Part 4: Microscopic nuclear structure and reaction theory

4.1 Nucleon-Nucleon (NN) interaction

There are several approaches to obtain the NN interaction potential. Most important are these:

a) construct $V_{NN}$ from general invariance principles. We will see that this predicts the spin- and isospin-dependence, but not the radial dependence of the potential.

b) from meson-exchange theories, the dominant term at large $r$ is the "one-pion exchange potential" (OPEP).

In 1941, Eisenbud and Wigner gave a very general discussion of the mathematical structure of $V_{NN}$ based on invariance principles in physics. They assumed a potential of the form

$$V_{NN}(\vec{r}_1,\vec{r}_2, \vec{p}_1, \vec{p}_2, \vec{s}_1, \vec{s}_2, \vec{t}_1, \vec{t}_2).$$

**Symmetry requirements:**

- Translational inv.: $\vec{r}_i \rightarrow \vec{r}_i + \vec{a}$, const. vector
- $V_{NN}$ depends only on relative distance vector
  $$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$
\[ V_{NN} = V_{NN} \left( \vec{r}, \vec{p}_1, \vec{p}_2 \right) \]

- Galilei inv.: 
  \[ \vec{p} \rightarrow \vec{p} + \vec{P}_0 \] 
  const. vector

\[ \Rightarrow \text{V}_{NN} \text{ depends only on rel. momentum} \]

\[ \vec{P} = \vec{p}_1 - \vec{p}_2 \]

\[ \Rightarrow \text{V}_{NN} = V_{NN} \left( \vec{r}, \vec{p}_1, \vec{p}_2, \vec{r}_1, \vec{r}_2 \right) \]

- approx. charge indep. of strong int. 
  V isospin scalar expts. reveal that for the strong part of \( V_{NN} \)

\[ V_{pp} \approx V_{nn} \approx V_{np} \]

\[ \Rightarrow \text{invar. under rotation in isospin space \ V isospin scalar. Possible scalars:} \]

\[ \begin{align*}
  \hat{\tau}_i^z = \hat{\tau}_i \cdot \tau_c \quad (i = 1, 2) \\
  = 3 \quad \text{(trivial constants)} \\
  \hat{\tau}_1 \cdot \hat{\tau}_2 
\end{align*} \]

\[ \Rightarrow \begin{align*}
  \left\{ \begin{array}{c}
  \text{const.} \\
  i = 1, 2
\end{array} \right\}
\end{align*} \]

Hence, \( V_{NN} \) has the form

\[ V_{NN} = v_1 \left( \vec{r}, \vec{p}_1, \vec{r}_1, \vec{r}_2 \right) + v_2 \left( \vec{r}, \vec{p}_1, \vec{r}_1, \vec{r}_2 \right) \hat{\tau}_1 \cdot \hat{\tau}_2 \]

- Rotational inv:

  The generators of rotations are the ang. momentum operators.
  Rot. inv. implies that all terms in \( v_{12} \left( \vec{r}, \vec{p}_1, \vec{r}_1, \vec{r}_2 \right) \) above must be constructed to have a total ang. momentum of zero, i.e., they must be scalars in the combined
(coordinate + spin)-space! There is a very large number of scalars that can be formed from \((\hat{r}, \hat{p}, \hat{s}_1, \hat{s}_2)\), e.g.
\[
\hat{r} \cdot \hat{r} = r^2, \quad \hat{s}_1 \cdot \hat{s}_2, \quad \hat{s}_1 \cdot \hat{r}, \quad \hat{s}_1 \cdot \hat{p}, \quad \hat{s}_2 \cdot (\hat{r} \times \hat{p}), \quad \hat{r} \cdot (\hat{s}_1 \times \hat{s}_2), \quad \hat{p} \cdot (\hat{s}_1 \times \hat{s}_2), \quad \text{etc.}
\]

These possible combinations can be reduced further by 2 additional requirements:

- **Parity Inv. \((\mathcal{P})\):**

  \[
  \Rightarrow v(\hat{r}, \hat{p}, \hat{s}_1, \hat{s}_2) \xrightarrow{\mathcal{P}} v(-\hat{r}, -\hat{p}, \hat{s}_1, \hat{s}_2)
  \]

- **Time-reversal Inv. \((\mathcal{T})\):**

  \[
  \Rightarrow \tilde{v}(\hat{r}, \hat{p}, \hat{s}_1, \hat{s}_2) \xrightarrow{\mathcal{T}} v(\hat{r}, -\hat{p}, -\hat{s}_1, -\hat{s}_2)
  
  \text{note that the spins must transform like } \hat{s} = \hat{r} \times \hat{p}.
  
  \]

This rules out the fall combinations above:

\[
\begin{align*}
\hat{s}_i \cdot \hat{p} & \quad \text{(not } \mathcal{P} \text{-inv.)} \\
\hat{r} \cdot \hat{p} & \quad \text{(not } \mathcal{T} \text{-inv.)} \\
\hat{r} \cdot (\hat{s}_1 \times \hat{s}_2) & \quad \text{(not } \mathcal{P} \text{-inv.)} \\
\hat{p} \cdot (\hat{s}_1 \times \hat{s}_2) & \quad \text{(violates both } \mathcal{P} \text{ and } \mathcal{T}).
\end{align*}
\]

This leaves us with these building blocks for \(v_1, v_2\):

\[
v(\hat{r}, \hat{p}, \hat{s}_1, \hat{s}_2) \propto \begin{cases} 
  r^2 \text{ v arb. fn in } f(r) \\
  \hat{s}_1 \cdot \hat{s}_2 \\
  \hat{s}_1 \cdot \hat{r} \\
  \hat{s}_1 \cdot (\hat{r} \times \hat{p}) \end{cases} \quad \text{for } i = 1, 2
\]
The last term contains the possibility of a strong spin-orbit interaction, because

\[ \mathbf{Z} \cdot \mathbf{S} = (\mathbf{r} \times \mathbf{p}) \cdot \frac{1}{2} (\mathbf{\hat{s}}_1 + \mathbf{\hat{s}}_2) \]

Argonne \( v_{18} \) NN-potential


This potential was developed by physicists at Argonne Nat. Lab. and it is composed of 18 terms — hence the name.

\[ V_{NN} (\mathbf{r}, \mathbf{p}, \mathbf{\hat{s}}_i, \mathbf{\hat{c}}_i) = \sum_{k=1}^{18} v_k (r) O_k \]

Note that one cannot obtain any information about the radial factors \( v_k (r) \) from invariance principles. One either needs to obtain these from meson exchange (see below) or from exp. N-N scattering data. Here are some of the important terms:

\[ O_1 = 1 \quad (" central component" \equiv c) \]
\[ O_2 = \mathbf{\hat{r}} \cdot \mathbf{\hat{c}} \quad (" isospin component" \equiv \mathbf{\hat{t}}) \]
\[ O_3 = \mathbf{\hat{s}}_1 \cdot \mathbf{\hat{s}}_2 \quad (" spin component" \equiv \mathbf{\hat{s}}) \]
\[ O_4 = (\mathbf{\hat{s}}_1 \cdot \mathbf{\hat{s}}_2) (\mathbf{\hat{r}} \cdot \mathbf{\hat{c}}) \quad (" spin-isospin component" \equiv \mathbf{\hat{t}} \cdot \mathbf{\hat{s}}) \]

The radial factors corresponding to these 4 terms, i.e. \( v_1, v_2, v_3, v_4 (r) \), are shown in fig. 6 of the above paper. ⇒ slide 2
Another possible combination that can be constructed from the invariant blocks discussed above is the "tensor operator" $S_{12}$ defined as

$$S_{12} = 3 \left( \hat{\sigma}_1 \cdot \vec{r} \right) \left( \hat{\sigma}_2 \cdot \vec{r} \right) - \hat{\sigma}_1 \cdot \hat{\sigma}_2$$

Note that this operator is a second-rank tensor in either the spin-space or in conf. space alone, but a scalar in the combined conf. + spin space!

We observe that this tensor op. has the same form as the interaction potential between 2 magnetic dipoles:

$$V_{\text{magn. dipole}} \propto \frac{1}{r^3} \left[ 3 \left( \hat{\mu}_1 \cdot \vec{r} \right) \left( \hat{\mu}_2 \cdot \vec{r} \right) - \hat{\mu}_1 \cdot \hat{\mu}_2 \right]$$

A simple "geometric" interpretation of $S_{12}$ is that it depends on the angle between the 2 spins ($\hat{\sigma}_1 \cdot \hat{\sigma}_2$) and on the angle between $\vec{r}$ and $\hat{\sigma}_i$:

One can show that the tensor force has a vanishing angular average, i.e. $\int d\Omega S_{12} = 0$.

We already mentioned in the section that the N-N interaction arises from virtual meson exchange, discuss slide 16 in this section.
and from the energy-time uncertainty relation \( \Delta E \cdot \Delta t \approx \hbar \), we found that the range of the force, \( \Delta r \), is of order of the Compton wavelength of the meson (\( \lambda_c \)):

\[
\Delta r \approx \lambda_c = \frac{\hbar c}{m c^2} = \frac{197.3 \text{ MeV fm}}{m c^2(\text{meson})}
\]

This implies the following:

a) at large distances, \( V_{\text{NN}} \) is dominated by the lowest-mass meson, the pion with \( m c^2 \approx 140 \text{ MeV} \), resulting in a range of

\[
(\Delta r)_\pi = \frac{197.3 \text{ MeV fm}}{140 \text{ MeV}} \approx 1.4 \text{ fm} = 2 \lambda_c
\]

b) at intermediate distance, we can either have two-pion exchange or \( f_0/\phi \)-meson exchange.

c) at short distances, we have heavier meson exchange (\( \rho, \omega, \phi \)).

The "OPEP" potential

From the Feynman diagram on the left one can determine the one-pion exchange potential (OPEP) in the static limit (details: Bjorken & Drell, "Rel. Quantum Mechanic").
one finds

\[ V_{0\text{pep}}(\hat{r}_1, \hat{e}_1, \hat{e}_2, \hat{\tau}_1, \hat{\tau}_2) = f_1(r)(\hat{e}_1 \cdot \hat{e}_2)(\hat{\tau}_1 \cdot \hat{\tau}_2) + f_2(r)(\hat{\tau}_1 \cdot \hat{\tau}_2) S_{12} \]

with \( f_1(r) = 0.088 \cdot \left( \frac{m_\pi c^2}{3} \right) \frac{e^{-r/\lambda_\pi}}{(r/\lambda_\pi)} \)

and \( f_2(r) = 0.088 \cdot \left( \frac{m_\pi c^2}{3} \right) \frac{e^{-r/\lambda_\pi}}{(r/\lambda_\pi)} \left[ 1 + \frac{3}{(r/\lambda_\pi)^2} + \frac{3}{(r/\lambda_\pi)^3} \right] \)

where the pion rest energy is \( m_\pi c^2 \approx 140 \text{ MeV} \) and \( \lambda_\pi \approx 1.4 \text{ fm} \).

The term

\[ \frac{e^{-r/\lambda_\pi}}{(r/\lambda_\pi)} \]

is the famous "Yukawa potential" predicted by Yukawa in 1935! We see that the first term in \( V_{0\text{pep}} \) contributes to the "spin-isospin component" \( O_4 = (\hat{e}_1 \cdot \hat{e}_2)(\hat{\tau}_1 \cdot \hat{\tau}_2) \) term (for

in the Argonne \( v_{18} \) potential, see slide 2).

We introduce now further terms in \( v_{18} \):

\[ O_5 = S_{12} \] ("tensor" part, labeled \( t \))
\[ O_6 = S_{12}(\hat{\tau}_1 \cdot \hat{\tau}_2) \] ("tensor-isospin" part \( \tilde{t} \tilde{c} \))
\[ O_7 = \vec{L} \cdot \vec{S} \] ("spin-orbit" part \( l s \))
\[ O_8 = \vec{L} \cdot S(\hat{\tau}_1 \cdot \hat{\tau}_2) \] ("spin-orbit isospin" part \( l s \tilde{c} \))

+ other terms

Apparently, \( V_{0\text{pep}} \) contributes also a tensor-isospin component to \( v_{18} \). The radial part of the \( O_5 \) and \( O_6 \) terms (labeled \( t \) and \( t \tilde{c} \)) are shown in slide 3. It also
displays the OPEP contribution to the "\( t \epsilon \)" term \( o_6 \).

Discuss slide \( 4 \): radial part for spin-orbit term \( O_7 \) \( (\vec{r} \cdot \vec{s}) \) and \( O_8 \) \( (\vec{r} \cdot \vec{s} (\vec{r} \cdot \vec{r}_i)) \) in Argonne \( v_{18} \) potential.

Can we understand the spin-orbit term on the basis of meson exchange theory? Yes! The following table gives the mesons and the type of interaction potential \( V_{NN} \) that they generate [Greiner & Hämmerle textbook, p. 213]:

<table>
<thead>
<tr>
<th>Meson ((J^P))</th>
<th>Name</th>
<th>( V_{NN} ) Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar ((0^+))</td>
<td>&quot;( \sigma )-meson&quot; ( f_0/5 )</td>
<td>( s_1, \vec{r} \cdot \vec{s} )</td>
</tr>
<tr>
<td>Pseudoscalar ((0^-))</td>
<td>( \pi, \eta, \eta' )</td>
<td>( s_{12} )</td>
</tr>
<tr>
<td>Vector ((1^-))</td>
<td>( \omega, \phi )</td>
<td>( s_{12}, \vec{r} \cdot \vec{s}, \vec{r}_1 \cdot \vec{r}_2 )</td>
</tr>
</tbody>
</table>

\( N-N \) quantum states and \( \text{WF}'s \)

We define the following quantities for the \( N-N \) pair:

\( \vec{r} = \vec{r}_1 - \vec{r}_2 = \text{rel. distance} \)

\( \vec{p} = \vec{p}_1 - \vec{p}_2 = \text{rel. momentum} \)

\( \vec{L} = \vec{r} \times \vec{p} = \text{rel. orbital \( \text{ang. mom.} \)} \)

\( \vec{S} = \frac{1}{2} (\vec{s}_1 + \vec{s}_2) = \text{spin of } N-N \text{ pair} \)

\( \vec{J} = \vec{L} + \vec{S} = \text{total \( \text{ang. mom.} \) of } N-N \text{ pair} \)

\( \vec{I} = \frac{1}{2} (\vec{I}_1 + \vec{I}_2) = \text{isospin of } N-N \text{ pair} \).