Section 9.2.1 Paraxial Approximation

Exercise: When dealing with a laser beam that consists of several transverse modes, it is frequently convenient to clean up the beam and suppress the higher-order modes by passing the beam through an aperture at the focus. This is called spatial filtering. Consider a circularly symmetric beam of radius \( w_0 \) at the focus passing through an aperture of radius \( a \) located at the focus. The incident beam consists of the fundamental mode with an admixture of the first higher mode, of the form

\[
\psi(r, z) = \frac{1}{w(z)} \left[ 1 + \epsilon \left( 1 - \frac{2r^2}{w^2(z)} \right) \right] \exp \left[ i \Phi(z) + i k \frac{r^2}{2R(z)} - \frac{r^2}{w^2(z)} \right]
\]

(a) Expand the beam beyond the aperture in the form

\[
\psi(r, z) = \sum_{n=0}^{\infty} \frac{b_n}{w(z)} I_n^0 \left( \frac{2r^2}{w^2(z)} \right) \exp \left[ i \Phi_n^0 (z) + i k \frac{r^2}{2R(z)} - \frac{r^2}{w^2(z)} \right]
\]

and show that the amplitude of the \( n^{th} \) mode in the transmitted beam is

\[
b_n = (1 + \epsilon) f_n^{(0)} (x_0) - \epsilon f_n^{(1)} (x_0)
\]

where

\[
I_n^{(0)} (x_0) = \int_0^{x_0} L_n^0 (x) e^{-x} dx
\]

\[
I_n^{(1)} (x_0) = \int_0^{x_0} L_n^0 (x) xe^{-x} dx
\]

and \( x_0 = 2a^2 / w_0^2 \).

(b) Show that

\[
I_0^{(0)} (x_0) = 1 - e^{-x_0}
\]

\[
I_1^{(0)} (x_0) = x_0 e^{-x_0}
\]

\[
I_n^{(0)} (x_0) = \frac{x_0 e^{-x_0}}{n} L_{n-1}^1 (x_0)
\]

\[
I_0^{(1)} (x_0) = 1 - (1 + x_0) e^{-x_0}
\]

\[
I_1^{(1)} (x_0) = (1 + x_0 x_0^2) e^{-x_0} - 1
\]

\[
I_{n>1}^{(1)} (x_0) = \frac{x_0^2 e^{-x_0}}{n(n-1)} \left[ (n-1) L_{n-1}^1 (x_0) - L_{n-2}^2 (x_0) \right]
\]
For $\varepsilon = 1$, the ratio $b_2 / b_1$ is plotted in the figure below, illustrating the effect of spatial filtering.