Adaptive Centering with Random Effects: An Alternative to the Fixed Effects Model

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National Conference on Value-Added Modeling

April 23, 2008

Background

- Time-varying exposures in education
- Designs
  - Nested designs
  - Cross-classified designs
  - Crossed-nested designs
- Fixed effects models
  - Remove time-invariant confounding
  - Remove between-cluster confounding

Claims

1. **Adaptive centering with random effects can replicate the fixed effects analysis of time-varying treatments in any dimension of clustering.**
2. **Adaptive centering with random effects has several advantages:**

    a. Incorporating multiple sources of uncertainty

    b. Modeling heterogeneity

    c. Modeling multi-level treatments

    d. Improved estimates of unit-specific effects

Outline

- What causal effects are fixed effects models estimating?
- A simple illustrative example
  - One-dimensional control
  - Two-dimensional control
- The General Theory
  - The *L*-level Model
  - Identification
- One-dimensional confounding
  - 2-level model
  - 3-level model
- Two-dimensinal confounding

What causal effects are fixed effects models estimating?

- Consider the model

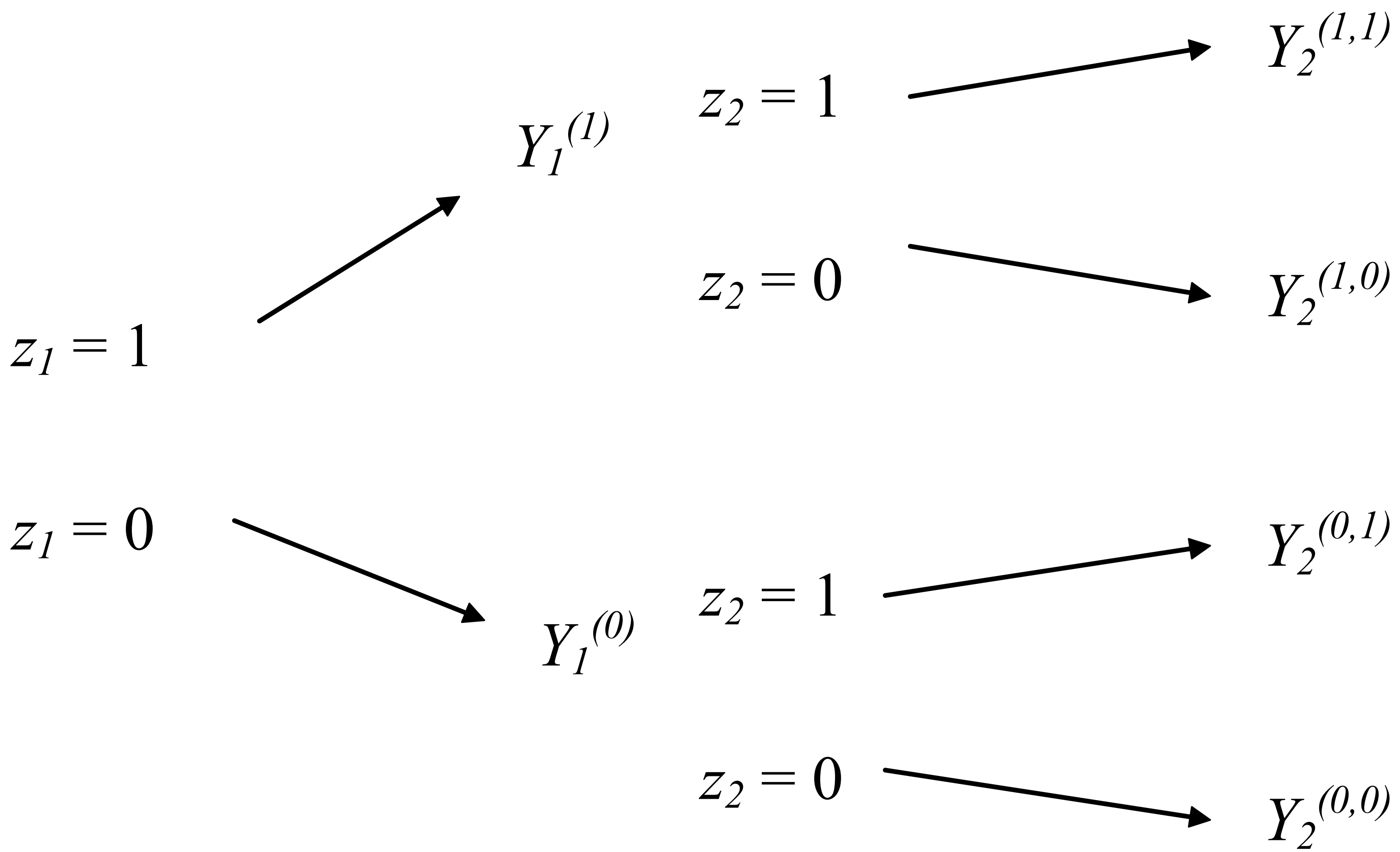
$$y_{ti} = \beta x_{ti} + u_i + e_{ti}$$

$$\textit{fixed effects } u_i \textit{ for } i = 1, \ldots n;$$

$$E(e_{ti} \mid x_{ti}) = E(e_{ti}) = 0$$

$$\textit{What is } \beta \,?$$

Figure 3 Potential Outcomes in a 2-year Study of Binary Treatments, Z_1 and Z_2



Causal Effects of Time-Varying Treatments

Grade-1 treatment on grade-1 outcome: $E[Y_1(1) - Y_1(0)] = \delta_{11}$

Grade-1 treatment on grade-2 outcome: $E[Y_2(1,0) - Y_2(0,0)] = \delta_{21}$

Grade-2 treatment on grade-2 outcome: $E[Y_2(0,1) - Y_2(0,0)] = \delta_{22}$

"Amplifying Effect:" $E[Y_2(1,1) - Y_2(0,0)] - (\delta_{21} + \delta_{22}) = \delta^*$

$$E[Y_1(z_1)] = E[Y(0)] + z_1\delta_{11}$$

$$E[Y_2(z_1, z_2)] = E[Y_2(0,0)] + z_1\delta_{21} + z_2\delta_{22} + z_1 z_2\delta^*$$

Cumulative effects model

$$\delta_{11} = \delta_{21} = \delta_{22} = \beta;$$

$$\delta^{*} = 0$$

$$x_{1i} = z_{1i}$$

$$x_{2i} = z_{1i} + z_{2i}$$

Ephemeral Effects Model

$$\delta_{11} = \delta_{22} = \beta$$

$$\delta_{21} = \delta^{*} = 0$$

$$x_{1i} = z_{1i}$$

$$x_{2i} = z_{2i}$$

A simple illustrative example

Table 1. Outcome data for 20 hypothetical kids by 9 teachers nested with 3 schools

	Teacher	1	2	3	4	5	6	7	8	9
	x	-1	0	1	-1	0	1	-1	0	1
w	Child									
0	1			-2.4102			2.4628			6.2245
1	2			3.6396		4.1441				11.0898
1	3		2.1827				10.1339			12.3134
0	4			-3170			3.6596		4.8397	
0	5		-0.0727			1.6280			6.0525	
0	6		-2.7852			1.4795				10.0131
0	7		0.2350				6.0839		7.5142	
0	8	-0.8803			3.5167					9.7337
0	9	-1.5147				5.8636				10.2860
0	10			2.6814			7.6954			10.0192
1	11	4.4966			9.5578			11.1152		
1	12	4.7195			8.2204				14.6855	
1	13	4.3609					12.6474		16.8547	
1	14	4.7778				11.9663				18.3998
1	15		8.5264			12.9066				18.6272
1	16		8.6820		11.8265				17.0661	
1	17		9.5595			13.8078			16.3071	
1	18	5.6075			12.7943					21.075
1	19	8.9094				13.5301				20.049
0	20	6.3465			7.3268			11.5147		

1. "True model"

$$y_{tijk} = \theta + \beta x_j + \gamma w_i + \delta(schoolid - 2)_k + \phi(childid)_i + \varepsilon_{tijk}$$

$$\beta = 2, \quad \gamma = 5, \quad \delta = 4, \quad \phi = 0.5, \quad \varepsilon_{tijk} \sim N(0,1)$$

Estimates of True Model Effects

Predictors	*Coeff.*	Std. Err.	t	p
(Constant)	-0.415	0.302	-1.375	0.175
x	2.171	0.200	10.866	0.000
w	4.799	0.278	17.294	0.000
schoolid-2	3.970	0.166	23.912	0.000
child id	0.539	0.027	20.001	

$\hat{\sigma}^2 = .934$

One-Dimensional Control: OLS Fixed Child Effects

$$y_{tijk} = \theta + \beta x_j + u_i + \varepsilon_{tijk},$$
$$\varepsilon_{tijk} \sim N(0, \sigma^2),$$
$$u_i \text{ for } i = 1, \ldots, 19 \text{ fixed}$$

Parameter	Estimate	Std. Error	t	Sig.
Intercept	13.894087	2.217045	6.267	0.000
x	**5.498095**	**0.865904**	**6.350**	**0.000**
[childid=1.00]	-17.299841	3.366029	-5.140	0.000
[childid=2.00]	-11.268353	3.227033	-3.492	0.001
[childid=3.00]	-9.349477	3.227033	-2.897	0.006
[childid=4.00]	-14.832045	3.227033	-4.596	0.000
[childid=5.00]	-11.358169	3.013434	-3.769	0.001
[childid=6.00]	-12.825538	3.108690	-4.126	0.000
[childid=7.00]	-11.115732	3.108690	-3.576	0.001
[childid=8.00]	-9.770723	3.013434	-3.242	0.002
[childid=9.00]	-9.015820	3.013434	-2.992	0.005
[childid=10.00]	-12.593491	3.366029	-3.741	0.001
[childid=11.00]	-0.006149	2.886346	-0.002	0.998
[childid=12.00]	-1.020260	2.900742	-0.352	0.727
[childid=13.00]	-0.773729	2.943507	-0.263	0.794
[childid=14.00]	-2.179455	3.013434	-0.723	0.474
[childid=15.00]	-2.373398	3.108690	-0.763	0.450
[childid=16.00]	0.463474	2.943507	0.157	0.876
[childid=17.00]	-0.669300	3.013434	-0.222	0.825
[childid=18.00]	1.097582	2.943507	0.373	0.711
[childid=19.00]	0.268870	3.013434	0.089	0.929
[childid=20.00]	0(a)	0	.	.

Estimates of Covariance Parameters

Parameter	Estimate
σ^2	12.496491

One-Dimensional Control: Child random effects with person-mean centered x

$$y_{tijk} = \theta + \beta(x_{t_{ik}} - \bar{x}_i) + u_i + \varepsilon_{tijk},$$

$$u_i \sim N(0, \tau^2), \quad \varepsilon_{tijk} \sim N(0, \sigma^2)$$

Note this gives the same coefficient, standard error, and residual variance estimate as the student fixed effects model.

Model Estimates

Parameter	Estimate	Std. Err.	df	t	Sig.
Intercept	8.029549	0.927088	19	8.661	0.000
$(x_{t_{ik}} - \bar{x}_i)$	5.498095	0.865904	39	6.350	0.000

Estimate of Covariance Parameters

Parameter	Estimate
σ^2	12.496491
τ^2	13.024353

Two dimensional controls: OLS fixed child and school effects

$$y_{tijk} = \theta + \beta x_j + u_i + s_k + \varepsilon_{tijk},$$
$$\varepsilon_{tijk} \sim N(0, \sigma^2),$$
$$u_i,\ i = 1, \ldots, 19 \text{ fixed}$$
$$s_k = 1,2 \text{ fixed}$$

Parameter	Estimate	Std. Error	df	T	Sig.
Intercept	14.642231	0.630345	37	23.229	0.000
X	2.573106	0.287937	37	8.936	0.000
[childid=1.00]	-11.449864	0.998365	37	-11.469	0.000
[childid=2.00]	-6.393372	0.946257	37	-6.756	0.000
[childid=3.00]	-4.474496	0.946257	37	-4.729	0.000
[childid=4.00]	-9.957064	0.946257	37	-10.523	0.000
[childid=5.00]	-8.433180	0.864876	37	-9.751	0.000
[childid=6.00]	-8.925554	0.901385	37	-9.902	0.000
[childid=7.00]	-7.215747	0.901385	37	-8.005	0.000
[childid=8.00]	-6.845734	0.864876	37	-7.915	0.000
[childid=9.00]	-6.090831	0.864876	37	-7.042	0.000
[childid=10.00]	-6.743514	0.998365	37	-6.755	0.000
[childid=11.00]	-0.006149	0.815539	37	-0.008	0.994
[childid=12.00]	-0.045263	0.821167	37	-0.055	0.956
[childid=13.00]	1.176263	0.837825	37	1.404	0.169
[childid=14.00]	0.745534	0.864876	37	0.862	0.394
[childid=15.00]	1.526586	0.901385	37	1.694	0.099
[childid=16.00]	2.413467	0.837825	37	2.881	0.007
[childid=17.00]	2.255688	0.864876	37	2.608	0.013
[childid=18.00]	3.047574	0.837825	37	3.637	0.001
[childid=19.00]	3.193858	0.864876	37	3.693	0.001
[childid=20.00]	0(a)	0	.	.	.
[schoolid=1.00]	-7.679293	0.367143	37	-20.916	0.000
[schoolid=2.00]	-3.340106	0.347120	37	-9.622	0.000
[schoolid=3.00]	0(a)	0	.	.	.

Estimates of Covariance Parameters

Parameter	Estimate
σ^2	0.997655

Table 3. Treatment Received

Teacher	1	2	3	4	5	6	7	8	9	
x	-1	0	1	-1	0	1	-1	0	1	$\bar{x}_i$
Child										
1			1			1			1	1
2			1		0				1	0.6667
3		0				1			1	0.6667
4			1			1		0		0.6667
5		0			0			0		0
6		0			0				1	0.3333
7		0				1		0		0.3333
8	-1			-1					1	0.3333
9	-1				0				1	0
10			1			1			1	1
11	-1			-1			-1			-0.3333
12	-1			-1				0		-0.6667
13	-1					1		0		0
14	-1				0				1	0.3333
15		0			0				1	0.3333
16		0		-1				0		-0.3333
17		0			0			0		0
18	-1			-1					1	-0.3333
19	-1				0				1	0
20	-1			-1			-1			-0.3333
x_k		-0.25			0			0.45		

Two-Dimensional Controls: Random child and school effects with interaction-contrast centering

$$y_{tijk} = \theta + \beta(x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}) + u_i + s_k + \varepsilon_{tijk},$$

$$\varepsilon_{tijk} \sim N(0,\sigma^2)$$

$$u_i \sim N(0,\tau^2),$$

$$s_k \sim N(0,\psi^2)$$

Model Estimates

Parameter	Estimate	Std. Err.	t	Sig.
Intercept	8.029463	2.851520	2.816	0.083
$x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}$	2.573106	0.287937	8.936	0.000

Estimates of Covariance Parameters

Parameter	Estimate
σ^2	0.997655
τ^2	16.857298
ψ^2	21.815022

Adaptive centering with random effects: advantages

a. Incorporating multiple sources of uncertainty

b. Modeling heterogeneity

c. Modeling multi-level treatments

d. Improved estimates of unit-specific effects

Getting the uncertainty right

- **Two-dimensional controls (kids and schools)**
  random effects of kids, teachers within schools, schools

$$y_{tijk} = \theta + \beta(x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}) + u_i + s_k + c_{j(k)} + \varepsilon_{tijk},$$

$$\varepsilon_{tijk} \sim (0, \sigma^2),$$

$$u_i \sim (0, \tau^2),$$

$$s_k \sim (0, \psi^2)$$

$$c_{j(k)} \sim (0, \lambda^2)$$

A natural way to incorporate uncertainty as a function of clustering

Note we are incorporating uncertainty associated with classrooms, which cannot be done using fixed effects if the treatment

is at that level.

A natural framework for modeling heterogeneity--

Heterogeneity is interesting;

A failure to incorporate heterogeneity leads to biased standard errors.

$$y_{tijk} = \theta + u_{0i} + s_{0k} + c_{0j(k)} + (\beta + u_{1ik} + s_{1k})(x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}) + \varepsilon_{tijk}$$

$$\begin{bmatrix} u_{0i} \\ u_{1ik} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}\right]$$

$$\begin{bmatrix} s_{0k} \\ s_{1k} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}\right]$$

$$c_{j(k)} \sim N(0, \lambda^2)$$

$$\varepsilon_{tijk} \sim N(0, \sigma^2)$$

3. We can study multilevel treatments and their interaction

$$y_{tijk} = \theta + u_{0i} + s_{0k} + c_{0j(k)} + (\beta + u_{1ik} + s_{1k})(x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}) +$$

$$+ \gamma_0 w_k + \gamma_1 w_k * (x_{t_{ik}} - \bar{x}_i - \bar{x}_k + \bar{x}) + \varepsilon_{tijk}$$

$$\begin{bmatrix} u_{0i} \\ u_{1ik} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}\right]$$

$$\begin{bmatrix} s_{0k} \\ s_{1k} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}\right]$$

$$c_{j(k)} \sim N(0, \lambda^2)$$

$$\varepsilon_{tijk} \sim N(0, \sigma^2)$$

Improved Unit-specific estimates

We estimate unit-specific effects as the posterior expected random effects: e.g.,

$$u^*_{1i} = E(u_{1ik} \mid y)$$

or

$$s^*_{0k} = E(s_{0ik} \mid y) - \beta \, \bar{x}_k$$

"OLS (fixed effects estimates) are inadmissible in dimension greater than two." – Lindley and Smith, *JRSS*, 1972.

General Theory

- Model and Estimation

$$\mathbf{Y} = \mathbf{X}\tilde{\mathbf{a}} + \mathbf{A}\mathbf{b} + \mathbf{e},$$

$$\mathbf{b} \sim (\mathbf{0}, \grave{\mathbf{U}})$$

$$\mathbf{e} \sim (\mathbf{0}, \mathbf{V}^{*})$$

$$Var(\mathbf{Y} \mid \mathbf{X}, \mathbf{A}) = \mathbf{V} = \mathbf{A}\boldsymbol{\Omega}\mathbf{A}^{T} + \mathbf{V}^{*}$$

$$\tilde{\mathbf{a}} = (\mathbf{X}^{T}\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}\mathbf{Y}$$

$$E(\tilde{\mathbf{a}} \mid \mathbf{X}, \mathbf{A}) = \tilde{\mathbf{a}} + (\mathbf{X}^{T}\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{V}^{-1}\mathbf{A}E(\mathbf{b} \mid \mathbf{X}, \mathbf{A})$$

$$If\ \ \mathbf{b} \perp \mathbf{X}, \mathbf{A},\ \ E(\mathbf{b} \mid \mathbf{X}, \mathbf{A}) = E(\mathbf{b}) = \mathbf{0}.$$

How Adaptive Centering Works

$$\begin{aligned}\mathbf{Y} &= \mathbf{X}\tilde{\mathbf{a}} + \mathbf{Ab} + \mathbf{e} \\ &= \mathbf{1}\theta + \mathbf{x}\hat{\mathbf{a}} + \mathbf{Ab} + \mathbf{e},\end{aligned}$$

$$Var(e) = \mathbf{V}_{L-1}$$

$$Var(Y) = \mathbf{V}_{L} = \mathbf{A}\grave{\mathrm{U}}\,\mathbf{A}^{T} + \mathbf{V}_{L-1}$$

$$\begin{pmatrix}\hat{\theta} \\ \hat{\mathbf{a}}\end{pmatrix} = \begin{bmatrix}\mathbf{1}^{T}\mathbf{V}_{L}^{-1}\mathbf{1} & \mathbf{1}^{T}\mathbf{V}_{L}^{-1}\mathbf{x} \\ \mathbf{x}^{T}\mathbf{V}_{L}^{-1}\mathbf{1} & \mathbf{x}^{T}\mathbf{V}_{L}^{-1}\mathbf{x}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1}^{T}\mathbf{V}_{L}^{-1}\mathbf{Y} \\ \mathbf{x}^{T}\mathbf{V}_{L}^{-1}\mathbf{Y}\end{bmatrix}$$

After centering

$$\begin{pmatrix}\hat{\theta} \\ \hat{\mathbf{a}}\end{pmatrix} = \begin{bmatrix}\mathbf{1}^{T}\mathbf{V}_{L}^{-1}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{x}^{T}\mathbf{V}_{L-1}^{*-1}\mathbf{x}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1}^{T}\mathbf{V}_{L}^{-1}\mathbf{Y} \\ \mathbf{x}^{T}\mathbf{V}_{L-1}^{*-1}\mathbf{Y}\end{bmatrix}$$

We solve

$$\mathbf{1}^T \mathbf{V}_L^{-1} \mathbf{x} = 0$$

$$\mathbf{1}^T \mathbf{V}_L^{-1} \mathbf{A} = 0$$

- One-dimensional confounding
  - 2-level model
  - 3-level model

Two-Level Random Intercept Model

$$\mathbf{Y}_i = (\mathbf{1}_i \quad \mathbf{x}_i)\begin{pmatrix} \theta \\ \hat{\mathbf{a}} \end{pmatrix} + \mathbf{1}_i u_i + \mathbf{e}_i, \quad \mathbf{V}_{2i} = \tau^2 \mathbf{1}_i \mathbf{1}_i^T + \mathbf{V}_{1i}, \quad \mathbf{V}_{1i} = \sigma^2 \mathbf{I}_i$$

$$\begin{pmatrix} \hat{\theta} \\ \hat{\mathbf{a}} \end{pmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \mathbf{1}_i^T \mathbf{V}_{2i}^{-1} \mathbf{1}_i & \sum_{i=1}^{n} \mathbf{1}_i^T \mathbf{V}_{2i}^{-1} \mathbf{x}_i \\ \sum_{i=1}^{n} \mathbf{x}_i^T \mathbf{V}_{2i}^{-1} \mathbf{1}_i & \sum_{i=1}^{n} \mathbf{x}_i^T \mathbf{V}_{2i}^{-1} \mathbf{x}_i \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} \mathbf{1}_i^T \mathbf{V}_{2i}^{-1} \mathbf{Y}_i \\ \sum_{i=1}^{n} \mathbf{Z}_i^T \mathbf{V}_{2i}^{-1} \mathbf{Y}_i \end{bmatrix}$$

After centering

$$\begin{pmatrix} \hat{\theta} \\ \hat{\mathbf{a}} \end{pmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \mathbf{1}_i^T \mathbf{V}_{2i}^{-1} \mathbf{1}_i & \mathbf{0} \\ \mathbf{0} & \sigma^2 \sum_{i=1}^{n} \mathbf{Z}_i^{*T} \mathbf{Z}_i^{*} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} \mathbf{1}_i^T \mathbf{V}_{2i}^{-1} \mathbf{Y}_i \\ \sigma^2 \sum_{i=1}^{n} \mathbf{Z}_i^{*T} \mathbf{Y}_i \end{bmatrix}$$

Centering

$$Z^{*}_{ti} = Z_{ti} - \bar{Z}_{.i}$$

$$\bar{Z}_{.i} = \sum_{t=1}^{T_i} Z_{ti} / T_i$$

Three-Level Random Intercept Model

$$\mathbf{Y}_k = (\mathbf{1}_k \;\; \mathbf{Z}_k)\begin{pmatrix}\theta \\ \hat{\mathbf{a}}\end{pmatrix} + \mathbf{1}_k \mathbf{s}_k + \bigoplus_{i=1}^{i} \mathbf{1}_{ik}\mathbf{u}_{ik} + \mathbf{e}_k, \quad \mathbf{V}_{3k} = \omega^2 \mathbf{1}_k \mathbf{1}_k^T + \bigoplus_{i=1}^{n_k} \mathbf{V}_{2ik}$$

$$\begin{pmatrix}\hat{\theta} \\ \hat{\mathbf{a}}\end{pmatrix} = \begin{bmatrix} \sum_{k=1}^{K} \mathbf{1}_k^T \mathbf{V}_{3k}^{-1} \mathbf{1}_k & \sum_{k=1}^{K} \mathbf{1}_k^T \mathbf{V}_{3k}^{-1} \mathbf{Z}_k \\ \sum_{k=1}^{K} \mathbf{Z}_k^T \mathbf{V}_{3k}^{-1} \mathbf{1}_k & \sum_{k=1}^{K} \mathbf{Z}_k^T \mathbf{V}_{3k}^{-1} \mathbf{Z}_k \end{bmatrix}^{-1} \begin{bmatrix} \sum_{k=1}^{K} \mathbf{1}_k^T \mathbf{V}_{3k}^{-1} \mathbf{Y}_k \\ \sum_{k=1}^{K} \mathbf{Z}_k^T \mathbf{V}_{3k}^{-1} \mathbf{Y}_k \end{bmatrix}$$

After centering

$$\begin{pmatrix}\hat{\theta} \\ \hat{\mathbf{a}}\end{pmatrix} = \begin{bmatrix} \sum_{k=1}^{K} \mathbf{1}_k^T \mathbf{V}_{3k}^{-1} \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \sum_{k=1}^{K} \sum_{i=1}^{n_k} \mathbf{Z}_{ik}^{*T} \mathbf{V}_{2ik}^{-1} \mathbf{Z}_{ik}^{*} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{k=1}^{K} \mathbf{1}_k^T \mathbf{V}_{3k}^{-1} \mathbf{Y}_k \\ \sum_{k=1}^{K} \sum_{i=1}^{n_k} \mathbf{Z}_{ik}^{T*} \mathbf{V}_{2ik}^{-1} \mathbf{Y}_{ik} \end{bmatrix}$$

Be Careful How You Center!

- Three-Level Case

  – Uncentered: $\mathbf{Z}_k = \{\mathbf{Z}_{ik}\}$

  – Centered $\mathbf{Z}_k^* = \{\mathbf{Z}_{ik} - \overline{\mathbf{Z}}_{.k}\}$

where $$\overline{\mathbf{Z}}_{.k} = \left(\sum_{i=1}^{n_k} (\tau^2 + \sigma^2 / n_{ik})^{-1} \right)^{-1} \sum_{i=1}^{n_k} (\tau^2 + \sigma^2 / n_{ik})^{-1} \mathbf{Z}_{ik}$$

Summary Table

Design	Example	Treatment	Dimension	Weights
2 level (nested)	Occasions within children	Within children	Single: within child	unity
3 level (nested)	Children within classes within schools	Teacher level	Single: within school	Precision: $(\tau^2 + \sigma^2 / n_{jk})^{-1}$
3 level (crossed)	Occasions within children by neighborhoods	Within children	Double: within child and within neighborhood	unity
4 level (crossed and nested)	Occasions within children crossed by teachers who, are nested within schools	Teacher level	Double: within child and within sfchool	Precision: $(\tau^2 + \sigma^2 / n_{jk})^{-1}$

Conclusions

Be sure you know what you are estimating

-- Cumulative effects, ephemeral effects, other

We must worry about time-varying confounding

Adaptive Centering with Random Effects

-- removes confounding at level L

-- Incorporates clustering at levels 1,…,*L*-1

-- can incorporate heterogeneous effects, multilevel treatments

-- provides estimators of unit-specific effects

-- can be used in nested, crossed, and crossed-nested designs