Technical Appendix to Accompany "Endogenous Growth Through Investment-Specific Technological Change"

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1 Derivation of various results

It is necessary to derive various formulae that are present in the text of the paper. This will be done in a series of results. First of all, it will be useful to let $\Omega = V_t I_t^{-\alpha}$, $\phi = V_t^{\theta-1} k_{t+1}$ and both of these are constant in the steady-state. Obviously there is a link between the two, and this can be written as follows:

$$\Omega = \phi^{-\alpha}(1+g)\left[(1+g)^{1/\alpha} - 1 + \delta\right]^{-\alpha}.$$

Throughout the following analysis it will be assumed that $\theta = \left(\frac{\alpha-1}{\alpha}\right)$, which is a necessary condition for balanced growth.

Then throughout remainder of this appendix, a series of the results will be derived.

Result #1

We have the following characterization for the price of capital

$$\begin{aligned} q_t &= B\left[(1-\theta)I_t^{\rho/(1-\theta)} + \theta V_t^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{\rho}{1-\theta}-1}\right) \\ &= B\left[(1-\theta) + \theta V_t^{\rho} I_t^{-\rho/(1-\theta)}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{\rho}{1-\theta}-1}\right)\left(I_t^{\frac{1}{1-\theta}-\frac{\rho}{1-\theta}}\right) \end{aligned}$$

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$$
\begin{aligned}
&= B\left[(1-\theta)+\theta\left(V_t I_t^{-\alpha}\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{\theta}{1-\theta}}\right) \\
&= B\left[(1-\theta)+\theta\left(\Omega\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{\theta}{1-\theta}}\right).
\end{aligned}
$$

Result #2
This latter result implies that

$$
\frac{q_{t+1}}{q_t}=\left(\frac{I_{t+1}}{I_t}\right)^{\frac{\theta}{1-\theta}}=\left[(1+g)^{1/\alpha}\right]^{\frac{\theta}{1-\theta}}=(1+g)^{\frac{\alpha-1}{\alpha}}=(1+g)^{\theta}.
$$

Result #3
We also have the following for the return to research spending

$$
\begin{aligned}
R_t &= -\theta B V_t^{\rho-1}\left[(1-\theta)I_t^{\rho/(1-\theta)}+\theta V_t^{\rho}\right]^{\frac{1-\rho}{\rho}} \\
&= -\theta V_t^{\rho-1} B\left[(1-\theta)+\theta V_t^{\rho} I_t^{-\rho/(1-\theta)}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{1}{1-\theta}-\frac{\rho}{1-\theta}}\right) \\
&= -\theta B\left[(1-\theta)+\theta\left(V_t I_t^{-\alpha}\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(I_t^{\frac{1-\rho}{1-\theta}}\right)V_t^{\rho-1} \\
&= -\theta B\left[(1-\theta)+\theta\left(V_t I_t^{-\alpha}\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(V_t I_t^{\frac{-1}{1-\theta}}\right)^{\rho-1} \\
&= -\theta B\left[(1-\theta)+\theta\left(V_t I_t^{-\alpha}\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(V_t I_t^{-\alpha}\right)^{\rho-1} \\
&= -\theta B\left[(1-\theta)+\theta\left(\Omega\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(\Omega\right)^{\rho-1}.
\end{aligned}
$$

Result #4
We also have

$$
\begin{aligned}
(\alpha A)\frac{k_{t+1}^{\alpha-1}}{q_t} &= k_{t+1}^{\alpha-1}\left(\frac{\alpha A}{B}\right)\left[(1-\theta)I_t^{\rho/(1-\theta)}+\theta V_t^{\rho}\right]^{\frac{\rho-1}{\rho}}\left(I_t^{\frac{-\rho}{1-\theta}+1}\right) \\
&= k_{t+1}^{\alpha-1}\left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta V_t^{\rho} I_t^{-\rho/(1-\theta)}\right]^{\frac{\rho-1}{\rho}}\left(I_t^{\frac{-\rho}{1-\theta}+1}\right)\left(I_t^{\frac{-1}{1-\theta}+\frac{\rho}{1-\theta}}\right) \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\left(V_t I_t^{-\alpha}\right)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left(I_t^{\frac{-\theta}{1-\theta}}\right)k_{t+1}^{\alpha-1} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\left(\Omega\right)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left(I_t^{1-\alpha}\right)k_{t+1}^{\alpha-1} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\left(\Omega\right)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[k_{t+1}-(1-\delta)k_t\right]^{1-\alpha}k_{t+1}^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[1-(1-\delta)/(1+g_k)\right]^{1-\alpha} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[(1+g_k)-(1-\delta)\right]^{1-\alpha}(1+g_k)^{\alpha-1} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{1-\alpha}(1+g)^{\frac{\alpha-1}{\alpha}} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{-\alpha}(1+g)\right]^{\frac{\alpha-1}{\alpha}} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[\phi^{\alpha}\phi^{-\alpha}\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{-\alpha}(1+g)\right]^{\frac{\alpha-1}{\alpha}} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[\phi^{-\alpha}\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{-\alpha}(1+g)\right]^{\frac{\alpha-1}{\alpha}}\phi^{\alpha-1} \\
&= \left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[\Omega\right]^{\frac{\alpha-1}{\alpha}}\phi^{\alpha-1}.
\end{aligned}
$$

Result #5
The investment-output ratio is calculated as follows:

$$
\begin{aligned}
\frac{\Psi(i_t,V_t)}{y} &= \frac{B\left[(1-\theta)I_t^{\frac{\rho}{1-\theta}}+\theta(V_t)^{\rho}\right]^{1/\rho}}{Ak_t^{\alpha}} \\
&= \frac{B\left[(1-\theta)+\theta(V_tI_t^{-\alpha})^{\rho}\right]^{1/\rho}I_t^{\alpha}}{Ak_t^{\alpha}} \\
&= \frac{B\left[(1-\theta)+\theta(\Omega)^{\rho}\right]^{1/\rho}\left[k_{t+1}-(1-\delta)k_t\right]^{\alpha}}{Ak_t^{\alpha}} \\
&= \frac{B\left[(1-\theta)+\theta(\Omega)^{\rho}\right]^{1/\rho}\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{\alpha}}{A}.
\end{aligned}
$$

Result #6
The research-output ratio is calculated as follows. Let

$$
\begin{aligned}
I_t &= k_{t+1}-(1-\delta)k_t \\
&= \left[(1+g)^{1/\alpha}-(1-\delta)\right]k_t
\end{aligned}
$$

so that

$$
\frac{V_t}{Ak_t^{\alpha}} = \frac{V_tk_t^{-\alpha}}{A}
$$

$$= \left[\frac{1}{A}\right]\left(V_t I_t^{-\alpha}\right)\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{\alpha}$$
$$= \left[\frac{\Omega}{A}\right]\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{\alpha}$$

Result #7

The ratio of research spending to output is calculated as follows. The law of motion for research knowledge is the following:

$$V_{t+1} = V_t + v_t$$

so

$$\frac{V_{t+1}}{V_t} = (1+g) = 1 + \frac{v_t}{V_t}$$

and hence

$$\frac{v_t}{V_t} = g$$

Therefore we have

$$\frac{v_t}{Ak_t^{\alpha}} = \frac{v_t}{V_t}\frac{V_t}{Ak_t^{\alpha}} = g\left[\frac{\Omega}{A}\right]\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{\alpha}.$$

2 The Model Without Taxes

Now the first optimization condition for the model without taxes can then be re-written as follows:

$$\left[i_t^{\left(\frac{\rho}{1-\theta}\right)-1}\right] B\left[(1-\theta)i_t^{\frac{\rho}{(1-\theta)}} + \theta(V_t)^{\rho}\right]^{(1/\rho)-1}(c_t)^{-\sigma} =$$

$$\beta(c_{t+1})^{-\sigma}\left[A\alpha k_{t+1}^{\alpha-1} + (1-\delta)\left[i_{t+1}^{\left(\frac{\rho}{1-\theta}\right)-1}\right] B\left[(1-\theta)i_{t+1}^{\frac{\rho}{(1-\theta)}} + \theta(V_{t+1})^{\rho}\right]^{(1/\rho)-1}\right]$$

or

$$(c_t)^{-\sigma} = \beta(c_{t+1})^{-\sigma}\left[\frac{A\alpha k_{t+1}^{\alpha-1}}{q_t} + (1-\delta)\left(\frac{q_{t+1}}{q_t}\right)\right]$$

where

$$q_t \equiv \frac{\partial \Psi(i_t, V_t)}{\partial i_t} = \left[i_t^{\left(\frac{\rho}{1-\theta}\right)-1} \right] B \left[(1-\theta) i_t^{\frac{\rho}{(1-\theta)}} + \theta (V_t)^{\rho} \right]^{(1/\rho)-1}$$

This can condition can then be written as

$$(1+g)^{\sigma} = \beta \left[\frac{A\alpha k_{t+1}^{\alpha-1}}{q_t} + (1-\delta) \left(\frac{q_{t+1}}{q_t} \right) \right]$$

Using results 2 and 4 yields

$$(1+g)^{\sigma} = \beta \left[\left(\frac{\alpha A}{B} \right) \left[(1-\theta) + \theta \left(\Omega \right)^{\rho} \right]^{\frac{\rho-1}{\rho}} \left[\Omega \right]^{\frac{\alpha-1}{\alpha}} \phi^{\alpha-1} + (1-\delta)(1+g)^{\theta} \right] \quad (1)$$

The second optimization condition can be written as follows:

$$\begin{aligned} (c_t)^{-\sigma} &= \sum_{i=1}^{\infty} \beta^i (c_{t+i})^{-\sigma} (-B\theta)(V_{t+i}^{\rho-1}) \left[(1-\theta) i_{t+i}^{\frac{\rho}{(1-\theta)}} + \theta (V_{t+i})^{\rho} \right]^{(1/\rho)-1} \\ &= \sum_{i=1}^{\infty} \beta^i (c_{t+i})^{-\sigma} R_{t+i} \end{aligned}$$

where

$$R_{t+i} = (-B\theta)(V_{t+i}^{\rho-1}) \left[(1-\theta) i_{t+i}^{\frac{\rho}{(1-\theta)}} + \theta (V_{t+i})^{\rho} \right]^{(1/\rho)-1}.$$

Using result 3, this can be further re-written as follows

$$1 = \sum_{i=1}^{\infty} \beta^i (1+g)^{-(i\sigma)} (-B\theta) \left[(1-\theta) + \theta \left(\Omega \right)^{\rho} \right]^{\frac{1-\rho}{\rho}} \left(\Omega \right)^{\rho-1}$$

or

$$1 = \frac{\beta (1+g)^{-\sigma} (-B\theta) \left[(1-\theta) + \theta \left(\Omega \right)^{\rho} \right]^{\frac{1-\rho}{\rho}} \left(\Omega \right)^{\rho-1}}{1 - \beta (1+g)^{-\sigma}}. \quad (2)$$

3 The Model With Taxes

The first equilibrium condition for the model with taxes can be written as follows:

$$q_t(1-\tau_i)(c_t)^{-\sigma} = \beta(c_{t+1})^{-\sigma}\left[r_{t+1}(1-\tau_k) + (\tau_k \delta q_t) + (1-\delta)(1-\tau_i)q_{t+1}\right].$$

This can be further re-written as follows:

$$(1+g)^{\sigma} = \beta\left[\frac{r_{t+1}(1-\tau_k)}{q_t(1-\tau_i)} + \left(\frac{\tau_k\delta}{1-\tau_i}\right) + (1-\delta)\left(\frac{q_{t+1}}{q_t}\right)\right].$$

Using results 2 and 4, this can be re-written as follows:

$$(1+g^{\sigma}) = \beta\left[\begin{array}{c} \left(\frac{A\alpha}{B}\right)\left(\frac{1-\tau_k}{1-\tau_i}\right)\phi^{\alpha-1}\Omega^{\frac{\alpha-1}{\alpha}}\left[(1-\theta)+\theta\Omega^{\rho}\right]^{1-1/\rho} \\ +\left(\frac{\tau_k\delta}{1-\tau_i}\right) + (1-\delta)(1+g)^{\theta} \end{array}\right] \tag{3}$$

Let

$$R_{t+i} = (1-\tau_i)\left(-B\theta\right)(V_{t+i}^{\rho-1})\left[(1-\theta)i_{t+i}^{\frac{\rho}{(1-\theta)}} + \theta(V_{t+i})^{\rho}\right]^{(1/\rho)-1}.$$

With this in mind, the other optimization condition can be written as

$$\begin{aligned}
(1-\tau_r) &= \sum_{i=1}^{\infty}\beta^i\left(\frac{c_{t+i}}{c_t}\right)^{-\sigma}R_{t+i} \\
&= \sum_{i=1}^{\infty}\beta^i\left[(1+g)^i\right]^{-\sigma}R_{t+i} \\
&= \sum_{i=1}^{\infty}\beta^i\left(1+g\right)^{-i\sigma}R_{t+i} \\
&= \frac{\beta\left(1+g\right)^{-\sigma}R}{1-\beta\left(1+g\right)^{-\sigma}} \\
&= \frac{\beta\left(1+g\right)^{-\sigma}\left(1-\tau_i\right)\left(-\theta B\right)\left[(1-\theta)+\theta\left(\Omega\right)^{\rho}\right]^{\frac{1-\rho}{\rho}}\left(\Omega\right)^{\rho-1}}{1-\beta\left(1+g\right)^{-\sigma}} \qquad (4)
\end{aligned}$$

4 Computing an equilibrium for the economy

The equilibrium conditions for the economy without distortions can be reduced to a few equations. Equation 1 is as follows:

$$(1+g)^{\sigma} = \beta\left[\left(\frac{\alpha A}{B}\right)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{\rho-1}{\rho}}\left[\Omega\right]^{\frac{\alpha-1}{\alpha}}\phi^{\alpha-1} + (1-\delta)(1+g)^{\theta}\right] \quad (5)$$

Equation 2 is as follows

$$1 = \frac{\beta(1+g)^{-\sigma}(-B\theta)\left[(1-\theta)+\theta\,(\Omega)^{\rho}\right]^{\frac{1-\rho}{\rho}}(\Omega)^{\rho-1}}{1-\beta(1+g)^{-\sigma}} \quad (6)$$

The condition defining Ω is again

$$\Omega = \phi^{-\alpha}(1+g)\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{-\alpha}. \quad (7)$$

From result 5, the ratio of investment to output is written as follows:

$$\frac{B\left[(1-\theta)+\theta(\Omega)^{\rho}\right]^{1/\rho}\left[(1+g)^{1/\alpha}-(1-\delta)\right]^{\alpha}}{A} = Iy^{*}. \quad (8)$$

Let g^*, and Iy^* respectively, denote the desired level of the output growth rate, and the investment to output ratio.

Now with the values of g^*, and Iy^* held constant, equations 5 through 8 become 4 equations in 4 unknowns: (A, B, Ω, ϕ).

In the version of the model with taxes, the values of the tax parameters, and of g^*, and Iy^* are taken as constants. Then equations 3, 4, 7, and 8 are 4 equation in the four unknowns (A, B, Ω, ϕ).