \alpha^2\delta - 6\alpha}{(6 - \delta - \alpha\delta)(1 - \alpha)}. \tag{A.41}$$

Differentiating (A.41) with respect to α for $\delta \in (\underline{\delta}(\alpha), \bar{\delta}(\alpha))$, we obtain

$$\begin{aligned}
\frac{\partial x_P}{\partial \alpha} &= \frac{-3(12 - 12\delta - 4\alpha\delta + (1 + \alpha)^2\delta^2)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)^2} & (A.42) \\
&\leq \frac{3(-12 + 12(1) + 4\alpha(1) - (1 + \alpha)^2(1)^2)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)^2} \\
&= \frac{3(4\alpha - (1 + \alpha)^2)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)^2} \\
&= \frac{-3(1 - 2\alpha + \alpha^2)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)^2} \\
&= \frac{-3(1 - \alpha)^2}{(6 - \delta - \alpha\delta)^2(1 - \alpha)^2} \\
&= \frac{-3}{(6 - \delta - \alpha\delta)^2} \\
&< 0,
\end{aligned}$$

where the first inequality follows because $\delta \leq 1$ and the last because neither α nor δ exceeds 1.

It then follows from the continuity of the equilibrium value of x_P in α that x_P is also decreasing in α when $\delta = \bar{\delta}(\alpha)$.

(c) For $\delta \in [\bar{\delta}(\alpha), 1]$, by Part (a) of Proposition 1,

$$x_N = \frac{-2\alpha + \alpha\delta + \delta}{3 + \alpha}. \quad (A.43)$$

Differentiating (A.43) with respect to α for $\delta \in (\bar{\delta}(\alpha), 1]$, we obtain

$$\begin{aligned}
\frac{\partial x_N}{\partial \alpha} &= \frac{2(-3 + \delta)}{(3 + \alpha)^2} & (A.44) \\
&\leq \frac{2(-3 + 1)}{(3 + \alpha)^2} \\
&= \frac{-4}{(3 + \alpha)^2} \\
&< 0,
\end{aligned}$$

where the first inequality follows because $\delta \leq 1$.

(d) For $\delta \in [\bar{\delta}(\alpha), 1]$, by Part (a) of Proposition 1,

$$x_S = \frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha}. \quad (A.45)$$

Differentiating (A.45) with respect to α for $\delta \in (\bar{\delta}(\alpha), 1]$, we obtain

$$\begin{aligned}\frac{\partial x_S}{\partial \alpha} &= \frac{3 + 6\alpha + \alpha^2 + 2\delta}{(3 + \alpha)^2} \\ &\geq \frac{3 + 6(0) + 0^2 + 2(0)}{(3 + \alpha)^2} \\ &= \frac{3}{(3 + \alpha)^2} \\ &> 0,\end{aligned}\tag{A.46}$$

where the first inequality follows because both α and δ are nonnegative.

For $\delta \in [0, \bar{\delta}(\alpha))$, by Part (d) of Proposition 1,

$$x_S = \frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta}.\tag{A.47}$$

Differentiating (A.47) with respect to α for $\delta \in (0, \bar{\delta}(\alpha))$, we obtain

$$\begin{aligned}\frac{\partial x_S}{\partial \alpha} &= \frac{12\delta}{(6 - \delta - \alpha\delta)^2} \\ &> 0,\end{aligned}\tag{A.48}$$

where the inequality holds because $\delta > 0$. Therefore, x_S is also increasing in α in this interval for δ .

It then follows from the continuity of the equilibrium value of x_S in α that x_S is also increasing in α when $\delta = \bar{\delta}(\alpha) \neq 0$. \square

Proof of Proposition 3. (a) For $\delta \in [\bar{\delta}(\alpha), 1)$, differentiating (A.37) with respect to δ , we obtain

$$\frac{\partial p}{\partial \delta} = \frac{\alpha(\alpha^2\delta^2 - 12\delta + 5\delta^2 + 12\alpha - 12\alpha\delta + 6\alpha\delta^2)}{2\delta^2(2 - \alpha\delta - \delta)^2}\tag{A.49}$$

For $0 < \alpha < 1$, the *Mathematica* derivations in Appendix B demonstrate that in order to establish that this derivative is negative, it is sufficient to show that

$$\delta > \frac{2\alpha}{1 + \alpha}.\tag{A.50}$$

Because $0 < \alpha < 1$,

$$\bar{\delta}(\alpha) = \frac{6\alpha}{(1 + \alpha)(2 + \alpha)} > \frac{2\alpha}{1 + \alpha}.$$

Therefore, because $\bar{\delta}(\alpha) > \underline{\delta}(\alpha)$, (A.50) holds. Because the derivative in (A.49) is negative for all $\alpha > 0$, p is decreasing in δ in this interval for δ .

(b) For $\delta \in [\bar{\delta}(\alpha), 1)$, differentiating (A.39) with respect to δ , we obtain

$$\begin{aligned}\frac{\partial x_P}{\partial \delta} &= \frac{1 + \alpha}{(3 + \alpha)(1 - \alpha)} \\ &\geq \frac{1 + 0}{(3 + \alpha)(1 - \alpha)} \\ &> 0,\end{aligned}\tag{A.51}$$

where the first inequality holds because $\alpha \geq 0$ and the second because $\alpha < 1$.

For $\delta \in [\underline{\delta}(\alpha), \bar{\delta}(\alpha))$, differentiating (A.41) with respect to δ , we obtain

$$\begin{aligned}\frac{\partial x_P}{\partial \delta} &= \frac{12(1 + \alpha)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)} \\ &\geq \frac{12(1 + 0)}{(6 - \delta - \alpha\delta)^2(1 - \alpha)} \\ &> 0,\end{aligned}\tag{A.52}$$

where the first inequality holds because $\alpha \geq 0$ and the second because $\alpha < 1$.

(c) For $\delta \in [\bar{\delta}(\alpha), 1)$, differentiating (A.43) with respect to δ , we obtain

$$\begin{aligned}\frac{\partial x_N}{\partial \delta} &= \frac{1 + \alpha}{3 + \alpha} \\ &\geq \frac{1 + 0}{3 + \alpha} \\ &> 0,\end{aligned}\tag{A.53}$$

where the both inequalities hold because $\alpha \geq 0$.

(d) For $\delta \in [\bar{\delta}(\alpha), 1)$, differentiating (A.45) with respect to δ , we obtain

$$\begin{aligned}\frac{\partial x_S}{\partial \delta} &= \frac{1 + \alpha}{3 + \alpha} \\ &> 0,\end{aligned}\tag{A.54}$$

where the inequality follows as in the proof of Part (c).

For $\delta \in (0, \bar{\delta}(\alpha))$, differentiating (A.47) with respect to δ , we obtain

$$\begin{aligned}\frac{\partial x_S}{\partial \delta} &= \frac{12(1 + \alpha)}{(6 - \alpha - \alpha\delta)^2} \\ &> 0,\end{aligned}\tag{A.55}$$

where the inequality holds because the derivative in (A.55) is a positive multiple of the derivative in (A.52). The value of x_S is 0 if and only if $\delta = 0$, so x_S is also increasing in δ at $\delta = 0$. \square

Proof of Proposition 4. We only show the derivations used to obtain the expressions for $E[x_1]$ as it is straightforward to derive the expressions for $E[x_3]$ from them using (19).

(a) Substituting the equilibrium values of p , x_P , x_N , and x_S in (17), we obtain

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[\frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{-4\alpha\delta + 2\delta^2 + 2\alpha\delta^2} \right. \\ &\quad + \left(1 - \frac{6\alpha - 7\alpha\delta - \alpha^2\delta + \delta^2 + \alpha\delta^2}{-4\alpha\delta + 2\delta^2 + 2\alpha\delta^2} \right) \left(1 - \frac{-2\alpha + \alpha\delta + \delta}{3 + \alpha} \right) \\ &\quad \left. + \frac{1}{2} \left(\frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \right) \right]. \end{aligned}$$

Simplifying,

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[\frac{-(\alpha + \delta)(6 - \delta - \alpha\delta)(2\alpha - \delta - \alpha\delta)}{-2\delta(2\alpha - \delta - \alpha\delta)(3 + \alpha)} + \frac{1}{2} \left(\frac{(1 + \alpha)(\alpha + \delta)}{3 + \alpha} \right) \right] \\ &= \frac{1}{3} \left[\frac{(\alpha + \delta)(6 - \delta - \alpha\delta) + (\alpha + \delta)\delta(1 + \alpha)}{2\delta(3 + \alpha)} \right] \\ &= \frac{1}{3} \left[\frac{6(\alpha + \delta)}{2\delta(3 + \alpha)} \right] \\ &= \frac{\alpha + \delta}{\delta(3 + \alpha)}. \end{aligned}$$

(b) Substituting the equilibrium values of p , x_P , x_N , and x_S in (18), we obtain

$$E[x_1] = \frac{1}{3} \left[1 + \frac{1}{2} \left(\frac{2\delta(1 + \alpha)}{6 - \delta - \alpha\delta} \right) \right].$$

Simplifying,

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[\frac{6 - \delta - \alpha\delta + \delta(1 + \alpha)}{6 - \delta - \alpha\delta} \right] \\ &= \frac{2}{6 - \delta - \alpha\delta}. \end{aligned}$$

□

Deriving (20)–(23).

For $\delta \in [\bar{\delta}(\alpha), 1]$, we can rewrite the expression for $E[x_1]$ in Part (a) of Proposition 4 as

$$E[x_1] = \frac{1}{3} \left[1 + \frac{3(\alpha + \delta)}{(3 + \alpha)\delta} - \frac{(3 + \alpha)\delta}{(3 + \alpha)\delta} \right].$$

Simplifying,

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[1 + \frac{(3\alpha + 3\delta - \alpha\delta - 3\delta)}{(3 + \alpha)\delta} \right] \\ &= \frac{1}{3} \left[1 + \frac{(3 - \delta)\alpha}{(3 + \alpha)\delta} \right], \end{aligned}$$

which is (20). Similarly, for $\delta \in [0, \bar{\delta}(\alpha))$, we can rewrite the expression for $E[x_1]$ in Part (b) of Proposition 4 as

$$E[x_1] = \frac{1}{3} \left[1 + \frac{6}{(6 - \delta - \alpha\delta)} - \frac{(6 - \delta - \alpha\delta)}{(6 - \delta - \alpha\delta)} \right].$$

Simplifying,

$$\begin{aligned} E[x_1] &= \frac{1}{3} \left[1 + \frac{(\delta + \alpha\delta)}{[6 - (1 + \alpha)\delta]} \right] \\ &= \frac{1}{3} \left[1 + \frac{(1 + \alpha)\delta}{[6 - (1 + \alpha)\delta]} \right], \end{aligned}$$

which is (22). The formulae in (21) and (23) are obtained by substituting (20) and (22) in (19).

Proof of Proposition 5. In light of (19), we only need to show how $E[x_1]$ responds to increases in α and δ .

(a) By rewriting (20) as

$$E[x_1] = \frac{1}{3} \left[1 + \frac{(3 - \delta)}{\left(1 + \frac{3}{\alpha}\right)\delta} \right],$$

it is clear that $E[x_1]$ is increasing in α when $\delta \in [\bar{\delta}(\alpha), 1]$ and $\delta \neq 0$. Because an increase in α decreases the denominator in the expression for $E[x_1]$ in Part (b) of Proposition 4, $E[x_1]$ is also increasing in α when $\delta \in (0, \bar{\delta}(\alpha))$.

(b) Because an increase in δ decreases the numerator and increases the denominator in the fraction in (20) when $\alpha > 0$, $E[x_1]$ is decreasing in δ when $\delta \in [\bar{\delta}(\alpha), 1]$. Because an increase in δ decreases the denominator in the expression for $E[x_1]$ in Part (b) of Proposition 4, $E[x_1]$ is increasing in δ when $\delta \in (0, \bar{\delta}(\alpha))$. \square

Appendix B. *Mathematica* Derivations

To implement our *Mathematica* derivations, we use a for α and d for δ . We let xPH denote the expression for x_P in (A.39), xPI denote the expression for x_P in (A.41), xSH denote the expression for x_S in (A.45), $xSIL$ denote the expression for x_S in (A.47), and xNH denote the expression for x_N in (A.43).

A partial derivative of the form $\partial f/\partial x$ is implemented by writing $D[f, x]$. The following notation for operators is used: $*$ for multiplication, $\&\&$ for conjunction, and $||$ for disjunction. The symbol $==$ is used to denote that two expressions are identically equal.

Plotting the graphs in Figure 1.

Input:

$$\text{Plot}\left[\left\{\frac{-3a-a^2}{2(1+a)} + \frac{1}{2}\sqrt{\frac{24a+33a^2+6a^3+a^4}{(1+a)^2}}, \frac{6a}{2+3a+a^2}\right\}, \{a, 0, 1\}\right]$$

Output:

See Figure 1.

Plotting the graphs in Figure 3.

Input:

$$\text{Plot}\left[\text{Evaluate@Table}\left[\text{Piecewise}\left[\left\{\left\{\frac{a+d}{d(3+a)}, d \geq \frac{-3a-a^2}{2(1+a)} + \frac{1}{2}\sqrt{\frac{24a+33a^2+6a^3+a^4}{(1+a)^2}}\right\}, \left\{\frac{2}{6-d-a*d}, d < \frac{-3a-a^2}{2(1+a)} + \frac{1}{2}\sqrt{\frac{24a+33a^2+6a^3+a^4}{(1+a)^2}}\right\}\right]\right], \{a, \{0, 0.1, 0.25, 0.5, 0.75\}\}, \{d, 0.5, 1\}, \text{PlotRange} \rightarrow \{0.3, 0.475\}\right]$$

Output:

See Figure 3.

Verifying (A.26).

Input:

$$\text{Reduce}[\{p == (6 * a - 7a * d - a^2 * d + d^2 + a * d^2)/(2d(d + a * d - 2a)), 0 \leq p \leq 1, 0 < a < 1, 0 \leq d \leq 1\}, \{d, p\}]$$

Output:

$$0 < a < 1 \ \&\& \ \frac{-3a-a^2}{2(1+a)} + \frac{1}{2}\sqrt{\frac{24a+33a^2+6a^3+a^4}{(1+a)^2}} \leq d \leq 1 \ \&\& \ p = \frac{6a-7ad-a^2d+d^2+ad^2}{-4ad+2d^2+2ad^2}$$

Verifying (A.27).

Input:

$$\text{Reduce}[\{p == (6 * a - 7a * d - a^2 * d + d^2 + a * d^2)/(2d(d + a * d - 2a)), p \geq 0, 0 < a < 1, 0 \leq d \leq 1\}, \{d, p\}]$$

Output:

$$0 < a < 1 \quad \&\& \quad \frac{2a}{1+a} \leq d \leq 1 \quad \&\& \quad p = \frac{6a-7ad-a^2d+d^2+ad^2}{-4ad+2d^2+2ad^2}$$

Verifying (A.38).

Input:

Simplify $[D[p, a]]$

Output:

$$\frac{6-a^2(-2+d)-5d-2ad}{-4ad+2d^2+2ad^2}$$

Verifying (A.40).

Input:

Simplify $[D[xPH, a]]$

Output:

$$\frac{-6+a^2(-2+d)+5d+2ad}{(-1+a)^2(3+a)^2}$$

Verifying (A.42).

Input:

Simplify $[D[xPI, a]]$

Output:

$$-\frac{3(12-4(3+a)d+(1+a)^2d^2)}{(-1+a)^2(-6+d+ad)^2}$$

Verifying (A.44).

Input:

Simplify $[D[xNH, a]]$

Output:

$$\frac{2(-3+d)}{(3+a)^2}$$

Verifying (A.46).

Input:

Simplify $[D[xSH, a]]$

Output:

$$\frac{3+6a+a^2+2d}{(3+a)^2}$$

Verifying (A.48).

Input:

Simplify $[D[xSIL, a]]$

Output:

$$\frac{12d}{(-6+d+ad)^2}$$

Verifying (A.49).

Input:

Simplify $[D[p, d]]$

Output:

$$\frac{a(a^2d^2+d(-12+5d)+6a(2-2d+d^2))}{2d^2(a(-2+d)+d)^2}$$

The sufficiency of (A.50).

Input:

Reduce $\left[\left\{ \frac{a(a^2d^2+d(-12+5d)+6a(2-2d+d^2))}{2d^2(a(-2+d)+d)^2} < 0, 0 \leq a < 1, 0 \leq d \leq 1 \right\}, d \right]$

Output:

$$0 < a < 1 \ \&\& \ \left(\frac{6}{5+a} - 2\sqrt{3}\sqrt{-\frac{-3+2a+a^2}{(1+a)(5+a)^2}} < d < \frac{2a}{1+a} \ || \ \frac{2a}{1+a} < d \leq 1 \right)$$

Verifying (A.51).

Input:

Simplify $[D[xPH, d]]$

Output:

$$\frac{1+a}{3-2a-a^2}$$

Verifying (A.52).

Input:

Simplify $[D[xPI, d]]$

Output:

$$-\frac{12(1+a)}{(-1+a)(-6+d+ad)^2}$$

Verifying (A.53).

Input:

Simplify $[D[xNH, d]]$

Output:

$$\frac{1+a}{3+a}$$

Verifying (A.54).

Input:

Simplify $[D[xSH, d]]$

Output:

$$\frac{1+a}{3+a}$$

Verifying (A.55).

Input:

Simplify $[D[xSIL, d]]$

Output:

$$\frac{12(1+a)}{(-6+d+ad)^2}$$