Sequentially Stable Coalition Structures *

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Abstract

In this paper, we examine the questions of which coalition structure is formed and how payoff is distributed among players in cooperative games with externalities. We introduce a sequential stability concept called a sequentially stable payoff configuration in a game with a coalition structure by extending the concept of the equilibrium binding agreements by Ray and Vohra (1997). Ray and Vohra capture explicitly the credibility of blocking coalitions, and then induce a recursive definition of the stable coalition structures in a game with externalities. However, in their definition, only breaking up is allowed for coalitions.

We propose a new concept of a sequential stable payoff configuration such that coalitions can both break up and merge into. The payoff configuration $z$ is said to sequentially dominate the payoff configuration $z'$ if there is a sequence of payoff configurations starting from $z$ to $z'$ such that (1) in each step, two coalitions may merge or one coalition may break into two coalitions, and (2) in each step, the members in the merging coalitions or the breaking coalition prefer the payoffs of the final configuration $z'$ to the present payoff.

*Preliminaries
A sequential stable payoff configuration is defined as a payoff configuration which sequentially dominates all other payoff configurations.

As an application of this stable concept, we study a common pool resource game, where each payoff configuration corresponds to one coalition structure. We show that if the number of players is between 3 and 48, then for some concave production function, the payoff configuration related to the grand coalition structure is sequentially stable in the common pool resource game.
1 Introduction

In this paper, we examine the questions of which coalition structure is formed and how payoff is distributed among players in cooperative games with externalities. We introduce a sequential stability concept called a sequentially stable payoff configuration in a game with a coalition structure by extending the concept of the equilibrium binding agreements by Ray and Vohra (1997). Ray and Vohra capture explicitly the credibility of blocking coalitions, and then induce a recursive definition of the stable coalition structures in a game with externalities. However, in their definition, only breaking up is allowed for coalitions.

We propose a new concept of a sequential stable payoff configuration such that coalitions can both break up and merge into. The payoff configuration \( z \) is said to sequentially dominate the payoff configuration \( z' \) if there is a sequence of payoff configurations starting from \( z \) to \( z' \) such that (1) in each step, two coalitions may merge or one coalition may break into two coalitions, and (2) in each step, the members in the merging coalitions or the breaking coalition prefer the payoffs of the final configuration \( z' \) to the present payoff.

A sequential stable payoff configuration is defined as a payoff configuration which sequentially dominates all other payoff configurations.

As an application of this stable concept, we study a common pool resource game, where each payoff configuration corresponds to one coalition structure. We show that if the number of players is between 3 and 48, then for some concave production function, the payoff configuration related to the grand coalition structure is sequentially stable in the common pool resource game.

2 Dominations and Some Basic Concepts

Let \( N = \{1, 2, ..., n\} \) be a set of players. A subset \( S \) of \( N \) is called a coalition. First we define a set of feasible payoff vectors under a coalition structure. We use the concept of a coalition structure to describe the coalitions may form among individuals. Here a coalition structure \( \mathcal{P} \) is a partition \( \{S_1, S_2, ..., S_k\} \) of \( N \), where \( S_1, S_2, ..., S_k \) in \( \mathcal{P} \) are pairwise disjoint and \( \bigcup_{j=1}^{k} S_j = N \). The set of partitions of \( N \) is denoted by \( \Pi(N) \).

The feasibility of the payoff distributions among players depends on the coalition structure. Then the set of feasible payoff vectors under \( \mathcal{P} \) is denoted by \( \mathcal{F}(\mathcal{P}) \subseteq \mathbb{R}^n \).

We give an example of a set of feasible payoff vectors.

Example 1. A game in partition function form \( (N, v) \) is defined by a pair of a set of players \( N \) and a partition function \( v \) which assigns to each partition \( \mathcal{P} \in \Pi(N) \) each coalition \( S \in \mathcal{P} \) and a real value \( v(S|\mathcal{P}) \). Given a game in partition function form, the set of feasible payoff vectors under \( \mathcal{P} \) is \( \{z \in \mathbb{R}^n | \sum_{i \in S} z_i \leq v(S|\mathcal{P}) \ \forall S \in \mathcal{P} \} \)
Example 2. A game in strategic form \((N, \{S\}_{i \in N}, \{f_i\}_{i \in N})\) is defined by a triple of a set of players \(N\), a set of strategies \(\{S_i\}_{i \in N}\) and a payoff function \(f_i\) which assigns to each \(n\)-tuple of strategies \((s_1, s_2, ..., s_n) \in S_1 \times S_2 \times \cdots \times S_n\) a payoff vector. We consider a mixed strategy and an expected payoff function. The mixed strategy of \(i\) is given by a probability distribution \(p_i\) on \(S_i\). The expected payoff of player \(i\) for \(p = (p_1, p_2, ..., p_n)\) is denoted by \(E_i(p)\).

Players in a coalition \(T\) can choose their joint mixed strategy \((p)_T\), which is given by a joint probability distribution on \(\Pi_{i \in T}S_i\). The set of joint mixed strategies of \(T\) is given by \(\Pi(T)\).

Given a game in strategic form and a partition \(\mathcal{P}\), the set of feasible payoff vectors under \(\mathcal{P}\) is \(\{z \in \mathbb{R}^n | z_i = E_i(p_1, p_2, ..., p_n), (p_i)_{i \in T} \in \Pi(T) \text{ for all } T \in \mathcal{P}\}\).

A pair \((z, \mathcal{P})\) is called a payoff configuration, where \(z = (z_1, z_2, ..., z_n) \in \mathcal{F}(\mathcal{P})\) is a feasible payoff vector and \(\mathcal{P}\) is a coalition structure.

We introduce two special types of coalition structures. \(\mathcal{P}^N = \{N\}\) is called a grand coalition structure, and \(\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}\) is called a singleton coalition structure or individual coalition structure. We also say that \(\mathcal{P}'\) is a finer coalition structure of \(\mathcal{P}\) (\(\mathcal{P}\) is a coarser coalition structure of \(\mathcal{P}'\)), if the coalition structure \(\mathcal{P}'\) is given by re-dividing the coalition structure \(\mathcal{P}\), that is, \(\forall S' \in \mathcal{P}', \exists S \in \mathcal{P}\) such that \(S' \subseteq S\) and \(|S'| > |P|\).

We introduce several stability concepts for a set of payoff configurations. This is an alternative way to define a core of a game with externalities. For this purpose, we define two simple concepts of dominations between two payoff configurations.

Definition 1. Let \(z \in \mathcal{F}(\mathcal{P})\) and \(z' \in \mathcal{F}(\mathcal{P}')\). We say that \((z, \mathcal{P})\) is dominated by \((y, \mathcal{P}')\) if

1. \(\mathcal{P}'\) is a finer coalition structure of \(\mathcal{P}\), and
2. there exists \(T \in \mathcal{P}'\) such that \(T \notin \mathcal{P}\) and \(y_T > z_T\).

Definition 2. Let \(z \in \mathcal{F}(\mathcal{P})\) and \(z' \in \mathcal{F}(\mathcal{P}')\). We say that \((z, \mathcal{P})\) is directly dominated by \((y, \mathcal{P}')\) under \(\mathcal{P}'\) if

3. \(\mathcal{P}'\) is a finer coalition structure of \(\mathcal{P}\), and \(|\mathcal{P}'| = |\mathcal{P}| + 1\),
4. there exists \(T \in \mathcal{P}'\) such that \(T \notin \mathcal{P}\) and \(y_T > z_T\).

We can define stable payoff configurations by these definitions of dominations.

The following definition is a natural extension of the credible core by Ray (1989) to games with externalities.

Definition 3. A credible coalition structure is given as follows:

1. \(\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}\) is credible.
2. For \(k\) \((k = n - 1, n - 2, ..., 1)\), \(\mathcal{P}\) with \(|\mathcal{P}| = k\) is credible if there exists \(z \in \mathcal{F}(\mathcal{P})\) such that \((z, \mathcal{P})\) is not directly dominated by any payoff configuration \((z', \mathcal{P}')\) where \(\mathcal{P}'\) is credible and \(|\mathcal{P}| = k - 1\).
We call this \((z, \mathcal{P})\) a credible payoff configuration. Moreover, the set of all credible payoff configurations is called a **credible core**, and is denoted by \(CC\).

This is a recursive definition. First, according to (1), \(\mathcal{P}^I\) is credible. Second, we can check whether or not each of a coalition structure of \((n - 1)\) coalitions is credible by using the fact \(\mathcal{P}^I\) is credible. Third, we can check whether or not each of a coalition structure of \((n - 2)\) coalitions is credible by using the fact in the second step, and so on.

Ray and Vohra (1997) extends the credible core concept by a different way. Their concept is called “equilibrium binding agreement”. The following definition of a modified credible coalition structure is the same as their concept properly, but it is expressed by a simpler way using a recursive definition.

**Definition 4.** A **modified credible** coalition structure is given as follows:

1. \(\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}\) is modified credible.
2. For \(k\) \((k = n - 1, n - 2, ..., 1)\), \(\mathcal{P}\) with \(|\mathcal{P}| = k\) is modified credible if there exists \(z \in \mathcal{F}(\mathcal{P})\) such that \((z, \mathcal{P})\) is not dominated by any payoff configuration \((z', \mathcal{P}')\) where \(\mathcal{P}'\) is modified credible and \(|\mathcal{P}'| > k\).

We call this \((z, \mathcal{P})\) a modified credible payoff configuration. Moreover, the set of all modified credible payoff configurations is called a **modified credible core**, and is denoted by \(MC\).

The difference between the two definitions is as follows: In a credible coalition structure, only the (credible) direct domination, that is, a deviation of only one coalition is considered, but in a modified credible coalition structure, every (credible) domination, that is, any deviation is considered.

Remark that a stability of coalition structures with respect to only deviation of coalitions but not for merge of coalitions is considered in both definitions.

### 3 Common Pool Resource Games

Here we apply two credible cores to an economy with externalities. Let the game of an economy with a common pool resource be described by a set of players \(N := \{1, 2, ..., n\}\). For any player \(i \in N\), let \(x_i \geq 0\) represent the amount of labour input of \(i\). Clearly, the overall amount of labour is given by \(\sum_{j \in N} x_j\). The technology that determines the amount of product is considered to be a joint production function of the overall amount of labour \(f : \mathbb{R}_+^{\aleph} \rightarrow \mathbb{R}_+\) satisfying

\[
f(0) = 0, \lim_{x \to \infty} f'(x) = 0, f'(x) > 0 \text{ and } f''(x) < 0 \text{ for } x > 0.
\]

The distribution of the product is supposed to be proportional to the amount of labour expended by players. In other words, the amount of the product assigned to player \(i\) is given by \(\frac{x_i}{\sum_{j \in N} x_j} f(\sum_{j \in N} x_j)\). The price of the product is normalized to be one unit of money and let \(q\) be a cost of labor per unit, and we suppose \(0 < q < f'(0)\).
Then individual $i$’s income is denoted by 

$$m_i(x_1, x_2, \ldots, x_n) = \frac{x_i}{x_N} f(x_N) - qx_i.$$ 

Coalition $S$’s total income is denoted by 

$$m_S \equiv \sum_{i \in S} m_i = \frac{x_S}{x_N} f(x_N) - qx_S,$$ 

where $x_S \equiv \sum_{i \in S} x_i$. We consider a game where each coalition is a player. It chooses its total labor input and its payoff is given by the sum of its members’ incomes. Naturally we can define a Nash equilibrium of that game.

**Definition 5.** $(x_{S_1}^*, x_{S_2}^*, \ldots, x_{S_k}^*)$ is an equilibrium under $\mathcal{P}$ $\iff$ 

$$m_{S_j}(x_{S_j}^*, x_{S_{-j}}^*) \geq m_{S_j}(x_{S_j}, x_{S_{-j}}^*) \quad \forall j, \quad \forall x_{S_j} \in \mathbb{R}_+.$$ 

**Proposition 1 (Funaki and Yamato(1999)).** For any $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, there exists unique equilibrium $(x_{S_1}^*, x_{S_2}^*, \ldots, x_{S_k}^*)$ under $\mathcal{P}$ which satisfies 

$$f'(x_N^*) + \frac{(k-1)f(x_N^*)}{x_N^*} = kq, \quad x_{S_j}^* = \frac{x_N^*}{k} \quad \forall j, \quad x_{S_j}^* > 0 \quad \forall j,$$ 

where $x_N^* = \sum_{j=1}^k x_{S_j}^*$.

Given a coalition structure $\mathcal{P} = \{S_1, \ldots, S_k\}$, let $(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$ be a unique equilibrium under $\mathcal{P}$ and let $x_N^*(\mathcal{P}) = \sum_{i=1}^k x_{S_i}^*(\mathcal{P})$. Moreover, let $m_{S_i}^*(\mathcal{P}) = m_{S_i}(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$ be the equilibrium income of coalition $S_i$ for $i = 1, \ldots, k$ and therefore $m_N^*(\mathcal{P}) = \sum_{i=1}^k m_{S_i}(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$. The following result is given by Funaki and Yamato (1999).

**Proposition 2 (Funaki and Yamato(1999)).** For two coalition structures $\mathcal{P}_k = \{S_1, S_2, \ldots, S_k\}$ and $\mathcal{P}'_k = \{S'_1, S'_2, \ldots, S'_k\}$ with $k < k'$,

$$x_N^*(\mathcal{P}_k) < x_N^*(\mathcal{P}'_k), \quad \frac{m_{N}^*(\mathcal{P}_k)}{n} > \frac{m_{N}^*(\mathcal{P}'_k)}{n},$$

$S \in \mathcal{P}_k$ and $S \in \mathcal{P}'_k$ $\implies m_{S}^*(\mathcal{P}_k) > m_{S}^*(\mathcal{P}'_k)$.

**Example 6.** For a common pool resource game, the set of feasible payoff vectors $\mathcal{F}(\mathcal{P})$ is given by $\{z \in \mathbb{R}^n | \sum_{i \in S_j} z_i \leq m_{S_j}^*(\mathcal{P}) \quad \forall S_j \in \mathcal{P}\}$ in Funaki and Yamato (1999).

For a common pool resource game, we have another possibility to define the set of feasible payoff vectors as follows: $\mathcal{F}(\mathcal{P}) = \{z \in \mathbb{R}^n | z_i = \frac{m_{S_i}^*(\mathcal{P})}{|S_i|} \quad \forall i \in n\}$. 

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S_j, \forall S_j \in \mathcal{P}\}. It is natural to consider this set because of the symmetry of players. We focus on this type of the set in this paper.

The following is an important lemma to obtain Theorems 1 and 2.

**Lemma 1.** In a common pool resource game, let a coalition structure \( \mathcal{P} \neq \mathcal{P}^f \) be given. Without loss of generality, denote the coalition structure by \( \mathcal{P} = \{S_1, S_2, ..., S_k\} \), where \( S_1 = \{1, 2, ..., r\}, 2 \leq r \leq n \), and \( 1 \leq k \leq n - r + 1 \).

Suppose that the coalition \( S_1 \) is divided into two subcoalitions, \( S'_1 = \{1, ..., \ell\} \) and \( S''_1 = \{\ell+1, ..., r\}, \) where \( 1 \leq \ell \leq r/2 \). All other players do not change their behavior in coalition formation. Denote this coalition structure \( \mathcal{P}' = \{S'_1, S''_1, S_2, ..., S_k\} \).

Let \( m^*_1(\mathcal{P}) \equiv m^*_{S_1}(\mathcal{P})/r \) and \( m^*_1(\mathcal{P}') \equiv m^*_{S'_1}(\mathcal{P}')/\ell \). Then \( m^*_1(\mathcal{P}') > m^*_1(\mathcal{P}) \) if \( k^2/(k+1)^2 \geq \ell/r \), in particular, if (i) \( r = n \) and \( n/\ell \geq 4 \), (ii) \( 3 \leq r \leq n - 1 \) and \( \ell/r \leq 4/9 \), or (iii) \( r = 2 \) and \( k \geq 3 \).

**Proof.** By Proposition 1,

\[
m^*_1(\mathcal{P}) = \frac{m^*_{S_1}(\mathcal{P})}{r} = \frac{f(x^*_{S_1}(\mathcal{P})) - q x^*_{S_1}(\mathcal{P})}{rk} = \frac{f(x^*_{S_1}(\mathcal{P})) - f'(x^*_{S_1}(\mathcal{P})) x^*_{S_1}(\mathcal{P})}{rk^2},
\]

\[
m^*_1(\mathcal{P}') = \frac{m^*_{S'_1}(\mathcal{P}')}{\ell} = \frac{f(x^*_{S'_1}(\mathcal{P}')) - q x^*_{S'_1}(\mathcal{P}')}{\ell(k+1)} = \frac{f(x^*_{S'_1}(\mathcal{P}')) - f'(x^*_{S'_1}(\mathcal{P}')) x^*_{S'_1}(\mathcal{P}')}{\ell(k+1)^2}.
\]

Therefore,

\[
m^*_1(\mathcal{P}') - m^*_1(\mathcal{P}) = \frac{rk^2\{f(x^*_{S_1}(\mathcal{P}')) - f'(x^*_{S_1}(\mathcal{P}')) x^*_{S_1}(\mathcal{P}')\} - \ell(k+1)^2\{f(x^*_{S'_1}(\mathcal{P}')) - f'(x^*_{S'_1}(\mathcal{P}')) x^*_{S'_1}(\mathcal{P}')\}}{r\ell k^2(k+1)^2}.
\]

Here, \( 0 < f(x^*_{S_1}(\mathcal{P})) - x^*_{S_1}(\mathcal{P}) f'(x^*_{S_1}(\mathcal{P})) < f(x^*_{S'_1}(\mathcal{P}')) - x^*_{S'_1}(\mathcal{P}') f'(x^*_{S'_1}(\mathcal{P}')) \) holds because \( f(x) - x f'(x) \) is increasing for \( x > 0 \), and \( x^*_{S_1}(\mathcal{P}) < x^*_{S'_1}(\mathcal{P}') \) by Proposition 2. Therefore, \( m^*_1(\mathcal{P}') > m^*_1(\mathcal{P}) \) if \( A \equiv rk^2 - \ell(k+1)^2 \geq 0 \), that is, \( k^2/(k+1)^2 \geq \ell/r \). This condition is satisfied in the following cases.

- **Case 1.** \( r = n \) and \( n/\ell \geq 4 \) Note that \( r = n \) if and only if \( k = 1 \). Hence, \( A = n - 4\ell \geq 0 \) if \( n/\ell \geq 4 \).

- **Case 2.** \( 3 \leq r \leq n - 1 \) and \( 4/9 \geq \ell/r \): Since \( r \neq n \), \( k \geq 2 \). Also, \( k^2/(k+1)^2 \) is increasing for \( k > 0 \). Therefore, \( k^2/(k+1)^2 \geq 4/9 \). Accordingly, if \( 4/9 \geq \ell/r \), then \( A \geq 0 \).

- **Case 3.** \( r = 2 \) and \( k \geq 3 \): Since \( r = 2 \), \( \ell = 1 \). Thus \( A = (k-1)^2 - 2 \geq 2 > 0 \). Q.E.D.

We apply the credibility concepts to this common pool resource game.

**Example 7.** In a common pool resource game, suppose a production function \( f(x) \) is given by \( f(x) = \sqrt{x} \).

(1) When \( n = 4 \), the singleton coalition structure \( \mathcal{P}^f \) and all coalition structures consisting of two coalitions are both credible and modified credible.
(2) When \( n = 5 \), all coalition structures consisting of odd number of coalitions are credible. All coalition structures consisting of odd number of coalitions except for \( \{ \{i\}, \{j\}, T\} \) (\(|T| = 3\)) are modified credible.

(3) When \( n = 6 \), all coalition structures containing even number of coalitions are credible. Only the grand coalition structure \( \mathcal{P}^N \), the singleton coalition structure \( \mathcal{P}^I \), \( \{Q, R\} \) (\(|Q| = |R| = 3\)) and \( \{\{i\}, \{j\}, T, U\} \) (\(|T| = |U| = 2\)) are modified credible.

The following theorem shows that if the number of players is odd, then coalition structures consisting of odd numbers of coalitions are credible, in particular, the grand coalition structure is credible and a credible core allocation exists. If the number of players is even, then coalition structures consisting of even numbers of coalitions are credible. In this case, although the grand coalition structure is not credible, coalition structures consisting of \((n-1)\) -person coalition and one-person coalition are credible. This result is rather simple, but for the modified credibility, it is not easy to get a general result.

**Theorem 1.** In a common pool resource game, let \( n \geq 4 \) and \( \mathcal{F}(\mathcal{P}) = \{z \in \mathbb{R}^n | z_i = \frac{m^S_j(\mathcal{P})}{|S_j|}, \forall i \in S_j, \forall S_j \in \mathcal{P}\} \). If \( n \) is odd, \( \mathcal{P} \) consisting of odd number of coalitions is credible, and \( \text{CC} (\mathcal{P}^N) \neq \emptyset \). If \( n \) is even, \( \mathcal{P} \) consisting of even number of coalitions is credible, and \( \text{CC} (\mathcal{P}^{N\setminus i}) \neq \emptyset \). Here \( \mathcal{P}^{N\setminus i} = \{N \setminus \{i\}, \{i\}\} \).

**Proof.** Consider the case \( n \geq 5 \) first. According to the proof of Theorem 1, for the payoff vector \( z \) in \( \mathcal{F}(\mathcal{P}) \) with \( \mathcal{P} \neq \mathcal{P}' \), \((z, \mathcal{P})\) is directly blocked by some \( T \) under some \( \mathcal{P}' \). Consider any coalition structure \( \mathcal{P} \) such that \(|\mathcal{P}| - 1 = |\mathcal{P}'|\) and \( \mathcal{P}' \) is finer than \( \mathcal{P} \). Since \( \mathcal{P}' \) is credible by definition, the above result implies that \((z, \mathcal{P})\) is directly blocked by finer credible coalition structure \( \mathcal{P}' \). This means that \( \mathcal{P} \) is not credible.. The set of such \( \mathcal{P} \) is denoted by \( \mathcal{P}^2 \). That is,

\[
\mathcal{P}^2 = \{\mathcal{P}||\mathcal{P}|-1=|\mathcal{P}'|\text{ and } \mathcal{P}' \text{ is finer than } \mathcal{P}\}.
\]

By a simple consideration, we have \( \mathcal{P}^2 = \{\mathcal{P}||\mathcal{P}|=n-1\} \). The above result directly implies that any \( \mathcal{P}' \in \mathcal{P}^2 \) is credible because any \( \mathcal{P} \in \mathcal{P}^2 \) is not credible, where

\[
\mathcal{P}^3 = \{\mathcal{P}'||\mathcal{P}'|-1=|\mathcal{P}| \text{ for some } \mathcal{P} \in \mathcal{P}^2 \text{ and } \mathcal{P} \text{ is finer than } \mathcal{P}'\}
\]

\[
= \{\mathcal{P}'||\mathcal{P}'|=n-2\}.
\]

This consideration implies that any \( \mathcal{P} \in \mathcal{P}^m \) is credible if \( m = n - 2k \) \((k = 0, 1, 2,...)\), and not credible if \( m = n - 2k - 1 \) \((k = 0, 1, 2,...)\). Since \( m = n - 2k \) is odd if \( n \) is odd, \( \mathcal{P} \) consisting of odd number of coalitions is credible, and \( \mathcal{P}^N \in \mathcal{P}^n \) is credible, that is, \( \text{CC} (\mathcal{P}^N) \neq \emptyset \). Since \( m = n - 2k \) is even if \( n \) is even, \( \mathcal{P} \) consisting of even number of coalitions is credible, and \( \mathcal{P}^{N\setminus i} \in \mathcal{P}^{(n-1)} \) is credible, that is, \( \text{CC} (\mathcal{P}^{N\setminus i}) \neq \emptyset \).
For the case \( n = 4 \), put \( r = 2 \) and \( k = 3 \) in Lemma 1. This implies \( \mathcal{P} \in \mathcal{P}^2 \) is not credible because \( \mathcal{P}^I \) is credible. Then \( \mathcal{P} \in \mathcal{P}^3 \) is credible. Put \( r = 3 \) and \( \ell = 1 \) in Lemma 1. This implies \( \mathcal{P} \in \mathcal{P}^4 \) is not credible because \( \mathcal{P} \in \mathcal{P}^3 \) is credible.

Q.E.D.

Unfortunately we cannot find a general property of a modified credible core of a common pool resource game.

**Example 8.** In a common pool resource game, let \( f(x) = x^\alpha \), and let \( n = 8 \).

When \( \alpha = 0.2, 0.5, 0.8 \), the grand coalition structure \( \mathcal{P}^N \) is both credible and modified credible. When \( \alpha = 0.001, 0.9, 0.995 \), the grand coalition structure \( \mathcal{P}^N \) is not modified credible but credible.

In both definitions of credible cores and modified credible cores, only breaking up is allowed for coalitions. In the next section, we propose another new concept of stability of payoff configurations such that coalitions can both break up and merge into.

## 4 Sequentially Stable Coalition Structure

In this section, we give a main concept of our solution called “Sequentially Stable Payoff Configuration” and apply this concept to the game with a common pool resource.

**Definition 6.** Let \( z \in \mathcal{F}(\mathcal{P}) \) and \( y \in \mathcal{F}(\mathcal{P}') \). We say that \( (z, \mathcal{P}) \) sequentially dominates \( (y, \mathcal{P}') \) if there is a sequence of payoff configurations \( \{(x^t, \mathcal{P}_t)\}_{t=0}^T \) with \( x^t \in \mathcal{F}(\mathcal{P}_t) \) such that

1. \( \mathcal{P}_T = \mathcal{P}, \mathcal{P}_0 = \mathcal{P}', x^0 = y \) and \( x^T = z \), and
2. for all \( t \) \((0 \leq t \leq T - 1)\), either \( \mathcal{P}_t \) is a finer coalition structure of \( \mathcal{P}_{t+1} \) with \( |\mathcal{P}_t| = |\mathcal{P}_{t+1}| + 1 \), or \( \mathcal{P}_{t+1} \) is a finer coalition structure of \( \mathcal{P}_t \) with \( |\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1 \), and
3. for all \( t \) \((0 \leq t \leq T - 1)\), for some \( S \in \mathcal{P}_{t+1} \) with \( S \not\in \mathcal{P}_t \),

\[
x^T_i < x^T_i \quad \forall i \in S.
\]

We also use a term *domination* and *sequential domination* for coalition structures when only one payoff vector is feasible under each coalition structure. That is, if \( (z^*, \mathcal{P}^*) \) (sequentially) dominates \( (z, \mathcal{P}^N) \), then we simply say that \( \mathcal{P}^* \) (sequentially) dominates \( \mathcal{P}^N \), and we use the following notation;

\[
\mathcal{P}_0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \rightarrow \ldots \rightarrow \mathcal{P}_T.
\]

The condition (3) shows that if \( \mathcal{P}_k \) is a finer coalition structure of \( \mathcal{P}_{k+1} \), for any member of two combining coalitions \( S \) and \( T \) such that \( S, T \in \mathcal{P}_k \) and
With out loss of generality, denote the coalition structure by \( \mathcal{P}_k \), for any member of one of the divided two coalitions \( S \) and \( T \) such that \( S, T \in \mathcal{P}_{k+1} \) and \( S \cup T \in \mathcal{P}_k \), his payoff \( x_i^k \) is smaller than his terminal payoff \( x_i^0 \).

**Definition 7.** Let \( z \in \mathcal{F}(\mathcal{P}^*) \). We say that \((z, \mathcal{P}^*)\) is a sequentially stable payoff configuration if for all feasible payoff configurations \((y, \mathcal{P})\) with \( \mathcal{P} \neq \mathcal{P}^* \), \((z, \mathcal{P}^*)\) sequentially dominates \((y, \mathcal{P})\).

The following lemma gives a necessary and sufficient condition that the payoff configuration in the grand coalition structure is preferable to the payoff configuration in another coalition structure for all players.

**Lemma 2.** In a common pool resource game, let a coalition structure \( \mathcal{P} \) be given. Without loss of generality, denote the coalition structure by \( \mathcal{P} = \{ S_1, S_2, S_3, ..., S_k \} \), where \( |S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k \). Let

\[
B(k) \equiv \left\{ f(x_N^*(\mathcal{P}))-f^r(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\right\}/[k^2\left\{ f(x_N^*(\mathcal{P}^N))-f^r(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\right\}],
\]

where \( \mathcal{P}^N = \{1, 2, ..., n\} \) is the grand coalition structure. Then for each \( i \in N \), \( m_i^*(\mathcal{P}) \geq m_i^*(\mathcal{P}^N) \) if and only if \( B(k) \geq 1/n \).

**Proof.** By Proposition 1,

\[
m_i^*(\mathcal{P}) = m_{S_j}^*(\mathcal{P})/r_j = [f(x_N^*(\mathcal{P}))-qx_N^*(\mathcal{P})]/(r_jk) = [f(x_N^*(\mathcal{P}))-f^r(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})]/(r_jk^2),
\]

for \( i \in S_j \) and \( j = 1, ..., k \). Notice that for the grand coalition structure \( \mathcal{P}^N \), \( k = 1 \) and \( r_1 = n \), so that \( m_i^*(\mathcal{P}^N) = [f(x_N^*(\mathcal{P}^N))-f^r(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)]/n \) for \( i \in N \). We also remark that a player belonging to the smallest coalition, \( S_1 \), obtains the highest payoff among all players, that is, the payoff of each player \( i, m_i^*(\mathcal{P}) \), is less than or equal to \( m_{S_j}^*(\mathcal{P})/r_j \). Therefore, each \( i \in N \), \( m_i^*(\mathcal{P}) \geq m_{S_j}^*(\mathcal{P})/r_j \) if and only if \( B(k) = \left\{ f(x_N^*(\mathcal{P}))-f^r(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\right\}/[k^2\left\{ f(x_N^*(\mathcal{P}^N))-f^r(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\right\}] \geq 1/n \). Q.E.D.

If we apply this lemma to a common pool resource game with a production function \( f(x) = x^\alpha \) \((0 < \alpha < 1)\), we have the followings:

By Proposition 1, for any \( \mathcal{P} \)

\[
x_N^*(\mathcal{P}) = (\alpha + k - 1)(x_N^*(\mathcal{P}))^{\alpha-1}/(kq) = \left(\frac{\alpha - 1 + k}{kq}\right)^{1/(1-\alpha)} ,
\]

\[
m_i^*(\mathcal{P}) = m_{S_j}^*(\mathcal{P})/r = [f(x_N^*(\mathcal{P}))-qx_N^*(\mathcal{P})]/(rk) = [f(x_N^*(\mathcal{P}))-f^r(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})]/(rk^2) = (1 - \alpha)(x_N^*(\mathcal{P}))^\alpha/(rk^2).
\]
Notice that if $\mathcal{P} = \mathcal{P}^N$, then $k = 1$ and $r = n$, so that
\[
x_N^*(\mathcal{P}^N) = \alpha(x_N^*(\mathcal{P}^N))^{q-1}/q = \left(\frac{\alpha}{q}\right)^{1/(1-\alpha)}.
\]
This implies
\[
f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N) = (1 - \alpha)(x_N^*(\mathcal{P}^N))^\alpha.
\]

We check the sequential stability of the grand coalition structure. First consider a case $n = 2^m$. We say $\mathcal{P}$ is a $k$-th stage coalition structure if $|\mathcal{P}| = k$.

**Theorem 2.** If $B(k) < 1/2^{k-1}$ for all $k(k = 2, \ldots, m, m + 1)$, the grand coalition structure is sequentially stable.

**Proof.** We have to show that every coalition structure other than the grand coalition structure $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. In the following, we denote a coalition structure $\mathcal{P} = \{S_1, S_2, S_3, \ldots, S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq \ldots \leq |S_k| = r_k$, by $\{r_1; r_2; r_3; \ldots; r_k\}$, because the payoff is determined by the sizes of all cotillions in a coalition structure.

Consider a coalition structure $\mathcal{P}^*$ consisting of the following $(m+1)$ coalitions: two 1-person coalitions, one 2-person coalition, one 4-person coalition, one 8-person coalition, ..., and one $2^{m-1}$-person coalition. This coalition structure is denoted by $\{1; 1; 2; 4; 8; \ldots; 2^{m-1}\}$.

The proof consists of four steps.

**(Step 1)** $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$:

Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{m}$ such that $\mathcal{P}_0 = \mathcal{P}^*, \mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, \ldots, m - 1$. This sequence is expressed by

\[
\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1}\} \to \mathcal{P}_1 = \{2; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1}\} \\
\to \mathcal{P}_2 = \{4; 4; 8; \ldots; 2^{m-2}; 2^{m-1}\} \\
\to \ldots \to \mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\} \to \mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\} \to \mathcal{P}_m = \mathcal{P}^N = \{2^m\}
\]

First, it follows from Lemma 2 that the 2nd stage coalition structure $\mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-1}/2^m = 1/2 > B(2)$ by the hypothesis.
Next, it follows from Lemma 1 that the 3rd stage coalition structure \( P_{m-2} = \{2^{m-2}, 2^{m-2}, 2^{m-1}\} \) is dominated by \( P^N \), since \( r_1/n = 2^{m-2}/2^m = 1/4 > B(3) \) by the hypothesis.

In general, for \( k = 2, \ldots, m, m + 1 \), it follows from Lemma 2 that the \( k \)-th stage coalition structure \( P_{m-k+1} = \{2^{m-k+1}, 2^{m-k+1}, 2^{m-k+2}, 2^{m-k+2}, \ldots, 2^{m-1}\} \) is sequentially dominated by \( P^N \), since \( r_1/n = 2^{m-k+1}/2^m = 1/2^{k-1} > B(k) \) by the hypothesis.

Therefore, the \((m+1)\)-th stage coalition structure \( P_0 = P^* = \{1; 1; 2; 4; \ldots; 2^{m-1}\} \) is sequentially dominated by \( P^N \).

**(Step 2)** Every \((m+1)\)-th stage coalition structure is sequentially dominated by \( P^N \):

Take any \((m+1)\)-stage coalition structure \( P \).

First we consider a sequence \( \{P_t\}_{t=0}^T \) such that
1) \( P_0 = P = \{r_1; r_2; r_3; \ldots; r_{m-1}; r_m; r_{m+1}\} \)
2) \( P_T = \{1; 1; 1; \ldots; 1; 2^m - m\} \), where \(|P_T| = m + 1\).
3) If \( t \) is zero or even, then the largest and the second largest coalitions in \( P_t \) are zero or even, then the largest and the second largest coalitions in \( P_t \) and \( P_{T-1} = \{1; 1; 1; \ldots; 1; \sum_{k=1}^{m+1} r_k - m + 1\} \) (\( m \)-th-stage)

\[ \rightarrow P_T = \{1; 1; 1; \ldots; 1; \sum_{k=1}^{m+1} r_k - m\} = \{1; 1; 1; \ldots; 1; 2^m - m\} \] (\((m+1)\)th-stage)

Next consider \( \{P_t\}_{t=T}^{T+T'} \) such that
1) \( P_T = \{1; 1; 1; \ldots; 1; 2^m - m\} \),
2) \( P_{T+T'} = P^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1}\} \),
3) If \( t = T + \lambda \) and \( \lambda \) is zero or even \((\lambda \leq T' - 2)\), then the smallest coalition of more than one members and a 1-person coalition in \( P_{T+\lambda} \) merge in \( P_{T+\lambda+1} \).
4) If \( t = T + \lambda \) and \( \lambda \) is odd (\( \lambda \leq T' - 2 \)), then \( 2^{m - \frac{\lambda + 1}{2}} \) persons in the coalition of \( 2^{m - \frac{\lambda + 1}{2}} - (m - \frac{\lambda + 1}{2}) \) persons in \( P_{T+\lambda} \) deviate and form a coalition in \( P_{T+\lambda+1} \). Note that \( 2^{m - \frac{\lambda + 1}{2}} - (m - \frac{\lambda + 1}{2}) \geq 1 \).

5) If \( t = T + T' - 1 \), then two one-person coalitions in \( P_{T+T'-1} \) merge in \( P_{T+T'} \).

This sequence \( \{P_t\}_{t=T}^{T+T'} \) of coalition structures is given by:

\[
P_T = \{1;1;1;1;\ldots;1;1;1;2^m-m\} \quad (m+1)\text{-th stage}
\]

\[
\rightarrow P_{T+1} = \{1;1;1;1;\ldots;1;1;1;2^m-m+1\} \quad (m)\text{-th stage}
\]

\[
\rightarrow P_{T+2} = \{1;1;1;1;\ldots;1;1;1;2^m-m+1-2^{m-1};2^{m-1}\}
\]

\[
= \{1;1;1;1;\ldots;1;1;1;2^{m-1}-m+1;2^{m-1}\} \quad ((m+1)\text{-th stage})
\]

\[
\rightarrow P_{T+3} = \{1;1;1;1;\ldots;1;1;2^{m-1}-m+2;2^{m-1}\} \quad (m)\text{-th stage}
\]

\[
\rightarrow P_{T+4} = \{1;1;1;1;\ldots;1;1;2^{m-1}-m+2-2^{m-2};2^{m-2};2^{m-1}\}
\]

\[
= \{1;1;1;1;\ldots;1;1;2^{m-2}-m+2;2^{m-2};2^{m-1}\} \quad ((m+1)\text{-th stage})
\]

\[
\rightarrow P_{T+5} = \{1;1;1;1;\ldots;1;2^{m-2}-m+3;2^{m-2};2^{m-1}\} \quad (m)\text{-th stage}
\]

\[
\rightarrow \ldots \rightarrow 
\]

\[
\rightarrow P_{T+T'-1} = \{1;1;1;1;4;8;\ldots;2^{m-3};2^{m-2};2^{m-1}\} \quad ((m+1)\text{-th stage})
\]

\[
\rightarrow P_{T+T'} = \{1;1;2;4;8;\ldots;2^{m-3};2^{m-2};2^{m-1}\} \quad (m)\text{-th stage}
\]

This sequence ends at the coalition structure \( P_{T'} = P^* \).

Hence if we combine two sequences \( \{P_t\}_{t=0}^{T} \) and \( \{P_t\}_{t=T}^{T+T'} \), we can get a sequence \( \{P_t\}_{t=0}^{T+T'} \) from any \((m+1)\)th stage coalition structure \( P \) to \( P^* \). Note that only \((m+1)\)th stage and \(m\)th stage coalition structures appear in this sequence.

Each member of any coalition in \((m+1)\)th stage coalition structure prefers the payoff under the grand coalition structure \( P^N \) to the payoff under the \((m+1)\)th stage coalition structure because of \( B(m+1) < 1/2^m \). Moreover any deviating coalition in the process from \(m\)th stage coalition structure to \((m+1)\)th stage coalition structure consists of at least two players. Each member of such a deviating coalition prefers the payoff in the grand coalition structure \( P^N \) to the payoff in the \(m\)th stage coalition structure, because of \( B(m) < 2/2^m = 1/2^{m-1} \) by Lemma 1.

Therefore if we combine this sequence \( \{P_t\}_{t=0}^{T'} \) and a sequence from \( P_{T+T'} = P^* \) to \( P^N \), every coalition structure in the sequence \( \{P_t\}_{t=T}^{T+T'} \) is sequentially dominated by \( P^N \). And so is the \((m+1)\)th stage coalition structure \( P \). This completes the proof of Step 2.

**Step 3** Every coalition structure \( P \) of less than \( m+1 \) coalitions other than the grand coalition structure \( P^N \) is sequentially dominated by \( P^N \).
First, we show that each member of a coalition of the maximal size in any coalition structure $\mathcal{P}$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}$. Denote $\mathcal{P}$ by $\mathcal{P} = \{S_1, S_2, S_3, \ldots, S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq \ldots \leq |S_k| = r_k$. Because $r_k \geq r_i$ for all $r_i$, $kr_k \geq \sum_{i=1}^{k} r_i = n$, that is, $r_k/n \geq 1/k$. Since $B(k) < 1/2^{k-1}$, it follows that $r_k/n \geq 1/k \geq 1/2^{k-1} > B(k)$. By Lemma 1, we have the desired result.

Take any coalition structure $\mathcal{P}$ of less than $m + 1$ coalitions other than $\mathcal{P}^N$. Consider the following sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m + 1)$-stage coalition structure $\mathcal{P}'$: one person in a coalition of the maximal size in $\mathcal{P}_t$ deviates and forms a 1-person coalition in $\mathcal{P}_{t+1}$. Notice that such a person in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 4) Every coalition structure $\mathcal{P}$ of more than $m + 1$ coalitions is sequentially dominated by $\mathcal{P}^N$.

Take any $k$-th stage coalition structure $\mathcal{P}$ of more than $m + 1$ coalitions. Since $B(k)$ is a decreasing function by Lemma 2, $B(k) < B(m + 1) < 1/2^m = 1/n \leq r_i/n$ holds for any $r_i \geq 1$. This together with Lemma 1 imply that each member of any coalition in $\mathcal{P}$ prefers her payoff under the grand coalition structure $\mathcal{P}^N$ to her payoff under $\mathcal{P}$.

Consider a sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m + 1)$-stage coalition structure $\mathcal{P}'$ such that two coalitions in $\mathcal{P}_t$ merge and form one coalition in $\mathcal{P}_{t+1}$. Notice that each member in these two coalitions in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

Q.E.D.

Next consider a case that $n = 2^m + l$ ($1 \leq l \leq 2^m - 1$).

**Theorem 3.** If $B(k) < \frac{2^{m-k+1}}{n}$ ($k = 2, \ldots, m, m+1$), the grand coalition structure is sequentially stable.

**Proof.** The basic idea of the proof is the same as that of the proof of Theorem 3. Consider a coalition structure $\mathcal{P}^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1} + l\}$ consisting of $(m + 1)$ coalitions.

(Step 1) $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$.

Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^m$ such that $\mathcal{P}_0 = \mathcal{P}^*, \mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, \ldots, m - 1$. This sequence is expressed by

$$\mathcal{P}^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1} + l\} \rightarrow \{2; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1} + l\}$$
First, it follows from Lemma 1 that the 2nd stage coalition structure \( P_{m-1} = \{2^{m-1}; 2^m + l\} \) is dominated by \( P^N \), since \( r_1/n = 2^{m-1}/n > B(2) \) by the hypothesis.

Next, it follows from Lemma 1 that the 3rd stage coalition structure \( P_{m-2} = \{2^{m-2}; 2^m + l\} \) is dominated by \( P^N \), since \( r_1/n = 2^{m-2}/n > B(3) \) by the hypothesis.

In general, for \( k = 2, ..., m, m+1 \), it follows from Lemma 1 that the \( k \)-th stage coalition structure \( P_{m-k+1} = \{2^{m-k+1}; 2^{m-k+2}; 2^{m-k+3}; ...; 2^{m-1} + l\} \) is sequentially dominated by \( P^N \), since \( r_1/n = 2^{m-k+1}/n > B(k) \) by the hypothesis. Therefore the \( (m+1) \)-th stage coalition structure \( P_0 = P^* = \{1; 1; 2; 4; ...; 2^{m-1} + l\} \) is sequentially dominated by \( P^N \).

We omit the rest of the proof that is similar to that of Proposition 3. Q.E.D.

**Theorem 4.** If \( B(k) < \frac{m-k+1}{n} \) \( (k = 2, ... , m) \), \( B(m+1) \geq \frac{1}{n} \) and \( B(m+2) < \frac{1}{n} \) the grand coalition structure is sequentially stable.

**Proof.** The important difference from Theorem 4 is that one person coalition in \((m+1)\)-th stage coalition structure does not like to move to \( P^N \), but members in coalitions with two or more players like to move to the destination coalition structure \( P^N \). We follow the procedures of the proof of Theorems 3 and 4.

Consider a coalition structure \( P^{**} = \{1; 1; 2; 2; 8; ...; 2^{m-2}; 2^{m-1} + l\} \) consisting of \( (m+2) \) coalitions instead of \( P^* = \{1; 1; 2; 4; ...; 2^{m-1} + l\} \) in Theorems 3 and 4.

**(Step 1)** \( P^{**} \) is sequentially dominated by \( P^N \).

Consider a sequence of coalition structures \( \{P_i\}_{i=0}^{m+1} \) such that \( P_0 = P^{**} \), \( P_{m+1} = P^N \).

\[
P^{**} = \{1; 1; 2; 2; 8; ...; 2^{m-2}; 2^{m-1} + l\}
\]

\[
\rightarrow P_1 = \{2; 2; 2; 2; 8; ...; 2^{m-2}; 2^{m-1} + l\} \quad (\text{\textit{m+1} stage})
\]

\[
\rightarrow P_2 = \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1} + l\} \quad (\text{\textit{m} stage})
\]

\[
\rightarrow P_3 = \{4; 4; 8; ...; 2^{m-2}; 2^{m-1} + l\} \quad (\text{\textit{m-1} stage})
\]

\[
\rightarrow \ldots \rightarrow \ldots
\]

\[
\rightarrow P_{m-1} = \{2^{m-2}; 2^{m-2}; 2^{m-1} + l\}
\]

\[
\rightarrow P_m = \{2^{m-1}; 2^{m-1} + l\} \rightarrow P_{m+1} = \{2^m + l\} = P^N
\]

We can prove that \( P_m = \{2^{m-1}; 2^{m-1} + l\} \), \( P_{m-1} = \{2^{m-2}; 2^{m-2}; 2^{m-1} + l\} \), ..., and \( P_2 = \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1} + l\} \) are sequentially dominated by \( P^N \) by the same argument as Step 1 in the proof of Theorem 4.
Also, it follows from Lemma 1 that $\mathcal{P}_1 = \{2; 2; 2; 2; 8; \ldots; 2^{m-2}; 2^{m-1} + l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2/n > B(m + 1)$ which is obtained from $2/n > B(m)$ by the hypothesis and $B(m + 1)$.

Moreover, it follows from Lemma 1 that $\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 2; 2; 8; \ldots; 2^{m-2}; 2^{m-1} + l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 1/n > B(m + 2)$ by the hypothesis. Therefore $\mathcal{P}^* = \{1; 1; 2; 2; 2; 8; \ldots; 2^{m-1} + l\}$ is sequentially dominated by $\mathcal{P}^N$.

**Step 2** Every $(m + 1)$-th stage coalition structure is sequentially dominated by $\mathcal{P}^N$.

Take any $(m + 1)$-stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^T$ such that
1) $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; \ldots; r_{m-1}; r_m; r_{m+1}\}$
2) $\mathcal{P}_T = \{1; 1; 1; \ldots; 1; 2^m + l - m\}$, where $|\mathcal{P}_T| = m + 1$.
3) If $t$ is zero or even, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.
4) If $t$ is odd, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^T$ of coalition structures is given by:
\[
\mathcal{P}_0 = \{r_1; r_2; r_3; \ldots; r_{m-1}; r_m; r_{m+1}\} \quad (m + 1)\text{-th stage})
\]
\[
\rightarrow \mathcal{P}_1 = \{1; r_1; r_2; r_3; \ldots; r_{m-1}; r_m; r_{m+1} - 1\} \quad (m + 2)\text{-th stage})
\]
\[
\rightarrow \mathcal{P}_2 = \{1; r_1; r_2; r_3; \ldots; r_{m-1}; r_m + r_{m+1} - 1\} \quad (m + 1)\text{-th stage})
\]
\[
\rightarrow \mathcal{P}_3 = \{1; 1; r_1; r_2; r_3; \ldots; r_{m-1}; r_m + r_{m+1} - 2\} \quad (m + 2)\text{-th stage})
\]
\[
\rightarrow \ldots \rightarrow \ldots
\]
\[
\rightarrow \mathcal{P}_{T-2} = \{1; 1; 1; \ldots; 1; \sum_{k=2}^{m+1} r_k - m + 1\} \quad (m + 1)\text{-th stage})
\]
\[
\rightarrow \mathcal{P}_{T-1} = \{1; 1; 1; \ldots; 1; \sum_{k=2}^{m+1} r_k - m\} \quad (m + 2)\text{-th stage})
\]
\[
\rightarrow \mathcal{P}_T = \{1; 1; 1; 1; \ldots; 1; \sum_{k=1}^{m+1} r_k - m\} = \{1; 1; 1; 1; \ldots; 1; 2^m + l - m\} \quad (m + 1)\text{-th stage})
\]

Next consider $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ such that
1) $\mathcal{P}_T = \{1; 1; 1; \ldots; 1; 2^m + l - m\}$,
2) $\mathcal{P}_{T+T'} = \mathcal{P}^* = \{1; 1; 2; 2; 2; 8; \ldots; 2^{m-2}; 2^{m-1} + l\}$.
3) If $t = 0$, $2^{m-1} - m$ persons in the coalition of $2^m + l - m$ persons in $\mathcal{P}_0$ deviate and form a coalition in $\mathcal{P}_1$.
4) If $t = T + \lambda$ and $\lambda$ is odd ($\lambda \leq T' - 3$), then the smallest coalition of more than one members and a 1-person coalition in $\mathcal{P}_{T+\lambda}$ merge in $\mathcal{P}_{T+\lambda+1}$.
5) If \( t = T + \lambda \) and \( \lambda \) is even \( (\lambda \leq T' - 3) \), then \( 2^{m-\frac{1}{2}} - (m - \frac{1}{2}) \) persons in the coalition of \( 2^{m-\frac{1}{2}} - (m - \frac{1}{2}) \) persons in \( P_{T+\lambda} \) deviate and form a coalition in \( P_{T+\lambda+1} \). Note that \( 2^{m-\frac{1}{2}} - (m - \frac{1}{2}) \geq 1 \).

6) If \( t = T + T' - 2 \), two one person coalitions in \( P_{T+T'-2} \) merge in \( P_{T+T'-1} \).

7) If \( t = T + T' - 1 \), one 4 person coalition in \( P_{T+T'-1} \) is divided into two 2 person coalitions in \( P_{T+T'} \).

This sequence \( \{P_t\}_{t=T}^{T+T'} \) of coalition structures is given by:

\[
P_T = \{1;1;1;\ldots;1;1;1;2^{m}+l-m\} \quad \text{(}(m+1)-\text{th stage})
\]

\[
\rightarrow P_{T+1} = \{1;1;1;\ldots;1;1;1;2^{m-1} - m;2^{m-1}+l\} \quad \text{(}(m+2)-\text{th stage})
\]

\[
\rightarrow P_{T+2} = \{1;1;1;\ldots;1;1;2^{m-1} - m + 1;2^{m-1}+l\} \quad \text{(}(m+1)-\text{th stage})
\]

\[
P_{T+3} = \{1;1;1;\ldots;1;1;2^{m-2} - m + 1;2^{m-2};2^{m-1}+l\} \quad \text{(}(m+2)-\text{th stage})
\]

\[
P_{T+4} = \{1;1;1;\ldots;1;1;2^{m-2} - m + 2;2^{m-2};2^{m-1}+l\} \quad \text{(}(m+1)-\text{th stage})
\]

\[
P_{T+5} = \{1;1;1;\ldots;1;2^{m-3} - m + 2;2^{m-3};2^{m-2};2^{m-1}+l\} \quad \text{(}(m+2)-\text{th stage})
\]

\[
\rightarrow \ldots \rightarrow \ldots
\]

\[
P_{T+T'-3} = \{1;1;1;1;1;2^{m-3};2^{m-2};2^{m-1}+l\} \quad \text{(}(m+1)-\text{th stage})
\]

\[
P_{T+T'-2} = \{1;1;1;1;4;8;\ldots;2^{m-3};2^{m-2};2^{m-1}\} \quad \text{(}(m+2)-\text{th stage})
\]

\[
P_{T+T'-1} = \{1;1;2;4;8;\ldots;2^{m-3};2^{m-2};2^{m-1}\} \quad \text{(}(m+1)-\text{th stage})
\]

\[
P_{T+T'} = \{1;1;2;2;2;8;\ldots;2^{m-3};2^{m-2};2^{m-1}+l\} \quad \text{(}(m+2)-\text{th stage}) = P^{**}
\]

Hence if we combine two sequences \( \{P_t\}_{t=0}^{T} \) and \( \{P_t\}_{t=T}^{T+T'} \), we can get a sequence \( \{P_t\}_{t=0}^{T+T'} \) from any \((m+1)\)-th stage coalition structure \( P \) to \( P^{**} \). Note that only deviation of a coalition with 2 or more members appears for all \((m+1)\)-th coalition structures in this sequence.

The rest part of the proof is the same as the proof of Theorem 4. Q.E.D.

**Theorem 5.** If \( f(x) = x^\alpha \) and either \( 3 \leq n \leq 48 \), then for some \( \alpha \), the grand coalition structure \( P^N \) is sequentially stable in the common pool game.

**Proof.** Note that \( B(k) \) is an increasing function of \( \alpha \), and \( \lim_{\alpha \to 0} B(k) = 1/k^2 \) for any \( k \). Hence for sufficiently small \( \alpha > 0 \), \( B(k) \) is very close to \( 1/k^2 \).

First of all, consider the case of \( n = 2^m \). In this case, it follows from Theorem 2 that for \( m \leq 5 \), that is, for \( n = 4,8,16,32 \), \( P^N \) is sequentially stable for a sufficiently small \( \alpha \), since \( \lim_{\alpha \to 0} B(k) = 1/k^2 < 1/2^{k-1} \) for \( k = 2,3,4,5,6 \): \( \lim_{\alpha \to 0} B(2) = 1/4 < 1/2 \), \( \lim_{\alpha \to 0} B(3) = 1/9 < 1/2^2 = 1/4 \), \( \lim_{\alpha \to 0} B(4) = 1/16 < 1/2^3 = 1/8 \), \( \lim_{\alpha \to 0} B(5) = 1/25 < 1/2^4 = 1/16 \), and \( \lim_{\alpha \to 0} B(6) = 1/36 < 1/2^5 = 1/32 \).

Next consider the case of \( n = 2^m + l \) \((1 \leq l \leq 2^m - 1)\). There are five subcases to examine:
1) $m = 1$. In this case, $n = 3$. Since $\lim_{\alpha \to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 1/3$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$.

2) $m = 2$. In this case, $n \in \{5, 6, 7\}$. Since $\lim_{\alpha \to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 2/n$ and $\lim_{\alpha \to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$.

3) $m = 3$. In this case, $n \in \{9, 10, \ldots, 14, 15\}$. Since $\lim_{\alpha \to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 4/n$, $\lim_{\alpha \to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha \to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$.

4) $m = 4$. In this case, $n \in \{17, 18, \ldots, 30, 31\}$. Suppose that $n \leq 24$. Since $\lim_{\alpha \to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 8/n$, $\lim_{\alpha \to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 4/n$, $\lim_{\alpha \to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha \to 0} B(5) = 1/25 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$ if $17 \leq n \leq 24$.

On the other hand, when $\lim_{\alpha \to 0} B(5) > 1/n$, that is, $n \in \{25, 26, \ldots, 31\}$, $\lim_{\alpha \to 0} B(6) = 1/36 < 1/n$. It follows from Theorem 4 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$ if $25 \leq n \leq 31$.

5) $m = 5$. In this case, $n \in \{32, 33, \ldots, 62, 63\}$. Suppose that $n \leq 35$. Since $\lim_{\alpha \to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 16/n$, $\lim_{\alpha \to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 8/n$, $\lim_{\alpha \to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 4/n$, $\lim_{\alpha \to 0} B(5) = 1/25 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha \to 0} B(6) = 1/36 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$ if $32 \leq n \leq 35$.

On the other hand, when $\lim_{\alpha \to 0} B(6) > 1/n$, if $\lim_{\alpha \to 0} B(7) = 1/49 < 1/n$, that is, $n \in \{36, 37, \ldots, 48\}$, it follows from Theorem 4 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small $\alpha$ if $32 \leq n \leq 48$.

Q.E.D.

5 Linear Public Goods Games

There are $n$ players and let $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) be the set of players. Each player $i \in N$ has one unit of initial endowment of a private good, and she faces a decision of splitting it between her contribution to a public good, $x_i \in [0, 1]$, and her own consumption of the private good, $1 - x_i$.

The level of the public good is the sum of the contributions of $n$ players, $\sum_{j \in N} x_j$. Player $i$’s payoff under the contribution vector $(x_1, \ldots, x_n)$ is given by
Corollary 1. Let \( u_i(x_1, \ldots, x_n) = a(\sum_{j \in N} x_j) + (1 - x_i) = (a - 1)x_i + 1 + a\sum_{j \neq i} x_j. \) Here we suppose \( 1 > a > \frac{1}{n}. \)

Then \( x_i^* = 0 \) is a dominant strategy.

Coalition \( S \subseteq N \) maximizes \( \sum_{i \in S} u_i. \) Here \( \sum_{i \in S} u_i(x) = (a|S| - 1)\sum_{i \in S} x_i + |S|a\sum_{j \in N \setminus S} x_j + |S|. \) When \( |S| > \frac{1}{a}, x_i^* = 1 \) \( \forall i \in S \) is a dominant strategy, and when \( |S| \leq \frac{1}{a}, x_i^* = 0 \) \( \forall i \in S \) is a dominant strategy.

Let \( F(P) = \{ (u_1(x^*(P)), u_2(x^*(P)), \ldots, u_n(x^*(P)) \} \), and for \( a \in (\frac{1}{n}, 1) \), let \( s^*(a) = \left[ \frac{1}{n} \right] + 1 \), which is the smallest integer greater than \( \frac{1}{a} \).

**Theorem 6.** Let \( a \in (\frac{1}{n}, 1) \) be given. The coalition structure \( P = \{ S_1, S_2, \ldots, S_k \} \) is both credible and modified credible if \( \max_{j \in \{1, \ldots, k\}} |S_j| = s^*(a) \).

**Proof:** Take any coalition structure \( P = \{ S_1, S_2, \ldots, S_k \} \) such that \( \max_{j \in \{1, \ldots, k\}} |S_j| = s^*(a) \). Let \( B \equiv \bigcup_{|S_j| = s^*(a)} S_j. \) For \( i \in B, x_i^*(P) = 1 \), and for \( i \notin B, x_i^*(P) = 0. \)

Also, for \( i \in B, u_i(x^*(P)) = a|B|, \) and for \( i \notin B, u_i(x^*(P)) = a|B| + 1. \)

Consider any \( P' \) that is finer than \( P. \) There are two cases to examine:

Case 1: There are \( S_j \in P \) and \( T_1, \ldots, T_\ell \in P' \) such that \( |S_j| = s^*(a) \) and \( \cup_{j=1, \ldots, \ell} T_j = S_j. \)

In this case, since \( |T_j| < s^*(a) \) for \( j = 1, \ldots, \ell, x_i^*(P') = 0 \) for \( i \in S_j. \) This implies \( \sum_{i \in N} x_i^*(P') \leq |B| - |S_j| = |B| - s^*(a) \). Therefore, \( \forall i \in B, u_i(x^*(P')) \leq a(|B| - s^*(a)) + 1 = a|B| + (1 - a \cdot s^*(a)) < a|B| = u_i^*(x^*(P)), \)

\( \forall i \notin B, u_i^*(x^*(P')) \leq a(|B| - s^*(a)) + 1 < a|B| + 1 = u_i^*(x^*(P)). \)

Case 2: Otherwise, \( x_i^*(P') = x_i^*(P) \) and \( u_i(x^*(P')) = u_i(x^*(P)) \) for all \( i \in N. \)

Q.E.D.

**Corollary 1.** Let \( a \in (\frac{1}{n}, 1) \) be given. The coalition structure \( P = \{ S_1^*, S_2^*, \ldots, S_k^* \} \) is both credible and modified credible if \( n \cdot \frac{s^*(a)}{a} = k \) is an integer and \( |S_j^*| = s^*(a) \) for \( j = 1, \ldots, k. \) Moreover, \( \sum_{i \in N} u_i(P) = \sum_{i \in N} u_i(P^N). \)

**Corollary 2.** Let \( a \in (\frac{1}{n}, 1) \) be given and \( n = k \cdot s^*(a) + m \) \( (0 \leq m < s^*). \) Then \( P = \{ S_1^*, S_2^*, \ldots, S_k^*, \{ i_1 \}, \{ i_2 \}, \ldots, \{ i_m \} \} \) is a credible and modified credible coalition structure, where \( |S_j^*| = s^*(a) \) for \( j = 1, \ldots, k. \) Moreover, as \( n \to \infty, \)

\[
\frac{\sum_{i \in N} u_i(P)}{\sum_{i \in N} u_i(P^N)} \to 1
\]

**Remark.** Theorem 6 holds for \( P = P^1 \cup P^2, \) where \( |S_p| \geq s^* \ \forall S_p \in P^1, \)

\( \sum_{S_p \in P^1} |S_p| = rs^*, (0 \leq r \leq k), \) and \( |S_q| < s^* \ \forall S_q \in P^2. \)

**References**

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