Bridging the gap between growth theory and the new economic geography: The spatial Ramsey model*

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Abstract

In this paper, we solve a Ramsey model in continuous time and space, with both time and space discounting. We extract the corresponding Pontryagin conditions and prove their sufficiency. Then, we consider a finite time sub-problem and study the existence of solutions and the behaviour of these solutions when the time horizon increases. We end up by proving at the same time, the existence of a solution to the original infinite time optimization problem and the convergence to a stationary solution of this problem. However, both results are only obtained for a non-zero measure set of initial conditions, but not for any initial condition.

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1 Introduction

The inclusion of the geographical space in economic analysis has regained relevance in the recent years. The emergence of a new economic geography discipline is indeed one of the major events in the economic literature of the last decade (see Krugman, 1991 and 1993, Fujita, Krugman and Venables, 1999, and Fujita and Thisse, 2002). Departing from the early regional science contributions which are typically based on simple flow equations (e.g., Beckman, 1952), the new economic geography models use a general equilibrium framework with a refined specification of local and global market structures, and some precise assumptions on the mobility of production factors. Their usefulness in explaining the mechanics of agglomeration, the formation of cities, the determinants and implications of migrations, and more generally, the dynamics of the distributions of people and goods over space and time, is undeniable, so undeniable that this new discipline has become increasingly popular in the recent years.

Two main characteristics of the new economic geography contributions quoted just above are: (i) the discrete space structure, and (ii) the absence of capital accumulation. Typically, the economic geographers use two-regions frameworks, mostly analogous to the two-country models usually invoked in trade theory. However, some continuous space extensions of these models have been already studied. In a continuous space extension of his 1993 two-region model, Krugman (1996) shows that the economy always displays regional convergence, in contrast to the two-region version in which convergence and divergence are both possible. Mossay (2003) proves that continuous space is not incompatible with regional divergence using a different migration scheme. In Krugman’s model, migration follows utility level differentials, which in turn implies that location real wages provides the only incentive for moving (predominant regional convergence force). In Mossay, migrations additionally depend on idiosyncrasies in location taste, inducing a divergence force, which can balance the utility gradient force mentioned before. As a consequence, regional divergence is a possible outcome in this model.

Both models, however, ignore the role of capital accumulation in migrations: They both assume zero (individual) saving at any moment. Indeed, the zero saving assumption is
a common characteristic to all the new economic geography contributions, independently of the adopted space structure, with the notable exception of Brito (2003). This strong assumption is done to ease the resolution of the models, which are yet very complex with the addition of the space dimension.

Nonetheless, as capital accumulation is not allowed, the new economic geography models are losing a relevant determinant of migrations, and more importantly, an engine of growth. While a large part of growth theory is essentially based on capital accumulation, the new economic geography has omitted this fundamental dimension so far. It seems however clear that many economic geography problems (e.g., uneven regional development) have a preeminent growth component, and vice versa. Thus, there is an urgent need to unify in some way the two disciplines, or at least to develop some junction models.

This paper follows exactly this line of research. We study the Ramsey model with space. Space is continuous and infinite, and optimal consumption and capital accumulation are space dependent. We study the model with one dimensional space, in which we relocated the geography location (note as \( x \in \mathbb{R} \)), such that, the rich regions are not far away from the central of the economy, that is, \( x = 0 \). In another word to say, a region, which is far away from \( x = 0 \), is a very poor region, where there is no capital flow. Furthermore, as mentioned by Ten Raa (1986), page 528–530, “to avoid simple but unrealistic boundary conditions”, we consider that space is continuous and infinite, and capital accumulation are space dependent. From macro-economic general equilibrium point of view one dimension for space is reasonable. Different from Ten Raa (1986) and Puu (1982), where they consider a fluid dynamics, hence obtained wave equations of income, in our case capital flowing follows heat equations because of decreasing return to capital: the regions with the largest capital endowments are those which display the lowest return, hence allowing capital to flow to less capital endowments regions.

In line with Mossay (2003), we shall allow for both dispersion and convergence forces. The convergence force is the well known neoclassical mechanism according to which the regions with less capital attract capital because of decreasing returns to this factor. The dispersion force comes from a specification of our objective function, which induces a kind
of preference for the center of space. Capital is assumed to be perfectly mobile across space. We abstract away from migrations, and consider an optimal control problem in which a benevolent planner has to find the optimal allocation of capital across time and space in order to maximize the utility from consumption at any time and place. This is far from a trivial optimal control problem as we shall see in the paper. 

The main difficulty comes from the fact that the state variable, capital, is shown to be governed by a parabolic partial differential equation. Establishing the Pontryagin conditions in such a case with infinite time and infinite space is not an easy task, as noted Brito (2003). Proving the existence of a solution is infinitely much harder, and we shall resort to the most recent advances in the related mathematical discipline to tackle it (among them, Raymond and Zidani (1998 and 2000), Lenhart and Yong, 1992). This will be our fundamental task.\footnote{This key issue is omitted by Brito (2003).} Incidentally, we shall also address the recurrent question in economic geography: assuming that the optimal control problem has a solution (which will be proved), do optimal capital flows assure space convergence in terms of capital? This would be a spatial version of the conditional convergence property inherent in the neoclassical growth model.

The paper is organized as follows. Section 2 presents the optimal control problem. Section 3 extracts the associated Pontryagin conditions and prove their sufficiency. Section 4 analyzes the stationary solutions associated with the latter conditions. Section 5 and 6 are devoted to prove the existence of solutions to the optimal control problem. Section 5 studies the finite time horizon counterpart of the original optimal control problem. Section 6 brings out the main existence result and the asymptotic properties of the solutions. Section 7 concludes.
2 The model

We consider the following central planner problem

$$\max_c \int_0^\infty \int_\mathbb{R} U(c(x,t), x) e^{-\rho t} dt \, dx,$$  \hspace{1cm} (2.1),

where $c(x,t)$ is the consumption level of a representative household located at $x$ at time $t$, $x \in \mathbb{R}$ and $t \geq 0$, $U(c(x,t), x)$ is the instantaneous utility function and $\rho > 0$ stands for the time discounting rate. For a given location $x$, the utility function is standard, i.e. $\frac{\partial U}{\partial c} > 0$, $\frac{\partial^2 U}{\partial c^2} < 0$, and checking the Inada conditions. Our specification of the objective function can be interpreted in two ways. First, the preferences depend on the location of the household, which is by no way inconsistent with the geography literature which typically report different attitudes towards consumption as we move from a region to another. Another plausible interpretation of the specification is the following. Suppose that $U(c, x)$ is separable, $U(c, x) = V(c) \psi(x)$, with $V(.)$ a strictly increasing and concave function, and $\psi(x)$ an integrable and strictly positive function such that $\int_\mathbb{R} \psi(x) = 1$. In such case, the presence of $x$ via $\psi(x)$ in the integrand of the objective function stands for the weight assigned to location $x$ by the central planner in a world of homogenous individual preferences. Again, this assumption is most acceptable if one has in mind that in many cases the governments’ concerns and actions are not uniform in space. For example, if the government is concerned with uneven regional development, she should assign more weight to the poor regions.

Further assumptions on the shape of preferences with respect to $x$ will be done along the way. We now turn to describe how capital flows from a location to another. Hereafter we denote by $k(x,t)$ the capital stock held by the representative household located at $x$ at date $t$.

2.1 The state equation

In contrast to the standard Ramsey model, the law of motion of capital does not rely entirely on the saving capacity of the economy under consideration: The net flows of
capital to a given location or space interval should also be accounted for. Suppose that
the technology at work in location \( x \) is simply \( y(x, t) = A(x, t)f(k(x, t)) \), where \( A(x, t) \)
stands for total factor productivity at location \( x \) and date \( t \), and \( f(\cdot) \) is the standard
neoclassical production function, which satisfies the following assumptions:

\((\text{A1})\) \( f(\cdot) \) is non-negative, increasing and concave;

\((\text{A2})\) \( f(\cdot) \) verifies the Inada conditions, that is,
\[
\begin{align*}
  f(0) &= 0, \quad \lim_{k \to 0} f'(k) = +\infty, \quad \lim_{k \to +\infty} f'(k) = 0.
\end{align*}
\]

Moreover we assume that the production function is the same whatever is the location. \( A(x, t) \) could be another heterogeneity factor. However, we will assume it is time inde-
dependent in the crucial parts of this paper, and hence, this heterogeneity could be omitted
from now on. The budget constraint of household \( x \in \mathbb{R} \) is
\[
\frac{\partial k(x, t)}{\partial t} = A(x, t)f(k(x, t)) - c(x, t) + \tau(x, t),
\]
where \( c(\geq 0) \) is the consumption and \( \tau \) is the household’s trade balance, which could be
positive or negative at any location and time. Since the economy is closed:
\[
\int_{\mathbb{R}} \left( \frac{\partial k(x, t)}{\partial t} - A(x, t)f(k(x, t)) + c(x, t) - \tau(x, t) \right) dx = 0.
\]
And if regions are considered as closed economies, then for any given region \( R \subset \mathbb{R} \):
\[
\int_{R} \left( \frac{\partial k(x, t)}{\partial t} - A(x, t)f(k(x, t)) + c(x, t) - \tau(x, t) \right) dx = 0.
\]
Capital flows searching regions with high marginal productivity. So that capital move-
ments tend to eliminate geographical differences. Since without inter-regional arbitrage
opportunities, the capital flows from regions with lower marginal productivity of capital to
the higher ones, that is equivalent to saying that capital flows from regions with abundant
capital toward the ones with relatively less capital. Suppose furthermore that there is no
institution barriers to capital flow (or do not consider the adjustment speed)\(^2\). Applying the fundamental theorem of calculus to region \(X = [a, b]\), we have
\[
\int_X \tau(x, t) dx = - \int_{\partial X} \frac{\partial k}{\partial x} dx = - \left( \frac{\partial k(b, t)}{\partial x} - \frac{\partial k(a, t)}{\partial x} \right) = - \int_X \frac{\partial^2 k}{\partial x^2} dx.
\]
Substitute the above equation into state equation, we have \(\forall X \subset \mathbb{R}, \forall t\)
\[
\int_X \left( \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} - (s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t)) \right) dx = 0.
\]
By Hahn-Banach Theorem, therefore the budget constraint can be written as:
\[
\frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} = s(x, t)A(x, t)f(k(x, t)) - \delta k(x, t), \forall (x, t).
\]
Moreover in the following, we always assume that the following assumption (A3)\(^3\) holds:
\[
(A3) \int_{\mathbb{R}} \frac{e^{-\frac{(y-x)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} [A(y, \tau)f(k(y, \tau)) - c(y, \tau)] dy d\tau \geq 0, \ x \in \mathbb{R}, \ t > 0.
\]
The initial distribution of capital, \(k(x, 0) \in C(\mathbb{R})\), is assumed to be known positive bounded function, that is, \(0 < k(x, 0) \leq K_0 < \infty\). Moreover, we assume that, if the\(^2\)

\[^2\]We could assume that there is institution barrier (or adjustment speed) to capital flow basing on location and time (see Ten Raa (1986) and Puu (1982)). But if we assume they are independent of capital \(k\) and consumption \(c\), then we can obtain a linear equation with coefficients in front of the Laplacean operator. After some affine transformation, we will get the similar result as below. But if the barriers (or adjustment speed) are functions of \(k\) and/or \(c\), we are facing nonlinear results, which we do not consider in this work.

\[^3\]From economy point of view, in term of investment \(i(x, t) = A(x, t)f(k) - c(x, t)\), it could be positive or negative for any location \(x\), at any time \(t(> 0)\). But we assume that the “average accumulated” investment in one economy at any time is nonnegative. Here “average” means in the sense of mathematical expectation with density function normal distributed. In deed, at location \(x\) and at time \(t\), in density function, we take \(x\) as mean and \(\sqrt{t-\tau}\) as volatility, where \(0 < \tau < t\). In another word, volatility is “paths independent”, which only depends on the “time to considering” (in the word of stochastic finance). As a result, we take density function as
\[
p(y, \tau) = \Gamma(x - y, t - \tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(y-x)^2}{4(t-\tau)}}, \ y \in \mathbb{R}, 0 < \tau < t.
\]
location is far away from the origin, there is no capital flow, that is

\[
\lim_{x \to \pm \infty} \frac{\partial k}{\partial x} = 0.
\]

### 2.2 The optimal control problem

We can now write our optimal control problem:

\[
\max_c \int_0^\infty \int_\mathbb{R} U((c(x,t), x)) e^{-\rho t} dx dt.
\]

subject to:

\[
\begin{aligned}
\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} &= A(x,t) f(k(x,t)) - c(x,t), \quad (x,t) \in \mathbb{R} \times [0, \infty), \\
k(x,0) &= k_0(x) > 0, \quad x \in \mathbb{R}, \\
\lim_{x \to \pm \infty} \frac{\partial k}{\partial x} &= 0.
\end{aligned}
\]

Here comes the definition of an optimal solution:

**Definition 1** A trajectory \((c(x,t), k(x,t))\), both functions in \(C^{2,1}(\mathbb{R} \times [0, \infty))\), is admissible if \(k(x,t)\) is a solution of the equation (2) with control \(c(x,t)\) on \(t \geq 0, \ x \in \mathbb{R}\), and if the integral objective function (1) converges. A trajectory \((c^*(x,t), k^*(x,t))\), \(t \geq 0, \ x \in \mathbb{R}\) is an optimal solution of problem (1) and (2) if it is admissible and it it is optimal in the set of admissible trajectories, ie. for any admissible trajectory \((c(x,t), k(x,t))\), the value of the integral (1) is not greater than its value corresponding to \((c^*(x,t), k^*(x,t))\).

It is not very hard to see that the shape of preferences is crucial for the convergence of the integral (1) when space is unbounded. In effect, suppose that the preferences are homogenous or that the central planner assigns the same weight to all locations, ie. the utility level at \(x\) only depends on the consumption level at \(x\), then the integral is clearly divergent for a uniformly distributed consumption. Brito (2003) also noticed this fact, and to get rid of it, he considered a different objective function, namely average utility
function in space instead of our Benthamian type functional. We prefer to take another approach, and notably to maintain the Benthamian functional as the natural extension of the original Ramsey model. We could have simplified our treatment by having space bounded but we finally prefer to address the pure case of infinite space and infinite time.

But considering that space is infinite just like time imposes a kind of symmetric handling of both to get admissible solutions. In particular, just like time discounting is needed to ensure the convergence of the integral objective function in the standard Ramsey model, we need a kind of space discounting. Hence, a natural choice of \( U(c, x) \) is to take it rapidly decreasing with respect to the second variable. That is, \( U(c, x) \), for any fixed \( c \), defined as,

\[
\{ U(c, \cdot) \in C(\mathbb{R}) | \forall m \in \mathbb{Z}_+, |x^m U(c, x)| \leq M_m, \forall x \in \mathbb{R}, M > 0 \}.
\]

A possible choice of \( U(c, x) \) checking the above mentioned characteristic is \( U(c, x) = V(c) \frac{\rho'}{2} e^{-\rho|x|} \), where \( V(.) \) is strictly increasing and concave in \( c \), and \( \rho' > 0 \). Whatever is the interpretation, heterogenous individual preferences or non-uniform weighting of homogenous preferences by the central planner, the specification above depicts a kind of preference for the center of the space. The idea of spatial discounting is not new in the economic geography literature. Fujita and Thisse (2002), chapter 7, present some firms problems with infinite space location and spatial discounting. Applied to our set-up, this idea amounts to assume that there is a single shopping center situated at \( x = 0 \) from where the consumption good can be bought. Travelling to \( x = 0 \) implies a loss in utility, and this loss can take the exponential form given above. Indeed, Fujita and Thisse refer to this property by spatially discounted accessibility.

To close this section, we notice that the other part of the definition of an optimal solution, ie. the existence of solutions to the state equation for given control and under the specified boundary constraints is no problem thanks to C.V. Pao(1992). As mentioned by Pao, page 296, that to ensure the uniqueness result we require for some growth condition when \( x \rightarrow \pm \infty \). We name as the following:
(A4) For any given finite $T$, if $(x, t) \in \mathbb{R} \times (0, T]$, there are some constants $A_0 > 0, C_0 > 0, K_0 > 0$ and $b < \frac{1}{4T}$, such that, as $x \to \pm\infty$

$$0 < A(x, t) \leq A_0 e^{b|x|^2}, \quad 0 < c(x, t) \leq C_0 e^{b|x|^2}, \quad 0 < k_0(x) \leq K_0 e^{b|x|^2}.$$  

(A4’) For $(x, t) \in \mathbb{R} \times (0, \infty)$, there are some positive constants $A_1 > 0, C_1 > 0$, such that, when $x \to \pm\infty$ and/or $t \to \infty$

$$0 < A(x, t) \leq A_1, \quad 0 < c(x, t) \leq C_1, \quad 0 < k_0(x) \leq K_0 e^{b|x|^2}.$$  

Theorem 1 Consider state equation (2.2), let assumption (A1), (A2) hold and $A, c \in C^{2,1}(\mathbb{R} \times (0, \infty))$.

(1) Suppose (A4) holds, then problem (2.2) has a unique solution $k \in C^{2,1}(\mathbb{R} \times (0, T])$, given by

$$k(x, t) = \int_{\mathbb{R}} \Gamma(x - y, t) k_0(y) dy$$

$$+ \int_0^t \int_{\mathbb{R}} \Gamma(x - y, t - \tau) [A(y, \tau) f(k(y, \tau)) - c(y, \tau)] dy d\tau.$$  

Moreover

$$|k| \leq Ke^{b'|x|^2}, \text{ as } x \to \infty,$$

where $K$ is a positive constant, which depends only on $A_0$, $K_0$, $C_0$, $T$, $b' \leq \min\{b, \frac{1}{4T}\}$ and

$$\Gamma(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{x^2}{4t}}, & t > 0, \\ 0, & t < 0. \end{cases}$$  

(2) Suppose that (A4’) holds, then problem (2.2) has a unique global solution $k \in C^{2,1}(\mathbb{R} \times (0, \infty))$, for any $(x, t) \in \mathbb{R} \times (0, \infty)$, the solution is given by (2.3).
(3) If (A3) holds also, then the above solutions in (1) and (2) are nonnegative. Furthermore, for any $x \in \mathbb{R}, t > 0$

$$k(x, t) \geq \inf_{x \in \mathbb{R}} k_0(x) > 0.$$ 

**Remark 1.** (2.3) is the result of “trade”, that is, at time $t$, at any location $x$, capital $k(x, t)$ not only depends on the history of this location, but also benefit from the whole economy and the history of this economy.

**Remark 2.** Assumption (A3) ensures us a positive capital accumulation process. In one economy, it is possible to invest (or consume) more than output at some locations, but this kind of investment (or consumption) should be not large enough to destroy the total economy. In another word, the probability of infinite investment (or consumption) is zero. More than that, this condition also promise the consumption equation to be well posed (this will be clear at the end of next section) and the convergence of the maximization problem (we will show this in section 5).

**Proof of Theorem 1.** (1) Let $\{x, t \} \in \mathbb{R} \times (0, T]$. Define a sequence $\{k^{(n)}\}, (n \geq 1)$ successively from the iteration process

$$\begin{cases}
\mathcal{L}k^{(n)} = k_t^{(n)} - k_{xx}^{(n)} = F(t, x, k^{(n-1)}) = A(x, t)f(k^{(n-1)}) - c(x, t), \text{ in } \mathbb{R} \times (0, T], \\
k^{(n)}(x, 0) = k_0(x), \text{ in } \mathbb{R},
\end{cases}
$$

with $k^{(0)}(x, t) = k_0(x)$. Then this sequence is well defined, if assumption (A1), (A4) hold. Due to Theorem 7.1.1 of Pao(1992), a unique solution sequence $\{k^{(n)}\} \in C^{2,1}(\mathbb{R} \times (0, T])$ exists and is given by

$$k^{(n)}(x, t) = \int_{\mathbb{R}} \Gamma(x - y, t) k_0(y) dy$$

$$+ \int_0^t \int_{\mathbb{R}} \Gamma(x - y, t - \tau) \left[ A(y, \tau) f(k^{(n-1)}(y, \tau)) - c(y, \tau) \right] dy d\tau,$$

where $\Gamma(x, t)$ is the fundamental solution to parabolic operator $\mathcal{L}$. It is well known that this fundamental solution takes form as

$$\Gamma(x, t) = \begin{cases}
\frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, & t > 0, \\
0, & t < 0.
\end{cases}$$

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(see for example, page 261–265, Ladyzenskaja et al.(1968) or page 14, Frideman (1983)). Furthermore, from Theorem 7.1.1, Pao(1992), for each \( n \), the solution satisfy the growth condition

\[ |k^{(n)}| \leq K'e^{b|x|^2}, \text{ as } x \to \pm \infty, \]

for some positive constant \( K' \). Notice that, the sequence staring from \( k_0 \), and then \( K' \) does not depend on \( n \). Hence, we obtain that for \( t \in (0, T] \), for any \( x \), there is estimate

\[ |k^{(n)}| \leq K''e^{b'|x|^2}, \]

for some positive constant \( K'' \).

Then there is a subsequence, denoted as \( k^{(n_j)} \), which converges to a function \( \tilde{k} \in C^{2,1}(\mathbb{R} \times (0, T]) \), and satisfies

\[ |\tilde{k}| \leq Me^{b|x|^2}, \quad x \in \mathbb{R}, \]

with some positive constant \( M \).

Due to the uniqueness solution of the linear equation (see Theorem 7.1.1, Pao(1992)), it is not difficult to prove that in fact the whole sequence converges to \( \tilde{k} \). In (2.5), taking limit for \( n \to \infty \) on both sides, we have that

\[
\tilde{k}(x, t) = \int_{\mathbb{R}} \Gamma(x - y, t)k_0(y)dy \\
+ \int_0^t \int_{\mathbb{R}} \Gamma(x - y, t - \tau) \left[ A(y, \tau)f(\tilde{k}(y, \tau)) - c(y, \tau) \right] dyd\tau.
\]

By the well known fundamental solution result, \( \tilde{k} \) is the solution of problem (2.2) for \((x, t) \in \mathbb{R} \times (0, T]\). Denote this above solution as \( k \), then \( k \) satisfies the growth condition

\[ 0 < k \leq Ke^{b|x|^2}, \text{ as } x \to \pm \infty. \]

(2) Notice that for generally used economy models with Cobb-Douglas or CES production functions, and iso-elastic utility functions, the solution can not blow up in finite time. See equation (1.9), Galaktionov and Vazquez (1995), page 228. Therefore the study of global solutions is allowed. That is, there is no \( T* < \infty \), such that,

\[
\lim_{t \to T^*} k(x, t) = \infty.
\]

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Hence the solution we obtain in (1) is global for $t$.

Finally, (3) is a direct result of (2.3), because of (A3). Q.E.D.

3 Necessary and sufficient conditions for the finite horizon optimization problem

The introduction of a finite horizon $T$ allows us to convert the system resulting from the Pontryagin condition into the form of a Cauchy problem. The objective function is given by

$$
\int_0^T \int_\mathbb{R} U(c(x,t),x)e^{-\rho t} dt dx + \int_\mathbb{R} \phi(k(x,T),x)e^{-\rho T} dx,
$$

where function $\phi(.)$ is taken continuously differentiable, strictly increasing with respect to its first argument, and for example rapidly decreasing with respect to its second argument to assure the convergence of the second integral term. We also require that the second integral term vanishes as $T$ tends to infinity so that the finite horizon problem above “converges” to a infinite problem.

Suppose that $(c^*, k^*)$ is the optimal solution of (2.1)-(2.2) and consider the perturbed path $(c, k)$, such that, for any arbitrary function $p$, $h$, and $\epsilon > 0$,

$$
p, h : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R},
$$

and

$$
c = c^* + \epsilon h,
$$

$$
k = k^* + \epsilon p.
$$

The corresponding value function $V$ is defined as

$$
V = \int_0^T \int_\mathbb{R} U(c(x,t),x)e^{-\rho t} dt dx + \int_\mathbb{R} \phi(k(x,T),x)e^{-\rho T} dx - \int_0^T \int_\mathbb{R} q(x,t) \left( \frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k}{\partial x^2}(x,t) - A(x,t)f(k(x,t)) + c(x,t) \right) dx dt.
$$
As usual, to achieve the maximum of $V$, the first order condition $\frac{\partial V}{\partial \epsilon} = 0$ should hold. Let us study first:

$$\int_0^T \int_{\mathbb{R}} q(x, t) \frac{\partial k(x, t)}{\partial t} dx dt,$$

and

$$\int_0^T \int_{\mathbb{R}} q(x, t) \frac{\partial^2 k(x, t)}{\partial x^2} dx dt.$$

Integration by parts yields

$$\int_0^T \int_{\mathbb{R}} q(x, t) \frac{\partial k(x, t)}{\partial t} dx dt = \int_{\mathbb{R}} kq|_{0}^{\infty} dx - \int_T^0 \int_{\mathbb{R}} \frac{\partial q}{\partial t} dx dt,$$

and

$$\int_0^T \int_{\mathbb{R}} q(x, t) \frac{\partial^2 k(x, t)}{\partial x^2} dx dt = \int_{\mathbb{R}} kq|_{-\infty}^{\infty} dx + \int_0^T \int_{\mathbb{R}} \frac{\partial^2 q}{\partial x^2} dx dt.$$

As a consequence,

$$\frac{\partial V}{\partial \epsilon} = \int_0^T \int_{\mathbb{R}} (U'_1(c, x)e^{-\rho t} - q) h dx dt - \int_0^T \int_{\mathbb{R}} p(x, t) \left( -\frac{\partial q}{\partial t} - \frac{\partial^2 q}{\partial x^2} \right) dx dt - \int_{\mathbb{R}} p(x, t)q(x, t)|_{-\infty}^{\infty} dx +$$

$$+ \int_0^T q \frac{\partial}{\partial \epsilon} \frac{\partial k}{\partial x}|_{-\infty}^{\infty} dx - \int_0^T p \frac{\partial q}{\partial x}|_{-\infty}^{\infty} + \int_0^T qAf'(k(x, t)) dx dt,$$

where $U'_1$ means derivative with respect to the first variable, $c$. Then, $\frac{\partial V}{\partial \epsilon} = 0$, provided

$$q(x, t) = e^{-\rho t}U'_1(c, x),$$

and for admissible $c(x, t)$ and $k(x, t)$ in $C^{2,1}(\mathbb{R} \times [0, T])$,

$$\begin{cases} \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + q(x, t)Af'(k(x, t)) = 0, \\ q(x, T) = e^{-\rho T}\phi'_1(k(x, T), x), \\ q(x, t) = e^{-\rho t}U'_1(c(x, t), x), \\ \lim_{x \to -\infty} \frac{\partial q}{\partial x} = \lim_{x \to -\infty} \frac{\partial q}{\partial x} = 0, \end{cases}$$
where \( \phi'_1(k(x,T),x) \) stands for the derivative of function \( \phi \) with respect to its first argument. One can make the variable change: \( \tilde{q}(x,t) = e^{\rho t}q(x,t) \), which results in the following transformed condition:

\[
\frac{\partial \tilde{q}}{\partial t} + \frac{\partial^2 \tilde{q}}{\partial x^2} + \tilde{q}(x,t) (Af'(k(x,t)) - \rho) = 0.
\]

**Theorem 2 (Pontryagin Conditions)** Suppose that Assumption (A1)–(A4) hold, \( A \in C^{2,1}(\mathbb{R} \times (0,T)) \) is nonnegative and has finite upper-bound, that is, \( 0 \leq A \leq M_A < \infty \). Suppose that \( c \in C^{2,1}(\mathbb{R} \times (0,T)) \) is an optimal control and \( k \in C^{2,1}(\mathbb{R} \times (0,T)) \) is its corresponding state. Then there exists a function \( q(x,t) \in C^{2,1}(\mathbb{R} \times [0,T]) \) satisfying the following adjoint equation (or co-state equation)

\[
\begin{cases}
\frac{\partial \tilde{q}}{\partial t} + \frac{\partial^2 \tilde{q}}{\partial x^2} + \tilde{q}(x,t) (Af'(k(x,t)) - \rho) = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty), \\
\lim_{t \to \infty} e^{-\rho t} \tilde{q}(x,t) = 0, \forall x, \\
\lim_{x \to \infty} \frac{\partial \tilde{q}}{\partial x} = \lim_{x \to -\infty} \frac{\partial \tilde{q}}{\partial x} = 0.
\end{cases}
\]  

Moreover there exists a unique function \( (U')^{-1}(\tilde{q}(x,t)) \) for any fixed \( t \geq 0 \), such that,

\[
\tilde{q}(x,t) = U'_1((U')^{-1}(\tilde{q}(x,t)),x)
\]

and

\[
c(x,t) = (U'_1)^{-1}(\tilde{q}(x,t),x),
\]

with

\[
\frac{\partial c}{\partial x} \frac{\partial^2 U(c,x)}{\partial c^2} + \frac{\partial^2 U(c,x)}{\partial c \partial x} = \frac{\partial \tilde{q}}{\partial x}.
\]

The proof of existence of solutions to equation (3.1) follows the same type of arguments as those work at part (1) of Theorem 1. See for example Raymond and Zidani (1998) and in press, in addition to the Lions (1971). The final claim comes from the implicit function theorem, using \( \frac{\partial^2 U(c,x)}{\partial c^2} \neq 0 \).

In fact, the above conditions are not only necessary, they are also sufficient (see in another context, Gozzi and Tessitore, (1998)).
Theorem 3 (Necessary and Sufficient Conditions) Provided that (A1)-(A4) hold, $q(x, t) > 0, (x, t) \in (\mathbb{R} \times [0, T])$, and the utility function is strictly increasing with respect to its first argument, then the Pontryagin conditions, obtained in Theorem 2, are also sufficient to the original optimal control problem with finite horizon.

Proof We write the above value function in a different way and separate the state variable $k$ and the optimal control variable $c$. Then we have

$$V = \int_0^T \int_{\mathbb{R}} [U(c(x, t), x)e^{-\rho t} - q(x, t)c(x, t)]dxdt$$

$$- \int_0^T \int_{\mathbb{R}} q(x, t) \left( \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} - A(x, t)f(k(x, t)) \right) dxdt$$

$$= \int_0^T \int_{\mathbb{R}} F(c, q)dxdt - \int_0^T \int_{\mathbb{R}} G(k, q)dxdt,$$

where $q(c, t)$ is the solution of the costate equation mentioned in last proposition and

$$F(c, q) = U(c(x, t), x)e^{-\rho t} - q(x, t)c(x, t),$$

and

$$G(k, q) = q(x, t) \left( \frac{\partial k(x, t)}{\partial t} - \frac{\partial^2 k(x, t)}{\partial x^2} - A(x, t)f(k(x, t)) \right).$$

Hence the control only appears in function $F$. It follows that

$$V \leq \int_0^T \int_{\mathbb{R}} \max_c F(c, q)dx dt - \int_0^T \int_{\mathbb{R}} G(k, q)dxdt.$$

So if for any $(x, t) \in (\mathbb{R} \times [0, T])$, we can find $c^*(x, t) \in C^{2,1}(\mathbb{R} \times [0, T])$, such that,

$$F(c^*, q) = \max_c F(c, q),$$

then the above $c^*$ is optimal. In fact we have that

$$\frac{\partial F(c, q)}{\partial c} = U'_1(c, x)e^{-\rho t} - q(x, t),$$

and

$$\frac{\partial^2 F(c, q)}{\partial c^2} = U''_1(c, x)e^{-\rho t}.$$
By assumption, $U''(c, x) < 0$ and $U'(c, x) > 0$, so function $F(c, q)$ is strictly concave. Hence there is unique maximum point, such that,

$$\frac{\partial F(c, q)}{\partial c} = 0,$$

or equivalently, we have

$$U'(c, x)e^{-\rho t} = q(x, t),$$

if and only if $q(x, t) > 0, (x, t) \in (\mathbb{R} \times [0, T)).$ Q.E.D

We now move to prove the most important result of this contribution, the existence of a solution to (2.1)-(2.2). By Theorem 3, this amounts to proving that the system of partial differential equations (2.2)-(3.1) (plus the associated boundary conditions) has a solution.

We will proceed as follows:

(i) We characterize the stationary solution of the system (2.2)-(3.1).

(ii) We then characterize the solutions of a finite time horizon counterpart of problem (2.2)-(3.1), say $P_T$, including existence and regularity.

(iii) We study the asymptotic behavior of the solutions to $P_T$ as the horizon $T$ increases, and prove that these solutions converge to the stationary solution of (2.1)-(2.2).

(iv)

(i) is done in the next section. (ii) is done in Section 5, and the remaining tasks (iii) and (iv) are undertaken in Section 6.

4 Steady State behavior

In this section, we are going to study the steady state solutions. Mainly, we will consider the stability of the steady state solution. Moreover the convergence to the steady state solution is also very important, which we will consider in the third subsection after investigating some examples in subsection 2.

First of all, let’s show some necessary definitions following the literature.
We define a stationary (or steady state) solution to (2.2) and (3.3) by the standard conditions \( \frac{\partial k(x,t)}{\partial t} = \frac{\partial \tilde{q}(x,t)}{\partial t} = 0. \)

**Definition 2** A steady state solution \( k_s(x) \) of (2.2) is said to be stable if given any constant \( \epsilon > 0 \), there exists a \( \delta > 0 \), such that,

\[ |k(x,t) - k_s(x)| < \epsilon, \quad \forall x \in \mathbb{R}, \text{whenever, } |k_0(x) - k_s(x)| < \delta, \quad \forall x \in \mathbb{R}, \]

where \( k(x,t) \) is the solution of (2.2). If, in addition,

\[ \lim_{t \to \infty} |k(x,t) - k_s(x)| = 0, \quad \forall x \in \mathbb{R}, \]

we say \( k(x,t) \) is asymptotically stable. If the steady state is not stable, then we call it is unstable.

**Definition 3** The set of initial functions \( k_0(x) \), whose corresponding solutions \( k(x,t) \) satisfy the above stability definition is called the stability region of the steady state solution \( k_s(x) \). If this is true for all the initial functions then \( k_s \) is said to be globally asymptotically stable.

The above definitions can be found on page 184, C.V. Pao(1992).

Before further study, noticing the following fact that, due to assumption (A3), (A4) or Remark 2 of Theorem 1, in this section we are going to assume that around steady state,

\[ 0 < m_c < c(x) \leq M_c < \infty, \tag{4.1} \]

where \( m_c, M_c \) are constants, satisfy that \( m_c \) is the minimum level for household to survive, and \( M_c \) comes from the Assumption (A4') and the Remark 2 of Theorem 1.

### 4.1 Aggregate Steady State Condition

Suppose that \( \tilde{q}_s \) and \( k_s \) are steady state solution of

\[ \tilde{q}_{xx} + \tilde{q}(Af'(k) - \rho) = 0, \tag{4.2} \]
and

\[-k_{xx} = A(x)f(k) - c(x), \quad (4.3)\]

respectively, which verify

\[
\lim_{x \to \pm \infty} \frac{\partial \tilde{q}}{\partial x} = 0. \quad (4.4)
\]

and

\[
\lim_{x \to \pm \infty} \frac{\partial k}{\partial x} = 0. \quad (4.5)
\]

In short, we call system (4.2),(4.3), with conditions (4.4),(4.5) as system \( P_s \).

By Theorem 7.11.1(Page 362), C.V. Pao(1992), we have the following results.

**Theorem 4** Suppose that around steady state \( A(x,t) = A(x) \), then at steady state, we have

1. The steady state condition is

\[
\int_{\mathbb{R}} \tilde{q}_s[A(x)f'(k_s(x)) - \rho]dx = 0.
\]

2. If furthermore \( c_s(x) \) check (4.1), then at steady state, we have

\[
\int_{\mathbb{R}} [A(x)f(k_s(x)) - c_s(x)]dx = 0.
\]

**Proof** Statement (1) comes from (4.4). For the last statement integration over \( \mathbb{R} \) of the steady state equation (4.2) and noticing the fact (4.5).

We finish the proof.

**Remark** It is easy to see that \( Af'(k_s) - \rho = 0 \) always checked statement (3) in Theorem 5. Also, \( k_s = f^{-1}(\frac{c}{A}) \) is a special solution to the state equation. In another word, the classical Ramsey model steady state conditions are special cases in our model. In a spatial economy, we name the above Theorem 5 as **aggregate steady state condition**.

As we can see from the aggregate steady state condition, in general, the stationary solutions computed are not promised to be unique (see also Brito(2003)). Instead of solving a parameterized case for which uniqueness conditions can be extracted, as Brito did, we keep on working in the general case in section 4.3. To this end, we need more concepts. The following definition is particularly important.
Definition 4 We say that \((k_*, \tilde{q}_*)\) (resp. \((k^*, \tilde{q}^*)\)) is a lower (resp. upper) solution of problem \((P_S)\), with \(\tilde{q}^* = q^*e^{pt}\), if

\[
\begin{cases}
\frac{\partial^2 k_*(x,t)}{\partial x^2} + Af(k_*(x)) - (U'_1)^{-1}\tilde{q}_*(x) \geq 0, \quad (\text{resp.} \leq 0), \\
\frac{\partial^2 \tilde{q}_*(x,t)}{\partial x^2} + \tilde{q}_*(x)(Af'(k_*(x)) - \rho) \geq 0, \quad (\text{resp.} \leq 0).
\end{cases}
\]

4.2 Special cases and examples study

Before generally study the convergence section, we consider some special cases basing on the aggregate steady state condition. In this subsection, we take output function is Cobb-Douglas form, that is, \(f(k) = k^\alpha(x)\), and utility function is separable, such that, \(U(c(x,t), x) = V(c(x))\psi(x) = \frac{c^{1-\sigma}}{1-\sigma}\psi(x)\), where \(\sigma > 0\) is elasticity of substitution. Noticing that \(\tilde{q}(x,t) = U'_1(c, x) = c^{-\sigma}\psi(x)\), and \(c = \left(\frac{\tilde{q}(x,t)}{\psi(x)}\right)^{-\frac{1}{\sigma}}\),

Then at steady state, \((k, \tilde{q})\) satisfy the following system

\[
\begin{cases}
\tilde{q}_{xx} + \tilde{q}(Af'(k) - \rho) = 0, \\
-k_{xx} = A(x)f(k) - \left(\frac{\tilde{q}(x,t)}{\psi(x)}\right)^{-\frac{1}{\sigma}},
\end{cases}
\] (4.6)

Obviously, \(\tilde{q} = 0\) is not a solution of this system due to assumption (4.1).

As to the aggregate steady state condition, there are several special cases need to study.

Case A.

\[
\rho = \alpha A(x)k^{\alpha-1}, \quad Ak^\alpha = \left(\frac{\tilde{q}(x,t)}{\psi(x)}\right)^{-\frac{1}{\sigma}} = c(x).
\]

That is,

\[
k_s(x) = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}(A(x))^{\frac{1}{1-\alpha}}, \quad c_s(x) = \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}(A(x))^{\frac{1}{1-\alpha}}.
\]

It is easy to get that, both capital accumulation and consumption depend on the technology. Moreover, at different location, though the capital and consumption may be different basing on different technology level \((A(x))\), the consumption-capital ration is a constant.
\( \frac{c_s(x)}{k_s(x)} = \frac{\rho}{\alpha} \). In another word to say, in this case, the consumption is a ratio of capital wherever the location, see this result also in a simple paper of Camacho and Zou (2004). Furthermore, if technology level is the same in all the location, that is, \( A(x) = \text{constant} \), so do \( k \) and \( c \). As a consequence, \( c_x = 0 \), and

\[
Af'(k_s) = \rho - \frac{\psi''(x)}{\psi(x)}.
\]

Therefore, \( k \) and \( c \) are constants, if and only if \( \rho - \frac{\psi''(x)}{\psi(x)} \) is constant.

Case B.

\[
\rho = \alpha A(x) k^{\alpha - 1}, \quad \int_{\mathbb{R}} [A(x) f(k_s(x)) - c_s(x)] dx = 0.
\]

From the first equation, we have again \( k_s(x) = \left( \frac{\alpha}{\rho} \right)^{\frac{1}{1-\alpha}} (A(x))^{\frac{1}{1-\alpha}} \), and substituting it into the second one, we have

\[
\int_{\mathbb{R}} c_s(x) dx = \left( \frac{\alpha}{\rho} \right)^{\frac{1}{1-\alpha}} \int_{\mathbb{R}} (A(x))^{\frac{1}{1-\alpha}} dx > 0.
\]

Moreover for any technology function \( A(x) \), such that, \( A(x) > 0, \int_{\mathbb{R}} A(x) dx < \infty \), we have that assumption (4.1) is checked. Furthermore,

\[
\left( \frac{\rho}{\alpha} \right)^{\frac{1}{1-\alpha}} \left( \int_{\mathbb{R}} (A(x))^{\frac{1}{1-\alpha}} dx \right)^{-1} \int_{\mathbb{R}} c_s(x) dx = 1. \tag{4.7}
\]

Therefore there are infinite continuous functions \( c_s(x) \) satisfy equation (4.7), for example all the continuous density functions after dividing by the coefficients in (4.7), could take as consumption functions \( c_s(x) \), though the capital is unique at different location.

Case C.

\[
\begin{align*}
A k^\alpha &= c(x) = \left( \frac{\bar{q}_s(x,t)}{\psi(x)} \right)^{-\frac{1}{\sigma}}, \\
\int_{\mathbb{R}} \bar{q}_s [A(x) f'(k_s(x)) - \rho] dx &= 0.
\end{align*} \tag{4.8}
\]

Specially, we follow the example in Camacho and Zou(2004), taking \( A(x) = e^{-ax^2} \) and taking \( \psi(x) = \frac{b}{\sqrt{2\pi}} e^{-(bx)^2/2} \), with \( 0 < a < \infty, 0 < b < \infty \) are undetermined constants.
We claim that there exists $\beta > 0$, such that the capital takes form,

$$ k_s(x) = e^{-\beta x^2}. $$

Direct calculation by putting all the functions above into (4.8), we obtain the following proposition.

**Proposition 1.** If parameters $\rho$ and $\alpha$ satisfy $\alpha < \rho < \sqrt{1 + \alpha}$, and if

$$ \psi(x) = \frac{b}{\sqrt{2\pi}} e^{-(bx)^2/2}, \quad A(x) = e^{-ax^2}, $$

where constants $a$, $b$ such that

$$ \begin{align*}
0 < a &\leq 1 - \alpha, \\
 b^2 &> 2a\sigma + (1 - \alpha - \sigma) \left(1 + \frac{1}{\rho^2 - \alpha^2}\right) > 0, 
\end{align*} $$

Then there exists a $\beta = \frac{1}{2a\sigma} \left(\frac{2(1 - \alpha - \sigma)}{\rho^2 - \alpha^2} + 2a\sigma - b^2\right) > 0$, such that

$$ k_s(x) = e^{-\beta x^2}, \text{ and } c_s(x) = e^{-(a+\alpha\beta)x^2}. $$

But this is not the unique solution to system (4.8). In deed we have the following result.

**Proposition 2.** Taking $A(x)$ and $\psi(x)$ as in the above proposition, suppose that the coefficients satisfy the following condition

$$ \rho > \alpha, \quad a > 0, \quad b > 0, \quad \frac{a}{b^2} < \frac{(1 - \alpha)(1 - \alpha + 2\sigma\alpha)}{2\sigma(\sigma\alpha + (1 - \alpha)(1 - \alpha + 2\sigma\alpha))}. $$

Then there exists $\xi > 0$, such that

$$ k_s(x) = e^{-\xi|x|}, \quad c_s(x) = A(x)k^\alpha = e^{-(ax^2+\alpha\xi|x|)}, $$

where $\xi$ satisfies,

$$ \frac{\rho}{\sqrt{b^2 - 2a\sigma}} e^{y^2} \Phi(y) = \frac{\alpha}{\sqrt{b^2 - 2a\sigma + 2a}} e^{z^2} \Phi(z), $$

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4.3 Convergence to the steady state

Now, let us introduce the following notations:

\[ g_1(x, t) = g_1(k(x, t), \bar{q}(x, T - t)) = A(x, t)f(k(x, t)) - (U'_1)^{-1}(q(x, T - t)), \]
\[ g_2(x, t) = g_2(k(x, t), \bar{q}(x, T - t)) = -\bar{q}(x, T - t) (A f'(k(x, t)) - \rho). \]

The following developments will connect the solutions of \( \mathcal{P}_T \) with the solution of the steady state system \( \mathcal{P}_S \), which will be crucial when we will have to connect the solutions of \( \mathcal{P}_T \) when \( T \) goes to infinity with the solutions of the original infinite time problem.

The following theorem is crucial.

**Theorem 5**  

i) Suppose \((k_1, \bar{q}_1)\) to be a lower (respectively upper) solution of \( \mathcal{P}_S \), then the corresponding solution \((k(t, k_1, \bar{q}_1), \bar{q}(t, k_1, \bar{q}_1))\) of \( \mathcal{P}_T \) is a non-decreasing (respectively increasing) function of \( t \).

ii) Suppose there exist a lower solution \((k_1, \bar{q}_1)\) and an upper solution \((k_2, \bar{q}_2)\) of \( \mathcal{P}_S \). Then the corresponding solutions \((k(t, k_1, \bar{q}_1), \bar{q}(t, k_1, \bar{q}_1))\) and \((k(t, k_2, \bar{q}_2), \bar{q}(t, k_2, \bar{q}_2))\) of \( \mathcal{P}_T \) are defined for any \( t \in [0, T) \) and satisfy

\[ k_1 \leq k(t, k_1, \bar{q}_1) \leq k(t, k_2, \bar{q}_2) \leq k_2, \]
\[ \bar{q}_1 \leq \bar{q}(t, k_1, \bar{q}_1) \leq \bar{q}(t, k_2, \bar{q}_2) \leq \bar{q}_2, \]

where \( t \in [0, T) \).

iii) Under the assumptions of ii) \([k(t, k_1, \bar{q}_1), \bar{q}(t, k_1, \bar{q}_1)], [k(t, k_2, \bar{q}_2), \bar{q}(t, k_2, \bar{q}_2)]\) converge as \( t \to \infty \) to limits \((k_\star, \bar{q}_\star), (k^\star, \bar{q}^\star)\) respectively, which are solutions of \( \mathcal{P}_S \) and satisfy
$k^* < k^*, \tilde{q}^* < \tilde{q}^*$. In addition, $(k^*, \tilde{q}^*), (k^*, \tilde{q}^*)$ are minimal, respectively maximal solutions of $P_S$ in space $K = \{ k \in L^\infty_+, \tilde{q} \in L^\infty_+ | k_1 \leq k \leq k_2, \tilde{q}_1 \leq \tilde{q} \leq \tilde{q}_2 \}$, where

$$L^\infty_+(\mathbb{R}) = \{ u \geq 0 : \sup_{x \in \mathbb{R}} |u(x)| < \infty \}.$$ 

The proof is long and complicated, we report it in details in the appendix. Theorem 5 is very important in that it sheds light on the behaviour of the solutions paths of system $P_T$ both in the short and long run. Property (i) implies that if the Cauchy system is initialized with an upper solution (to $P_S$) then the corresponding solution path is non-increasing, which already provides monotonic solutions to the system and suggests convergence as time increases. Property (ii) is a step further towards establishing convergence: if the Cauchy systems are initialized with an upper or lower solution to $P_S$, then the obtained solutions paths lay between the lower and upper solutions. Property (i) gives monotonicity and Property (ii) gives boundedness. This allows to obtain relatively general convergence results as depicted in the result (iii). Effectively, it is finally shown that the path obtained with an initial condition equal to an upper (resp. lower) solution to $P_S$ will converge as the time horizon increases to a maximal (resp. minimal) solution of the set of stationary solutions.

Therefore, Theorem 5 provides a very promising basis to establish the convergence of the solutions of $P_T$ to the solutions of the original infinite time problem. However, the main result, namely Property (iii) is only established when the initial conditions are either lower or upper solutions to $P_S$. Clearly, a more general convergence result is needed. This is done in the next section.

All results shown in the previous sections are independent of the sign of functions $g_1$ and $g_2$. This may not be the case in this section. We shall prove that the result (iii) of Theorem 5 can be obtained for a much larger class of initial conditions. The main result obtained in this section is Theorem 7. But before, we need some intermediate results.

At first, we need to identify some general conditions under which the lower and upper solutions to $P_S$ exist (recall that Theorem 5 assumes their existence).
Lemma 1 Let us write

\[
(P_S) \equiv \begin{cases} \\
\Delta k + g_1(k, \bar{q}) = 0, \\
\Delta \bar{q} + g_2(k, \bar{q}) = 0,
\end{cases}
\]

for any \(x \in \mathbb{R}\). Then for any \(x \in \mathbb{R}\), and its some neighborhood \(V \subseteq \mathbb{R}\), there exist nontrivial nonnegative upper (or lower) solution series \((k_\eta, q_\eta)_\eta\) of \(P_S\), where \(\eta \in (0, \bar{\eta})\), \(\bar{\eta}(>0)\) is a constant, such that

\[
k_\eta(x) > 0, \quad \bar{q}_\eta(x) > 0,
\]

\(\supp\{k_\eta\} \subseteq V, \supp\{\bar{q}_\eta\} \subseteq V\), and,

\[
|k_\eta|_\infty \to_{\eta \to 0} 0, \quad |\bar{q}_\eta|_\infty \to_{\eta \to 0} 0,
\]

where

\[
\supp\{k_\eta\} = \{x \in \mathbb{R} | k_\eta \neq 0\}.
\]

Moreover, if we define,

\[
M_1(V) = \{(k_\eta, \bar{q}_\eta) | g_1(k_\eta, \bar{q}_\eta) \geq \lambda_0, g_2(k_\eta, \bar{q}_\eta) \geq \lambda_0\},
\]

and

\[
M_2(V) = \{(k_\eta, \bar{q}_\eta) | g_1(k_\eta, \bar{q}_\eta) \leq \lambda_0, g_2(k_\eta, \bar{q}_\eta) \leq \lambda_0\},
\]

where \(\lambda_0\) is one eigenvalue and \(\xi_0\) is the corresponding eigenfunction of the Laplacian in any fixed interval \(B \subseteq \mathbb{R}\), with homogenous Dirichlet conditions, namely,

\[
\Delta \xi_0 + \lambda_0 \xi_0 = 0,
\]

where \(\xi_0 > 0\) in \(B\), \(\xi_0 = 0\) on \(\partial B\) and \(|k|_\infty\) is

\[
|k|_\infty = |k|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |k(x)|.
\]

If \((k_\eta, \bar{q}_\eta)_\eta \subseteq M_1(V)\), then, it can be proved that \((k_\eta, \bar{q}_\eta)_\eta\) is a series of lower solutions of \((P_S)\). If \((k_\eta, \bar{q}_\eta)_\eta \subseteq M_2(V)\), then, it can be proved that \((k_\eta, \bar{q}_\eta)_\eta\) is a series of upper solutions of \((P_S)\).
The proof is in the appendix. The next theorem establishes the existence of a minimal and maximal element of the set of solutions to $P_S$, a property also assumed in Theorem 5.

**Theorem 6** The set $S = \{(k, \tilde{q}) | (k, \tilde{q})$ is a solution of $P_S$ and $k > 0, \tilde{q} > 0\}$ has a nontrivial minimal element $(k_m, \tilde{q}_m)$ and a nontrivial maximal element $(k_M, \tilde{q}_M)$.

**Proof:** We have already established in Section 4 that $P_S$ admits at least one non-trivial solution, $S \neq \emptyset$. Then, it has a minimal element. On the other hand, by Kuratowski-Zorn Lemma, we obtain that it has also a maximal element. Q.E.D

Then the property (iii) of Theorem 5 together with Lemma 1 above can be reformulated as follows. Let us call $\omega_m(x,t) = (k_m(x,t), \tilde{q}_m(x,t))$ the minimal solution of $P_S$, $\omega_M(x,t) = (k_M(x,t), \tilde{q}_M(x,t))$ the maximal solution of $P_S$. Then, $[\omega_*, \omega^*]$ attracts all solutions of $P$ $\omega(., \omega_0) = (k(., k_0, \tilde{q}_0), \tilde{q}(., k_0, \tilde{q}_0)) \in M_1 \cup M_2$, such that $\omega_0 \neq (0,0)$, in the $(L^\infty)^2$ sense.

We are now ready to prove our general existence result.

**Theorem 7** $(k(., k_0, q_0), q(., k_0, q_0)) \in M_1 \cup M_2$ with $k_* < k_0 < k^*$ and $q_* < q_0 < q^*$, converges to a stationary solution as $t \to \infty$.

**Proof:** We only need to prove that $(k(., k_0, q_0), q(., k_0, q_0))$ are non-decreasing or non-increasing. If $(k(., k_0, q_0), q(., k_0, q_0)) \in M_1$, then, as in lemma ??:

\[
\Delta k + g_1(x,k,q) \geq 0, \\
\Delta q + g_2(x,k,q) \geq 0.
\]

This implies that $k_t \leq 0$ and $q_t \leq 0$. That is, the solution is non increasing. Consequently, by Theorem 5, Theorem 6 and Lemma 1, the system converges.

If $(k(., k_0, q_0), q(., k_0, q_0)) \in M_2$, then, as in lemma ??:

\[
\Delta k + g_1(x,k,q) \leq 0, \\
\Delta q + g_2(x,k,q) \leq 0.
\]
This implies that $k_t \geq 0$ and $q_t \geq 0$. That is, the solution is non increasing. The system converges by the same arguments as above. Q.E.D

Our demonstration ends with establishing the link between the solutions to $P_T$ when $T$ increases and the solutions to the original infinite time optimization problem, which is roughly obvious.

**Corollary 1** Any solution of $P_T$, $(k, q)$ converges asymptotically to a solution of the original infinite horizontal system $P_\infty$.

**Proof:** We will prove that $P_T$ converges to $P_\infty$. So that a solution of the former converges to a solution of the later. Assume that $(k, q)$ is a solution of $P_T$ and $(k_\infty, q_\infty)$ is a solution of $P_\infty$. Looking at these programs, the only difference concerns the limit condition on the shadow price of capital. That is, $q(x, T) = e^{-\rho T} \phi'_1(k(x, T), x)$, for $P_T$ and $\lim_{t \to \infty} e^{-\rho t} \phi'_1(k_\infty(x, t), x) = 0$, for $P_\infty$. We already know that a solution of $P_T$, under the hypothesis mentioned in the previous section, converges to a solution of the steady state problem. We only need to verify if this solution verifies:

$$\lim_{t \to \infty} e^{-\rho t} \phi'_1(k(x, t), x) = \lim_{t \to \infty} e^{-\rho t} \phi'_1(k(x), x) = \phi'_1(k(x), x) \lim_{t \to \infty} e^{-\rho t} = 0.$$  

Q.E.D.

Naturally, Theorem 7 does not establish convergence for any initial condition. However, it enlarges the bassin of convergence found out in Theorem 5. Convergence to the steady state is assured if the initial conditions are taken between the minimal and the maximal solution of $P_S$. Establishing a more global convergence and existence result sounds a daunting task by now.

5 Convergence of the original maximization problem

In this section, we would like to show that though our starting point is the finite horizontal problem, the solutions we obtain indeed promise the convergence of the original
maximization problem with infinite horizon. Following central planner problem (2.1), and suppose that \( U(c, x) \) is separable, \( U(c, x) = V(c) \psi(x) \).

We claim that there exists \( M_c > 0 \), such that,

\[
0 < c(x, t) < M_c, \quad \forall x \in \mathbb{R}, \ t > 0.
\]  

(5.1)

Proof of the above claim. Due to (A4), there exists \( X > 0 \), such that, when \( |x| > X \),

\[
0 < t < T < \infty, \quad 0 < c(x, t) \leq C_0 e^{b|x|^2},
\]  

(5.2)

with \( b < \frac{1}{4T} \). Because of Assumption (A4') or (4.1), it is not difficult to see that the only unbounded domain of \( c \) is \( \Omega := (-Z, Z) \times (0, T) \), where \( c(x, t) = C_0 e^{b|x|^2} \), with \( 0 < b < \frac{1}{4T} \).

But in fact that can not happen, otherwise, recalling Theorem 1, we then have

\[
k(x, t) = K e^{b|x|^2},
\]

which, if \( 0 < b < \frac{1}{4T} \), is a contradiction to the fact that,

\[
\lim_{x \to \pm \infty} \frac{\partial k}{\partial x} = 0.
\]

As a result, (5.1) is proved, with \( M_c = \max\{C_0, C_1\} \).

With the fact that in the utility function, \( V(\cdot) \) is strictly increasing and concave, and \( \psi(x) \) is integrable and strictly positive, such that \( \int_{\mathbb{R}} \psi(x) = 1 \), hence the convergence of (2.1) is obvious.

6 Conclusion

In this paper, we have solved a Ramsey model in continuous time and space, with both time and space discounting. We extract the corresponding Pontryagin conditions and prove their sufficiency. Starting with a finite time counterpart of the model, we prove at the same time, the existence of a solution to the optimization problem and the convergence
to a stationary solution of this problem. However, both results are only obtain for a non-zero measure set of initial conditions. Further studies look indeed necessary not only to extend the convergence and existence basin but also to characterize more deeply the structure of the steady state solutions of the spatial Ramsey model, which is crucial as one can infer from our paper. This task is in our agenda.

7 Appendix

A. Proof of Theorem 7, Section 4.3

Here, we follow the demonstration strategy in Bandle, Pozio and Tesei (1987), and de Mottoni, Schiaffino and Tesei (1984). In order to prove the theorem, we need the next results first.

**Definition 5** Space $C^\infty_0(\Omega)$ defines as,

$$C^\infty_0(\Omega) = \{ f \in C^\infty(\Omega) : \lim_{x \to \partial \Omega} D^\alpha f(x) = 0, \forall \alpha \in \mathbb{N} \},$$

where $\Omega$ could be any space in $\mathbb{R}$, and $\partial \Omega$ is the boundary of $\Omega$, if $\Omega = \mathbb{R}$, then $\partial \Omega$ take $\pm \infty$.

**Lemma 2** Let $(k^*, \tilde{q}^*)$ be an upper solution of (P) on $[0, T]$ with initial data $(k^*_0, \tilde{q}^*_0)$ and $(k^*_s, \tilde{q}^*_s)$ a lower solution with initial data $(k^*_0, \bar{q}^*_0)$. Then $\forall \lambda > 0$ and $0 \leq t \leq T$

$$e^{\lambda t} \int_\mathbb{R} (k^*_s(t) - k^*(t))^+ \leq \int_\mathbb{R} (k^*_0 - k^*_0)^+ + \int_0^t \int_\mathbb{R} e^{\lambda s} \left( g_1(k^*_s, \tilde{q}^*_s) - \bar{g}_1(k^*, \tilde{q}^* + \lambda (k^*_s - k^*)) \right),$$

and

$$\int_\mathbb{R} e^{\lambda t} (\tilde{q}^*_s(t) - \tilde{q}^*(t))^+$$

$$\leq \int_\mathbb{R} (\tilde{q}^*_0 - \tilde{q}^*_0)^+ + \int_0^T \int_\mathbb{R} e^{\lambda s} \left( g_2(k^*_s, \tilde{q}^*_s) - \bar{g}_2(k^*, \tilde{q}^* + \lambda (\tilde{q}^*_s - \tilde{q}^*)) \right).$$
**Proof:** Let $\varphi \in C_0^\infty(Q_t)$, $\varphi \leq 0$ in $Q_t$, where $Q_t = \mathbb{R} \times [0, t]$ and let us multiply $P$ by $\varphi$,

$$
\varphi k_t - \varphi \Delta k - g_1(k, \bar{q})\varphi = 0, \\
\varphi \bar{q}_t - \varphi \bar{q}_{xx} - g_2(k, \bar{q})\varphi = 0.
$$

(1) (2)

Integrating (we make the complete development for $k$ being that of $\bar{q}$ identical) over $Q_t$,

$$
\int \int_{Q_t} k_t \varphi = \int \int_{Q_t} k_{xx} \varphi + \int \int_{Q_t} g_1(k, \bar{q})\varphi,
$$

$$
\int_{\mathbb{R}} [k\varphi]_0^t - \int \int_{Q_t} k\varphi_t = \int_{0}^{t} [k\varphi]_{\partial \mathbb{R}} - \int \int_{Q_t} k_x \varphi_x + \int \int_{Q_t} g_1(k, \bar{q})\varphi,
$$

$$
\int_{\mathbb{R}} k(t) \varphi(t) - \int \int_{Q_t} k (\varphi_t + \varphi_{xx}) = \int_{\mathbb{R}} k(x, 0) \varphi_x(x, 0) + \int \int_{Q_t} g_1(k, \bar{q})\varphi.
$$

And so for $\bar{q}$,

$$
\int \int_{Q_t} \bar{q}(t) \varphi(t) - \int \int_{Q_t} \bar{q}(\varphi_t + \varphi_{xx}) = \int_{\mathbb{R}} \bar{q}(x, 0) \varphi_x(x, 0) + \int \int_{Q_t} g_2(k, \bar{q})\varphi.
$$

These two equalities are verified with sign ($\leq$) by a lower solution and with sign ($\geq$) by an upper solution. This means that

$$
\int_{\mathbb{R}} (k_s - k^*) \varphi - \int \int_{Q_t} (k_s - k^*) (\varphi_t + \varphi_{xx}) \leq \int_{\mathbb{R}} (k_s(x, 0) - k^*(x, 0)) \varphi(x, 0) + \int \int_{Q_t} (\varphi_1 - \varphi_1)\varphi,
$$

where $\varphi_1 = g_1(k_s, \bar{q}_s)$ and $\varphi_1 = g_1(k^*, q^*)$.

Then, if we denote $\varphi_2 = g_2(k_s, \bar{q}_s)$ and $\varphi_2 = g_2(k^*, \bar{q}^*)$, we obtain,

$$
\int_{\mathbb{R}} (k_s - k^*) \varphi - \int \int_{Q_t} (k_s - k_s) (\varphi_t + \varphi_{xx}) \leq \int_{\mathbb{R}} (k_s(x, 0) - k^*_0) \varphi(x, 0) + \int \int_{Q_t} (\varphi_1 - \varphi_1)\varphi, \quad (3)
$$

$$
\int_{\mathbb{R}} (\bar{q}_s - \bar{q}^*) \varphi - \int \int_{Q_t} (\bar{q}_s - \bar{q}_s) (\varphi_t + \varphi_{xx}) \leq \int_{\mathbb{R}} (\bar{q}_s(x, 0) - \bar{q}^*_0) \varphi(x, 0) + \int \int_{Q_t} (\varphi_2 - \varphi_2)\varphi. \quad (4)
$$

To go on with the proof, the following lemma is required.

**Lemma 3** Let $\varphi_n \in C_0^\infty$ be a sequence of test functions such that:

$$
\varphi_{n,t} + \varphi_{n,xx} = \lambda \varphi_n, \\
\varphi_n(x, T) = \chi(x), \\
\varphi_n(x, t) \leq 0, \quad \forall t \in [0, T], \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \mathbb{R}. \quad (I.e. \ a \ \text{backward problem is proposed for} \ \varphi_n). \quad \text{Then},
$$
i) $0 \leq \varphi_n \leq e^{\lambda(t-T)}$, in $\overline{Q}_T$;

ii) $\int_{Q_T}(\varphi_n,xx)^2 < c$;

iii) $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (\varphi_n,x)^2(t) < c$.

**Proof:** See for example Ladyzhenskaja, Solonnikov and Ural’ceva (1968) or Fridman (1983).

Back to the proof of Lemma (??), if we set $t = T$ and $\varphi = \varphi_n$ in (??),

$$
\int_{\mathbb{R}} (k_*(T) - k^*(T)) \chi - \int_{Q_T} (k_* - k^*)(\varphi_n,t + \varphi_n,xx) \leq \int_{\mathbb{R}} (k_*(x,0) - k^*(x,0))\varphi_n(x,0)
+ \int_{Q_T} (\varphi_n - \varphi_1)\varphi_n,
$$

$$
\int_{\mathbb{R}} (\tilde{q}_*(T) - \tilde{q}^*(T)) \chi - \int_{Q_T} (\tilde{q}_* - \tilde{q}^*)(\varphi_n,t + \varphi_n,xx) \leq \int_{\mathbb{R}} (\tilde{q}_*(x,0) - \tilde{q}^*(x,0))\varphi_n(x,0)
+ \int_{Q_T} (\varphi_n - \varphi_n)\varphi_n.
$$

Then

$$
\int_{\mathbb{R}} (k_*(T) - k^*(T)) \chi \leq \int_{\mathbb{R}} (k_*(x,0) - k^*(x,0))\varphi_n(x,0) + \int_{\mathbb{R}} (\varphi_n - \varphi_1 + \lambda(k_* - k^*))\varphi_n
\leq \int_{\mathbb{R}} (k_*(x,0) - k^*(x,0))^+ e^{-\lambda T} + \int \int_{Q_T} (\varphi_n - \varphi_1 + \lambda(k_* - k^*))e^{\lambda(s-T)}.
$$

Taking

$$
\chi(x) = \begin{cases} 
1, & \text{if } (k_*(x,T) - k^*(x,T)) \geq 0, \\
0, & \text{if } (k_*(x,T) - k^*(x,T)) < 0.
\end{cases}
$$

Then we have,

$$
\int_{\mathbb{R}} (k_*(T) - k^*(T))^+ \leq \int_{\mathbb{R}} (k_*(x,0) - k^*(x,0))^+ e^{-\lambda T} + \int \int_{Q_T} (\varphi_n - \varphi_1 + \lambda(k_* - k^*))e^{\lambda(s-T)},
$$

and

$$
\int_{\mathbb{R}} (\tilde{q}_*(T) - \tilde{q}^*(T))^+ \leq \int_{\mathbb{R}} (\tilde{q}_*(x,0) - \tilde{q}^*(x,0))^+ e^{-\lambda T} + \int \int_{Q_T} (\varphi_n - \varphi_2 + \lambda(\tilde{q}_* - \tilde{q}^*))e^{\lambda(s-T)}.
$$
Theorem 8  i) Let \((k, \tilde{q})\) and \((\hat{k}, \hat{\tilde{q}})\) be solutions of \((P_S)\) on \([0, T]\) with initial conditions \((k_0, \tilde{q}_0), (\hat{k}_0, \hat{\tilde{q}}_0)\) respectively. Then
\[
\|k(t) - \hat{k}(t)\|_{L^1(\mathbb{R})} \leq e^{K t} \|k_0 - \hat{k}_0\|_{L^1(\mathbb{R})}
\]
\[
\|\tilde{q}(t) - \hat{\tilde{q}}(t)\|_{L^1(\mathbb{R})} \leq e^{K t} \|\tilde{q}_0 - \hat{\tilde{q}}_0\|_{L^1(\mathbb{R})}
\]
where \(K\) is defined afterwards and function space \(L^1(\mathbb{R})\) is defined as follows, we say \(f \in L^1(\mathbb{R})\), if
\[
\int_{\mathbb{R}} |f(x)| dx < \infty,
\]
and the norm in \(L^1(\mathbb{R})\) is defined as
\[
||f||_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| dx.
\]

ii) Let \((k_*, \tilde{q}_*)\) be a lower solution of \((P_T)\) and \((k^*, \tilde{q}^*)\) be an upper solution, with initial data \((k_{*0}, \tilde{q}_{*0})\) and \((k^*_0, \tilde{q}^*_0)\) respectively. Then, if \(k_{*0} \leq k^*_0\) and \(\tilde{q}_{*0} \leq \tilde{q}^*_0\), it follows that,
\[
k_* \leq k^*,
\]
\[
\tilde{q}_* \leq \tilde{q}^*.
\]

Proof: Let us use the results in Lemma (??), setting \(\lambda = K\). Since \(g_1\) and \(g_2\) are derivative continuous, then there exists \(K_1 > 0\), such that,
\[
|g_1(k_*, \tilde{q}_*) - g_1(k^*, \tilde{q}^*)| < K_1 |k_* - k^*|.
\]

Similarly for \(g_2\). Defining \(h = (h_1, h_2)\) as:
\[
h_1(t) = e^{k_1 t} \int_{\mathbb{R}} (k_* - k^*)^+
\]
and
\[
h_2(t) = e^{k_2 t} \int_{\mathbb{R}} (\tilde{q}_* - \tilde{q}^*)^+.
\]

And by Lemma (??):
\[
h_1(t) \leq h_1(0) + 2K_1 \int_{\mathbb{R}} h_1(s) ds,
\]
\[ h_2(t) \leq h_2(0) + 2K_2 \int_{\mathbb{R}} h_2(s)ds. \]

By Gronwall’s lemma, we have,
\[ h_1(t) \leq h_1(0)e^{2K_1t}, \]
\[ h_2(t) \leq h_2(0)e^{2K_2t}, \]

or,
\[ \int_{\mathbb{R}} (k^* - k)^+ \leq e^{K_1t} \int_{\mathbb{R}} (k^*0 - k^*_0)^+ \]
\[ \int_{\mathbb{R}} (\bar{q}^* - \bar{q})^+ \leq e^{K_2t} \int_{\mathbb{R}} (\bar{q}^*_0 - \bar{q}^*_0)^+. \]

Taking \( K = \max\{K_1, K_2\} \), ii) obtains. i) follows by writing \((k^* - k)^+\).

**Lemma 4** Let \((k^*_s, \bar{q}^*_s)\) and \((k^*, \bar{q}^*)\) be a lower and upper solution respectively of problem \((P_S)\), such that \(0 \leq k_s \leq k^*\) and \(0 \leq \bar{q}_s \leq \bar{q}^*\). Let \((k_0, \bar{q}_0)\) verify \(k_s \leq k_0 \leq k^*\) and \(\bar{q}_s \leq \bar{q}_0 \leq \bar{q}^*\). Then, the solution \((k(t, k_0, \bar{q}_0), \bar{q}(t, k_0, \bar{q}_0))\) of \((P_T)\) is defined for any \(t \geq 0\) and satisfies:
\[
\begin{align*}
&k_s \leq k(t, k_0, \bar{q}_0) \leq k^*, \\
&\bar{q}_s \leq \bar{q}(t, k_0, \bar{q}_0) \leq \bar{q}^*,
\end{align*}
\]
with \(t \geq 0\).

**Proof:** It follows from the proof of the previous theorem, which stated that under the theorem’s assumptions, we have
\[
\begin{align*}
&\int_{\mathbb{R}} (k_s - k^*)^+ \leq e^{K_1t} \int_{\mathbb{R}} (k^*_0 - k^*_0)^+, \\
&\int_{\mathbb{R}} (\bar{q}_s - \bar{q}^*)^+ \leq e^{K_2t} \int_{\mathbb{R}} (\bar{q}^*_0 - \bar{q}^*_0)^+.
\end{align*}
\]

Since \((k, \bar{q})\) is a solution of \((P_S)\), it is a lower and an upper solution. Then if we take \((k, \bar{q})\) as upper solution and by Theorem (??):
\[
\begin{align*}
&k_s \leq k_0 \Rightarrow (k_s(0) - k_0)^+ = 0 \Rightarrow k_s(t, k_0, \bar{q}_0) \leq k(t, k_0, \bar{q}_0), \\
&\bar{q}_s \leq \bar{q}_0 \Rightarrow (\bar{q}_s(0) - \bar{q}_0)^+ = 0 \Rightarrow \bar{q}_s(t, k_0, \bar{q}_0) \leq \bar{q}(t, k_0, \bar{q}_0).
\end{align*}
\]

The other side of the inequality is obtained considering \((k, \bar{q})\) as a lower solution.
Lemma 5  Let \((k_*, \tilde{q}_*)\) be a lower solution of \((P_S)\), where \(k_*, \tilde{q}_* \in L^\infty_+(\mathbb{R})\). Then the corresponding solution of \((P_T)\), \([k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*)]\) is nondecreasing with respect to \(t\). (In the case of an upper solution, the result holds with non-increasing).

Proof: Let \((k_*, \tilde{q}_*)\) be a lower solution of \((P_S)\), and \([k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*)]\) the unique solution of \((P_T)\) with initial condition \((k_*, \tilde{q}_*)\) on some interval \([0, T]\). Then, from theorem ??,

\[
k(t, k_*, \tilde{q}_*) \geq k_*,
\]

\[
\tilde{q}(t, k_*, \tilde{q}_*) \geq \tilde{q}_*,
\]

\(\forall t \in [0, T]\). This in turn implies, using the same theorem, that

\[
k(s, k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*)) \geq k(t, k_*, \tilde{q}_*),
\]

\[
\tilde{q}(s, k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*)) \geq \tilde{q}(t, k_*, \tilde{q}_*),
\]

where \(t, s \geq 0\) and \(t + s \leq T\). By uniqueness,

\[
\begin{pmatrix}
  k(s, k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*)) \\
  \tilde{q}(s, k(t, k_*, \tilde{q}_*), \tilde{q}(t, k_*, \tilde{q}_*))
\end{pmatrix}
= 
\begin{pmatrix}
  k(t + s, k_*, \tilde{q}_*) \\
  \tilde{q}(t + s, k_*, \tilde{q}_*)
\end{pmatrix}.
\]

Hence, we have,

\[
\begin{cases}
  k(t + s, k_*, \tilde{q}_*) \geq k(t, k_*, \tilde{q}_*) \\
  \tilde{q}(t + s, k_*, \tilde{q}_*) \geq \tilde{q}(t, k_*, \tilde{q}_*).
\end{cases}
\]

The proof is similar for the upper solution. Q.E.D.

Proof of Theorem 5.

i) It is the content of Lemma (??).

ii) It follows from Theorem (??) and Lemma (??).
iii) Observe that
\[
k(t, k_*, \tilde{q}_*) \leq k^*,
\]
\[
\tilde{q}(t, k_*, \tilde{q}_*) \leq \tilde{q},
\]
and that both \(k(t, k_*, \tilde{q}_*)\) and \(\tilde{q}(t, k_*, \tilde{q}_*)\) are non-decreasing. Hence, \(k(t, k_*, \tilde{q}_*)\) and \(\tilde{q}(t, k_*, \tilde{q}_*)\) converge almost everywhere in \(\mathbb{R}\) as \(t\) diverges. Let:
\[
\begin{pmatrix}
  k_* \\
  \tilde{q}_*
\end{pmatrix} = \lim_{t \to \infty} \begin{pmatrix}
  k(t, k_*, \tilde{q}_*) \\
  \tilde{q}(t, k_*, \tilde{q}_*)
\end{pmatrix}.
\]

And let us prove now that \((k_*, \tilde{q}_*)\) is a solution of \((P_S)\). Consider \(\varphi\) as a test function, multiplying \(P_T\) by \(\varphi\) and integrating as in the proof of Lemma (??), specially taking \(\varphi(x, t) = a(x)b(t)\), where \(a(x) \in C_0^\infty(\mathbb{R})\), and \(\lim_{t \to \infty} b(t) = b_\infty \leq \infty, b_\infty \neq 0\), we obtain,

\[
b(t) \int_{\mathbb{R}} k(t, k_*, \tilde{q}_*) a - b(0) \int_{\mathbb{R}} k_* a - \int \int_{Q_t} k(s, k_*, \tilde{q}_*) ab'(s) ds = \int \int_{Q_t} g_1 ab + \int \int_{Q_t} kba_{xx}.\]

we obtain,

\[
\frac{1}{t} \left[ b(t) \int_{\mathbb{R}} k(t, k_*, \tilde{q}_*) a - b(0) \int_{\mathbb{R}} k_* a - \int \int_{Q_t} k(s, k_*, \tilde{q}_*) ab'(s) ds \right] = \frac{1}{t} \left[ \int \int_{Q_t} b (g_1 a + ka_{xx}) \right].
\]

So that the left hand side of the expression above is zero by

\[
\lim_{t \to \infty} b(t) = b_\infty \leq \infty,
\]

and L’Hopitale’s rule. Remark that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}} b(x, s) (a(x)g_1(k, \tilde{q}) + k(x, s)a_{xx}(x)) dx ds = b_\infty \int_{\mathbb{R}} ag_1(k_*, \tilde{q}_*) + k_*a_{xx},
\]

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by L’Hopitale’s rule again. Notice that, \( b_\infty \neq 0 \), therefore the equality holds if and only if the right hand side is zero, that is,

\[
\int_\mathbb{R} (ag_1(k_+, \tilde{q}_*) + k_+ a_{xx}) = 0.
\]

I.e.

\[
(k_+)^{xx} + g_1(k_+, \tilde{q}_*) = 0.
\]

And similarly,

\[
(\tilde{q}_+)^{xx} + g_2(k_+, \tilde{q}_*) = 0.
\]

Therefore, \((k_+, \tilde{q}_*)\) is a solution of \(P_S\).

**B. Proof of Lemma 1, Section 4:** In interval \([0, 1]\), the following Laplacian

\[
\Delta \xi_0 + \lambda_0 \xi_0 = 0
\]

with homogenous Dirichlet conditions

\[
\xi_0(0) = \xi_0(1) = 0,
\]

\[
\xi'_0(0) = \xi'_0(1) = 0,
\]

can be solved in space \(C^2([0, 1])\), following K. C. Chang (p 155 or p 243). The eigenvalues of this Laplacian are \(\lambda_n = (2n\pi)^2\) and the associated eigenfunctions are \(\sin(2\pi nx)\) and \(\cos(2\pi nx)\), for \(n = 0, 1, 2, \ldots\). We chose \(B = [0, 1/2]\) and \(n = 1\), so that \(\lambda_0 = (2\pi)^2\) and \(\xi_0 = \sin(2\pi x)\). It can be easily checked that \(\xi_0\) is positive in \(B\) and zero on the two boundary points.

Let us define

\[
(k_\eta(x), \tilde{q}_\eta(x)) = \begin{cases} 
(\eta \xi_0(x), \eta \xi_0(x)), & \text{if } x \in B, \\
(0, 0), & \text{if } x \in \mathbb{R} \setminus B.
\end{cases}
\]

Obviously, \(supp\{k_\eta(x)\} \subset [0, 1]\).
For any function $\zeta(x) \in C_0^\infty(B)$, such that, $\zeta \geq 0$ in $\mathbb{R}$, we have
\[
\int_{\mathbb{R}} (k_\eta \Delta \zeta + k_\eta g_1(k_\eta, \bar{q}_\eta)\zeta) = \int_{B} (\Delta k_\eta + k_\eta g_1(k_\eta, \bar{q}_\eta)\zeta) \\
= \int_{B} (-\lambda_0 + g_1(k_\eta, \bar{q}_\eta)) k_\eta \zeta,
\]
and similarly for $q_\eta$,
\[
\int_{\mathbb{R}} (\bar{q}_\eta \Delta \zeta + \bar{q}_\eta g_2(k_\eta, \bar{q}_\eta)\zeta) = \int_{B} (\Delta \bar{q}_\eta + \bar{q}_\eta g_2(k_\eta, \bar{q}_\eta)\zeta) \\
= \int_{B} (-\lambda_0 + g_2(k_\eta, \bar{q}_\eta)) \bar{q}_\eta \zeta.
\]

Then, various possibilities arise,

i) If $-\lambda_0 + g_i(k_\eta, \bar{q}_\eta) \geq 0$ for $i = 1, 2$, i.e. $(k_\eta, \bar{q}_\eta) \in M_1(V)$, then one can build a series of lower solutions.

ii) If $-\lambda_0 + g_i(k_\eta, \bar{q}_\eta) \leq 0$ for $i = 1, 2$, that is $(k_\eta, \bar{q}_\eta) \in M_2(V)$, then what one can obtain is a series of upper solutions.

For general interval, by dilation and translation, we can always study in interval $[0, 1]$ and $[0, 1/2]$. Q.E.D.
Reference


Brito P. (2003), The dynamics of distribution in a spatially heterogeneous world, in press.


