Exercise: Since the mirror at the end of an optical resonator has a finite diameter, some part of the incident beam diffracts beyond the edge of the mirror and is lost. This represents a loss in the resonator that must be compensated by the gain in the laser medium. A rough approximation to the loss can be computed in the following way.

(a) Compute the intensity distribution of a zero-order Gauss-Laguerre mode and show that for a mirror of diameter \( D = 2a \) the fractional energy loss for a mode with radius \( w \) is

\[
\delta = e^{-2a^2/w^2}
\]

(b) Consider a symmetric resonator (\( g_1 = g_2 = g \)) and show that the one-way loss (due to one mirror) is

\[
\delta = \exp\left(-2\pi N_F \sqrt{1-g^2}\right)
\]

where \( N_F = a^2 / \lambda L \) is the Fresnel number of the resonator. The losses are smallest for a so-called confocal resonator (\( g = 0 \)), in which the foci of the mirrors are both at the center of the resonator. In fact, at high Fresnel numbers the mode adjusts its shape to avoid the edge of the mirrors, and the losses can be much smaller than this rough estimate. For a symmetric confocal resonator, better approximations are given by

\[
\delta = \pi^2 2^4 N_F \exp\left(-4\pi N_F\right), \quad \text{for } N \geq 1
\]

\[
\delta = 1 - (\pi N_F)^2, \quad \text{for } N_F << 1
\]

The figure below compares the various approximations as a function of the Fresnel number. As shown, the rough estimate is not bad for low Fresnel numbers (large losses), but greatly exaggerates the losses (by orders of magnitude) at high Fresnel numbers.

Exercise: The ray-transfer matrix

\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

for a round-trip through an optical cavity has eigenvalues \( \lambda_i \) and eigenvectors \( \mathbf{x}_i \) that satisfy the equation
\[ \mathbf{M} \mathbf{x}_i = \lambda_i \mathbf{x}_i \]

Since it is a 2-by-2 matrix, \( \mathbf{M} \) is trivial to diagonalize.

(a) Show that in terms of the stability parameters, the eigenvalues are

\[ \lambda_\pm = m \pm \sqrt{m^2 - 1} \]

where

\[ m = 2g_1g_2 - 1 \]

and that

\[ \lambda_+ \lambda_- = 1 \]

(b) Draw a stability diagram and indicate the regions where the eigenvalues are real. Are they positive or negative?

(c) A ray that corresponds to an eigenvector with a real eigenvalue becomes, after one round trip through the cavity, the same ray except that it is multiplied by the factor \( \lambda \). If \( |\lambda| > 1 \) the ray diverges from the axis after many round trips, and for \( |\lambda| < 1 \) the ray converges to the optical axis. An arbitrary ray can be expanded in eigenvectors of the cavity. Show that for a cavity with real eigenvalues, a ray always diverges after a large enough number of round trips.

(d) Explain the meaning of complex eigenvalues?