Physics 221 — Lecture 30 — “Atoms and the Planck Radiation Law”
March 29, 1999

Reading
Meyer-Arendt Ch. 20; Hecht, Ch. 13; Young, Ch. 4.

Demonstrations
Planck.nb in Mathematica demonstration

Problems
Example 2, Ch. 20, problems 1, 2, 3, 6.

Reminders
See WWW for course information.

1. The end of the nineteenth century: the ultraviolet catastrophe. During the next several weeks we will be exploring quantum optics, that part of optics which is governed by the laws of atoms, molecules and solids at the smallest size scales. Quantum electrodynamics, which underlies the phenomena of quantum optics, is perhaps the most successful theory of the twentieth century, predicting with extraordinary accuracy essentially all the properties of electrons in atoms. The origins of this theory lie almost a century back in time:

a. Using the classical theory of the electron as an harmonic oscillator, Lorentz and others had formulated a theory of classical electrodynamics, which seemed to make perfect sense but which was seriously inconsistent with a number of experimental observations of radiation from atomic systems.

b. The solution to this problem turned out to be the idea of quanta (Greek: “a little bit”) introduced by Max Planck, a professor at the Technical University of Berlin at the end of the nineteenth century. The idea of quantization is central to our understanding of all phenomena at the atomic level.

c. By the end of the nineteenth century, primitive infrared detectors were becoming available, and physicists were starting to measure the distribution of intensity as a function of wavelength or frequency from light sources ranging from the sun to arc lamps (gas discharges). It was in this context that Planck’s theory of quanta was first elaborated.

d. Today, we will explore the origins of continuum radiation, and the impact of energy quantization on this radiation. Specific concepts to be discussed are:

i. Radiation from a charged particle in classical electrodynamics

ii. Radiation emitted by a hot body in thermal equilibrium (e.g., the sun)

iii. The distribution function for the number of radiative modes at given λ

2. The classical theory of radiation of an oscillating dipole. In general, every symmetry component of a system of accelerated charges — dipole, quadrupole, octupole, hexadecapole (!) — has a unique pattern of electric and magnetic field lines associated with it. However, in many physical situations, the classical picture of the oscillating electric dipole describes the most important component of the radiation. Indeed, in this case, the classical picture gives the same results as those derived from quantum electrodynamics. As a motivational exercise, we will take a quick peek at
radiation from an accelerated charge, normally a tour of a few weeks in a classical electrodynamics course. Let us consider what some fancy mathematics would tell us about the electric field of an accelerated charge.

a. Suppose we have a charge distribution \( \rho(\mathbf{r},t) \) which can be in motion. At some point \( \mathbf{r}' \), the scalar potential due to that charge distribution is:

\[
\Phi(\mathbf{r},t) = \int_V \frac{\rho(\mathbf{r},t)}{r' r} d^3r \Rightarrow \int_V \left[ \frac{\rho(\mathbf{r},t)}{r' - r} \right] d^3r
\]

where the bracket around the \( r(\mathbf{r},t) \) under the second integral is intended to indicate that the integral is to be evaluated at the *retarded* time \( (t - r'/c) \), since no signal can get to the observer faster than that and the charge may have moved during that interval.

b. There is an analogous expression for the so-called *vector potential* due to the moving currents in the volume of integration, and this gives rise to the magnetic field. (Sorry folks, this is only the five-cent tour!)

\[
\mathbf{A}(\mathbf{r}',t) = \frac{\mu}{c} \int_V \left[ \mathbf{J}(\mathbf{r},t) \right] d^3r
\]

c. In books on electromagnetic theory,\(^1\) it is shown that taking the gradient of the scalar potential and the curl of the vector potential gives rise to the electric field and magnetic field of the moving charge distribution; there are in fact two sets of fields, called the velocity and acceleration fields, the former varying as \( 1/r^2 \) and the latter as \( 1/r \). The \( 1/r \) fields are the radiation fields, and are the only ones which are important at large distances.

---

\(^1\) See, for example, J. B. Marion, *Classical Electromagnetic Radiation* (New York: Academic Press, 1965), Chapter 8 for a derivation of the Lenard-Wiechert potentials.
Radiation field from an accelerating charge in the classical picture. At left are the field lines from the charge undergoing uniform acceleration. At right is the “kink” in the electric field — the wavelike disturbance — propagating outward from the moving charge. This is what is responsible for the radiation field. From Hecht, Optics (2nd edition), pp. 47 ff.

d. Consider the special case of a charged particle executing simple harmonic motion along the x axis, so that its position as a function of time is given by \( x = x_0 \cos(\omega t) \). Differentiating this expression twice to get the acceleration \( a \) which we need to find the dependence of the radiated power on frequency, we find that \( a = \ddot{x}(t) = -\omega^2 x_0 \cos(\omega t) \). This gives a radiant flux ("intensity") or the rate of emission of radiant energy by an accelerated charged particle as a function of frequency which looks like the following:

\[
\Phi_e = -\frac{dQ_e}{dt} = \frac{2q^2a^2}{3c^3} = \frac{q^2\omega^4x_0^2}{3c^3}
\]

3. Radiation and quanta. The crisis in physics at the end of the nineteenth century was inaugurated by measurements with the spectrograph (containing a diffraction grating), and the first relatively primitive infrared detectors. The first step in developing a quantum theory of radiation was understanding continuum radiation as exemplified by radiation emitted by hot bodies in thermal equilibrium — so-called blackbody radiation (German: Hohlraumstrahlung).

a. Measurements of solar spectra had shown that the solar radiation had a peak in the visible region of the spectrum, but contained also ultraviolet and
infrared radiation. The maxima of these intensity distributions vs wavelength were found to be purely a function of temperature, as summarized in Wien’s displacement law, \( \lambda_{\text{max}} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} \).

b. What is a blackbody? Such a source exists only as an ideal emitter-absorber; it is one for which all incident radiation at every wavelength is totally absorbed. [Many practical sources approach the blackbody, and are referred to as "greybody" sources.] These sources were much studied in the nineteenth century because they represented hot matter in thermal equilibrium, and in fact it was the study of blackbody radiation which led directly to Planck’s radiation law and the birth of the photon concept. The irradiance produced by an object at a temperature \( T \) was found by Stefan and Boltzmann to be proportional to the fourth power of the temperature: \( I = \epsilon \sigma T^4 \).

c. This empirically verified relationship is known as Stefan’s law, after its discoverer (for once!), and the universal constant \( \sigma \) is called the Stefan-Boltzmann constant. The quantity \( \epsilon \) is known as the emissivity of the surface, and takes on values between 0 and 1 for bodies in thermal equilibrium at a temperature \( T \). For a true blackbody, the emissivity is 1 by definition. Note that the irradiance here is the total power radiated by the blackbody, and is thus an integral over all wavelengths.

d. Classical radiation theory also predicted that the intensity of light from a body at temperature \( T \) would become extremely large at short wavelengths, in contradiction to experiment. (This was the so-called “ultraviolet catastrophe.”) This was because, if radiation was in a cavity, it had to be built up of standing waves, and, the shorter the wavelength, the more waves could be fit into the cavity. Given that each such cavity standing mode had \( k_B T \) energy associated with it, this would lead to an infinite energy content in the uv!

4. Planck and the “quantum hypothesis.” Although he considered it only a working hypothesis, Max Planck suggested around 1889 that the results could be explained by assuming that the oscillators giving rise to the radiation could emit only discrete amounts of energy, where the integer \( n \) is a quantum number characteristic of the internal state of the oscillator.

a. According to Planck’s quantum hypothesis, the only oscillator energies allowed would be \( E = nh\nu \) where \( n \) is an integer, \( h \) is a constant \( [6.64 \times 10^{-34} \text{ J} \cdot \text{s}, \text{with units of action}] \) to be determined by experiment, and \( \nu \) is the frequency of the radiator.

b. If one accepts this hypothesis, and sums up the contributions over all the different wavelengths, one arrives at the Planck distribution law for the spectral radiance \( I(\lambda, T) \) of a black body:
\[ I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5 \left[ e^{\frac{hc}{\lambda k_B T}} - 1 \right]} \]

This distribution law can fit the observed \( I(\lambda,T) \) curves. On the other hand, it stands all of classical physics on its head.

Note that the maximum wavelengths in the top graphs are different, to show the general shape of the \( I(\lambda) \); plotting the spectrum on the same wavelength scale (in the bottom row of graphs) shows how the maximum shifts toward the blue with increasing temperature in accordance with Wien’s law.

d. If one integrates the Planck distribution law over all possible temperatures, one finds that the total energy emitted by a collection of radiators is

\[ I(\nu) = \langle E(\nu) \rangle \rho(\nu) = \frac{8\pi \nu^2}{c^3} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \quad k_B = 1.38 \cdot 10^{-23} J \cdot K^{-1} \quad (4) \]

The quantity \( \rho(\nu) \) is called the density of states per unit frequency, and we shall meet it (and variations of it) again and again as we talk about quantum physics.

5. Impact of the Planck hypothesis. The spectral irradiance distribution was a puzzle which nineteenth-century physicists strove in vain to reconcile with classical models of irradiation. Planck’s radical idea was that the wavelength distribution of blackbody radiation was quantized (the word means a "little bit" of something) rather than continuous. His spectral distribution properly accounts both for the long- and short-wavelength dependence of the spectrum. For room temperature, it peaks at a wavelength of about 10 µm -- nearly the wave-length of the CO\(_2\) laser. The dull red glow you see for a body heated below incandescence, say, to temperatures of order 700-800 K, is the visible or short-wavelength tail of a blackbody distribution which is still peaked in the infrared.
a. The Wien displacement law, which expresses the shift in the peak wavelength of the blackbody curve with temperature, can be derived by looking for the maximum of the curve as a function of wavelength.

\[ \lambda_{\text{max}} \cdot T = \frac{hc}{5k} = 2.88 \cdot 10^{-3} \text{ (µm} \cdot \text{K)} \]

b. The Stefan-Boltzmann law can also be derived from the Planck distribution law by integrating over wavelength: Changing variables to \( x = \frac{hc}{\lambda kT} \), one finds that:

\[ x \equiv \frac{hc}{\lambda kT} \Rightarrow \lambda = \frac{hc}{x kT} \quad \text{and} \quad d\lambda = -\frac{hc}{x^2 kT} \]

Integrating \( M_\lambda \) from \( \lambda = 0 \) to \( \lambda = \infty \), we obtain:

\[ I(T) = \int_0^\infty M_\lambda d\lambda = -2\pi hc \left( \frac{k_B T}{hc} \right) \int_0^\infty \frac{1}{xe^x - 1} \frac{ hc }{ k_B T } x^2 d\lambda \]

Continuing on with the integration, cancelling common factors, we have:

\[ I(T) = \frac{2\pi}{h^3 c^2} (k_B T)^4 \int_0^\infty \frac{ x^3 }{ (e^x - 1) } dx \equiv \sigma \cdot T^4 \]

This, of course, is precisely the form of the Stefan-Boltzmann law.

b. Even though Planck was not certain how seriously to take his quanta, others (notably Niels Bohr) quickly picked it up and applied it to the vexing problem of regularities in line emission from atoms. The initial idea was that electron waves had to be organized in space rather like standing acoustic waves on a string or a drumhead. Later this idea would be substantially refined by Schrödinger, Max Born and Werner Heisenberg into the concept of quantum states which satisfied the appropriate wave equation.

b. Homework assignment 2: By calculating the Planck function for a number of different wavelengths, show empirically that Wien’s displacement law correctly describes the shift of the peak as a function of wavelength.

c. Thought question: Suppose an electron is trapped in an extremely deep potential well. Can you sketch the optical absorption spectrum it would exhibit if you shined a white light on it?