Ambiguity in Macroeconomics: The Implications for the “Big Push” and Multiplier

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Abstract

By modelling individual’s preference under ambiguity as Choquet integral with respect to a Neo-additive capacity, the paper investigates the effects of ambiguity and agents’ ambiguity attitude on the process of economic industrialization. The results are presented with Keynesian flavor: Coordinating sufficient optimism can create "Big Push" to help the economy to achieve self-fulfilled Pareto-optimal equilibrium and economic boom; however, sufficient pessimism result simply in coordination failure and cause economy "stuck" in an inef-
icient state. In general context, the comparative static analysis give rise to two results: firstly, more pessimism decrease the level of equilibria efforts, and optimism has opposite effects. Secondly, sufficient ambiguity reduce multiplier effects of government conventional policies which we suggest are only helpful by building up confidence (in the case of prevailing pessimism) or cooling down the fever (in the case of prevailing optimism).

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1 Introduction

1.1 Keynesian Macroeconomics and “Big Push”

In his classical work, “The General Theory”, Keynes argued that expectations and uncertainty are important motive forces on macroeconomic activities due to their influence on the speed of real investment and on liquidity preference. Sudden shifts in the psychological forces behind uncertainty can produce booms and slumps in economy. In his view, aggregate investment, the engine of macro economy, leads to an increase in macroeconomic activity via the multiplier process. Businessmen’s expectations play a vital role in determining the level of their investment. Therefore
Keynesian policy suggestions tell us that it should be crucially helpful for government to intervene actively in the marketplace when there is recession.

Although Keynesian views on uncertainty have been known for a long time, it has not been clear how to formally model them. It was difficult to formalise Keynes ideas on the role of uncertainty in a formal economic model. As the core of Keynes’ conception of economic society, uncertainty is impossible to express in exact probability and seemingly incompatible with the traditional notion of equilibrium. Meanwhile, accepting Keynes means at least accepting multiplicity of equilibria, which creates difficulties for traditional comparative static analysis.

Motivated by these intuitively appealing ideas and challenge, this paper aims to apply the techniques recently developed in decision theory to model Keynesian ideas of uncertainty rigorously. Especially, we hope that the technology used here could make some contribution to microeconomic methodological premises of Keynesian assumption, which in Keynes’ famous phrase is: animal spirits.

We start with a broad-studied model of “Big Push” in economy industrialization which we found suitable to address essential Keynesian characteristics and to fix our ideas. Firstly introduced by Rosenstein-Rodan (1943), the “Big Push” can be interpreted as the switch of economy from one equilibrium to a better equilibrium path without any exogenous improvement. Comparing with previous “big push” literatures which all relied crucially on the importance of externalities, the emphasis of this paper is on the effects of ambiguity, specifically, the impact of confidence on the “big push”. We construct the model in line with the investment model in Metrick (Feb., 1997), whilst acknowledging that the expectations and ambiguity play
a role in the estimates of the profit-stream of any newly-to-be-purchased plant or machinery. We represent industrial agents’ beliefs as neo-additive capacities and show a new equilibrium decision rules. Our results show a strong flavor of Keynes: In an ambiguous situation, sufficient pessimism will result in the lowest level of economic activities and depression. Conversely, sufficient optimism by themselves can cause an economic boom, dot com boom maybe the one to name. When sufficient numbers of agents are optimistic enough and higher strategies are simultaneously played, “Big Push” can arise and the “economics of euphoria” could self-fulfilled.

To bring about more methodological message and deepen the insights, we present further an abstract model and comparative static analysis. Again, the results suggest that sufficient ambiguity will lead to determined and unique equilibrium and optimism and pessimism will separately lead to Pareto-superior and Pareto-inferior equilibria. The comparative static analysis is to firstly evaluate the consequences of a change of ‘animal spirits’ in an equilibrium path. The result shows that the increased optimism will increase the level of strategy played in highest and lowest equilibrium, but the increased pessimism has reversed effects. The investigation on the effects of changing economic parameters results in an opinion on Keynesian multiplier effects and policy predictions. The standard multiplier story tells us that the aggregate response to a shock exceeds the individual response. In the paper we confirm that Keynesian multiplier is a psychologically based effects rather than automatic or mechanical ones. When there is sufficient ambiguity with prevailing pessimism or optimism, the multiplier effects will disappear. The Bank of Japan’s fail in the early 1990s can be the case for us. Some argue that the bank of Japan’s mistake is of not cutting
interest rate far enough, we would like to suggest the conventional policies only work if it affects individual’s expectations, which is, in economic depression, to increase investor’s confidence and to cool them down in the case of boom..

1.2 Ambiguity and Coordination Games

Although it was normally ignored in practice, as early as in Knight (1921), ambiguity is distinguished from risk as a situation where probabilities are unknown or imperfectly known in economic theory. The dominant theory of decision-making under ambiguity is Subjective Expected Utility (SEU henceforth) (Savage (1954)), which reduces all problems of ambiguity to ones of risk under a subjective probability. However, experimental evidence, in which Ellsberg Paradox (Ellsberg (1961)) is a best-known example, does not support SEU. A significant amount of research has been conducted on alternative theories since 60s. Amongst many quantitative, psychological approaches, most commonly used one is Choquet expected utility (henceforth CEU) (Choquet (1953)), which are also supported by having a strong axiomatic foundation and reasonable experimental evidence. According to CEU theory, agents maximize their utility with respect to their beliefs represented as non-additive probabilities (or capacities). Non-additive probabilities are unique and subjective, whereas they do not satisfy all the properties of mathematical probabilities. CEU preferences have been axiomatized by Schmeidler (1989), Gilboa (1987) and Sarin and Wakker (1992). Recently Chateauneuf-Eichberger-Grant (2002) studied a special case of non-additive probabilities, known as neo-additive capacities. This allows us to model decision
maker more easily not only as an ambiguity-aversion (pessimism), but also could be an ambiguity-lover (optimism).

We adopt the the notion of equilibrium proposed in Dow and Werlang (1994) for games under ambiguity (“ambiguous games” henceforth), which is extended in Eichberger and Kelsey (2000). In ambiguous games, decision-makers are CEU maximizer with respect to their neo-additive capacity. Comparing to Nash equilibrium, this new concept relaxes the assumption of full rationality by allowing individuals react either “carefully” or “impulsively” in the situation involving ambiguity. Because the strategy sets in economic applications are usually continuum variables, such as prices, quantities and investment expenditures, we investigate the existence of such equilibria to ambiguous games with continuous strategy space and applied the result into the model.

The model is constructed in the framework of coordination games (Cooper (1999)). This type of games present strategic complementarities and multiple equilibria which quite match Keynesian features modelling (Cooper and John (1988)). Adding another Keynesian feature, uncertainty, it is particularly of our interest in this paper to apply the combination of the theories in ambiguity and coordination games to Keynesian macroeconomic study.

There have been various attempts at resolving multiplicity of equilibrium in coordination games, for instance, the literature on so-called global games, a class of games with strategic complementarities, arbitrary numbers of players and actions, and slightly noisy payoff signals (see Morris and Shin (2003) ). The global-game approach addresses the equilibrium selection through introducing incomplete infor-
mation. With more precise private information relative to public information, as the signal noise vanishes, the game has a unique strategy profile that survives iterative dominance. Rather than clarifying heterogeneity due to asymmetric information in global games, we underline the modified agents’ preference under ambiguity by abandoning the assumption of rational expectation. We take this different modelling approach to capture the essential aspects of reality that expectations and confidence play a role in the situation involving ambiguity. By doing this, we would like to argue that our approach can be carried over in a more straightforward way to a broad class of models including global games.

The paper is organized as follows. In Section 2, we introduce the concept of Choquet integral with Neo-additive capacity. Section 3 develops the equilibrium concept for general ambiguous games and examines the existence of such equilibria. In section 4, we present the application in the industrialization “Big Push” model. Section 5 investigates the uniqueness of equilibrium in more general structure of coordination games with continuous strategy space. Comparative statics examine separately the effects of ambiguity and its attitudes on the equilibrium strategy level and multiplier. Finally, concluding remarks appear in Section 6. The appendix provides the relevant proofs for reference.

2 CEU Preference with Neo-additive Capacity

In this section, we present the concept of Choquet integral, an expected value of a function with individual’s belief represented by Neo-additive capacity when there is
exogenous ambiguity. Neo-additive capacity is an useful case of capacity which models both optimistic and pessimistic attitudes towards ambiguity. The representation in this paper has an axiomatic foundation in the work of Chateauneuf, Eichberger, and Grant (2002).

2.1 Definition of Neo-additive Capacity

A capacity generalizes the notion of probability and assigns non-additive weights to subsets of $S_i$. Formally, capacities are defined as follows:

**Definition 2.1** A capacity on $S_i$ is a real-valued function $v : s_i \rightarrow \mathbb{R}$ which satisfies the following properties:

1. $A, B \subseteq S_i, A \subseteq B$ implies $v(A) \leq v(B)$, monotonicity

2. $v(\emptyset) = 0$ and $v(S_i) = 1$. normalization

The capacity $v$ is called convex if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ holds, the concave capacity is defined by substituting $\leq$ with $\geq$. In preferences represented by a Choquet integral with respect to capacity, a convex capacity interprets pessimistic attitudes and conversely, concave capacity reflects optimism. The Neo-additive capacity is the useful combination of both.

**Definition 2.2** For a pair of real numbers such that $\lambda \geq 0, \gamma \geq 0$, and $\lambda + \gamma \leq 1$ and a given probability $\pi(A)$, a neo-additive capacity is defined as:
\[ v(A) = \begin{cases} 
1 & \text{for } A = S \\
\lambda + (1 - \lambda - \gamma) \pi(A) & \text{for } \phi \subsetneq A \subsetneq S \\
0 & \text{for } A = \phi
\end{cases} \]

One can see the Neo-additive capacity is a convex combination of an additive capacity and capacities on two extreme outcomes, one is complete ignorance with objective probability of 1 and one is complete ambiguity with objective probability of 0. The parameters \( \lambda \) and \( \gamma \) are respectively used to represent the degrees of optimism and pessimism. Intuitively the weight \( (1 - \lambda - \gamma) \) given to \( \pi \), subjective beliefs of decision maker, measures confidence in beliefs.

The below illustrates the property of neo-additive capacity.

**Definition 2.3** Pessimism refer to the belief which overweight the bad outcomes and optimism refer to the belief which overweight the good outcomes.

**Lemma 2.1** A Neo-additive capacity is pure optimistic if \( \gamma = 0 \), pure pessimistic if \( \lambda = 0 \).

### 2.2 Choquet integral with Neo-additive Capacity

If beliefs are represented by capacities, the expected utility of real-valued function \( f \) can be defined by Choquet integral.

**Definition 2.4** An expectation of utility with respect to a capacity is defined as an integrate function, Choquet integral.

\[
\int u dv = \int_{0}^{\infty} v(\{x : u(x) \geq t\}) \, dt + \int_{-\infty}^{0} [v(\{x : u(x) \geq t\}) - 1] \, dt
\]
Choquet integral with respect to neo-additive capacities forms a general case and allows a response to ambiguity by over-weighting either bad outcomes or good outcomes. (Chateauneuf, Eichberger, and Grant (2002))

**Definition 2.5** The Choquet expected value of a real valued function $f : S \rightarrow R$ with respect to a Neo-capacity $v$ is defined as:

$$CEU(v) = \int f dv = \gamma \cdot \inf_{s \in S} (f) + \lambda \cdot \sup_{s \in S} (f) + (1 - \lambda - \gamma) E_{\pi} (f)$$

It says that payoff of any strategy is valued by a weighted average of the expected payoff and maximal, minimum payoffs for given $S$. This can therefore help to explain many economic observations confirmed in laboratory experiments. Such as economic behavior in depressions or bubbles; paradox of people buying insurance and gambling, etc.

**Definition 2.6** A player is a pure pessimist if he uses the choice criterion: $CEU(v) = \int f dv = \gamma \cdot \inf_{s \in S} (f) + (1 - \gamma) E_{\pi} (f)$ and a pure optimist if he uses $CEU(v) = \int f dv = \lambda \cdot \sup_{s \in S} (f) + (1 - \lambda) E_{\pi} (f)$ as the choice criterion.

Intuitively, we are thinking of an agent as a pessimist if, in the presence of ambiguity, he overweights lower payoffs, and as an optimist if, instead, he overweights higher payoffs. The higher the ambiguity, the higher the weights on extreme outcomes, which mean the higher $\gamma$ or $\lambda$ is.
3 Equilibrium for Ambiguous Games

To study ambiguity in the strategic games, we obviously need an alternative solution concept to Nash equilibrium, in which players’ beliefs are represented by Neo-additive capacity and their preference are CEU. Many attempts have been made to solve such games (we call ambiguous games henceforth) but all are based on representing player’s beliefs as convex capacities (ambiguity aversion) (such as Dow and Werlang (1994), Marinacci (2000)). In this paper we adopt the recent concept proposed in Eichberger.J. and B.Schipper (2003) to model not only ambiguity aversion but also ambiguity preference by representing players’ belief as Neo-additive capacity.

Definition 3.1 A generalized ambiguous game, \( G(\gamma, \lambda) \), is a game in which the presence of both optimist and pessimist are allowed, and players’ beliefs are represented as capacities.

According to the concept of equilibrium, it is crucial to assume that players will focus their beliefs on strategies which are best responses of the opponents. As is well known, such set of strategies are defined as the support of beliefs. As proposed in Eichberger.J. and B.Schipper (2003), the support of beliefs represented by capacities is defined as the below.

Definition 3.2 Let \( v \) be a capacity on a set \( S \). The support, \( \text{supp}_K v \), of \( v \) is defined by \( \text{supp} v = \text{supp} \pi \).

This definition has similar interpretation with the ones in Dow and Werlang (1994), and Marinacci (2000), however, it has power to discriminate between the
states in which the decision-maker believes and ones in which (s)he does not believe even if (s)he gives all states positive capacity. Therefore, in the case with the possibility of optimism, unlike the other definitions of supports, this concept can work if all states may get positive capacity because of optimism. Similar to the supports defined in Marinacci (2000), the problem with this concept is that such supports are unique but do not always exist. A simple example is the capacity of complete uncertainty, defined by \( v(A) = 0 \) for all \( A \subseteq S \). However, we argue that uniqueness of the support is a desirable property of a capacity. The non-existence of a support will not be a problem because we believe only a capacity having a support is suitable to represent equilibrium beliefs in a game.

Based on the concept of supports, the equilibrium in ambiguous games is formally defined as below.

**Definition 3.3** Let \( R_i(v_i) = \{ x_i : P_i(x_i, v_i) \geq P_i(s_i, v_i) \text{ for all } s_i \in S_i \} \) denote the best response of player \( i \) given his(her) beliefs \( v_i \). Notation \( R_i(v_i) \) represents the set of strategies which maximizes (Choquet) expected utility.

**Definition 3.4** An equilibrium in ambiguous games is a belief system \( (v_1^*, ..., v_I^*) \) where \( v_i^* \) is a Neo-additive capacity on \( S \), if for all \( i \in I \) there exists \( \text{supp} v_i^* \) such that \( \text{supp} v_i^* \subseteq \times_{j \neq i} R_j(v_j^*) \)

Note that equilibrium for ambiguous game is a modification of Nash equilibrium to allow for ambiguity. This new concept is close in spirit to Nash equilibrium in the sense that it still represents a situation where every player ‘believes’ that her (his)
opponents will only choose best response given their beliefs. So every player is assumed to have optimizing behavior and maximize her(his) expected utility preference given their beliefs. The difference is that ambiguities are incorporated and reflected into the modeling of their beliefs. By representing player’s beliefs as Neo-additive capacity and preference as Choquet integral, players could behave either “cautiously” or “impulsively”.

There are many useful advantages of new defined equilibrium solution. Foremost, this equilibrium concept allows a weak deviation of a player’s beliefs from her(his) opponent’s behavior, while holding the definition of the support of capacity in the sense that no player expects other players to choose actions that are not best responses given their beliefs. This explain well that a set of strategies with capacity zero is not necessarily Savage-null. (Savage (1954)) Secondly by relaxing the assumption of full rationality, this concept not only maintains the same equilibrium solution in the normal case but also provides intuitive results when the consequences of this strategy are particularly extreme or players have any doubt on the correctness of the equilibrium predictions. In other words, a standard Nash equilibrium is the special case of an equilibrium for ambiguous games where no player overweights the extreme results, which mean both $\gamma$ and $\lambda$ equal zero. Equilibrium for ambiguous games can be applied wherever Nash equilibrium can. We would like to argue that the extension of the concept of equilibrium to ambiguous games can lead to a better understanding in situations involving ambiguity and it could be helpful to resolve some of the problems which arise in game theory from assuming full rationality.
Meanwhile, as noted above, there is no requirement in such definition that two players have to hold consistent beliefs in regard to the other players’ behavior. So equilibrium beliefs can not determine precisely which strategy will be actually played except when the support of the belief for each player consists of a single strategy. Furthermore, when players’ beliefs are represented by sets of Neo-additive capacities, it is technically difficult to model expected payoff while allowing players to choose mixed strategies. Thirdly, in economic applications, players’ strategy sets are mostly continuous variables. In such situations, pure strategy equilibria exist. Therefore, here we only investigate pure equilibria for ambiguous games with continuous strategy space.

**Definition 3.5** An equilibrium for ambiguous games, \((v^*_1, \ldots, v^*_I)\) is pure if \(\text{supp} \ v^*_i\) contains a single element of \(S_i\) for \(i \in N\).

**Proposition 3.1** If in an ambiguous game \(G(\gamma, \lambda)\), for all players \(i \in I\), the strategy sets \(S_i\) are closed, bounded, convex and continuous, and if the payoff functions \(p_i(s_i, s_{-i})\) are continuous in \(s\) and quasi-concave in each player’s own strategy \(s_i\), then there is a pure strategy equilibrium for \(G\), in which player \(i\) has degree of ambiguity-aversion \(\gamma_i\) and ambiguity-preference \(\lambda_i\).

The proof is straightforward from the definition of supports of neo-additive capacities and continuous and quasi-concave utility function.

Next, we will show that how these techniques can be used to examine the effects of ambiguity on economic behavior, in particularly, on the macroeconomic phenomena with the presence of strategic complementarities.
4 "Big Push" in Economic Industrialization

We take as application example the “big push” model in economy industrialization. As argued in Kevin M. Murphy and Vishny (1989), there are two important factors in “big push”. Firstly, the precondition of coexistence of multiple, Pareto-ranked equilibria for “big push” requires that the economy be capable of sustaining two alternative levels of industrialization. Secondly, under fixed preferences, endowments and available technologies, strategic complementarity and possible coordinated action across economic sectors could create a “big push” to achieve self-sustaining Pareto-preferred industrialization equilibrium. It is noteworthy that strong positive spillover across the sectors through profits is sufficient but not necessary for the “big push”. In other words, even each industrialized firm can not break even individually with its investment, the strategic complementarities across firms through routes other than profits imply the simultaneously coordinated investment of sufficiently many firms will make industrialization self-sustained.

By modeling psychological phenomena such as excessive optimism and pessimism with the presence of ambiguity. The results show that the anticipated scale of industrialization could be larger by optimism or smaller by pessimism. If optimism or pessimism is sufficiently high, equilibrium will be unique. We firstly present a 2 × 2 coordination games as an example, we show that there exist threshold values of optimism or pessimism. For pessimism, if γ is above its threshold value, no sectors industrialize for lack of confidence, then small demand determines the small market
size, which is not enough for industrialization to be done. Thus non-industrialization will be the equilibrium in economy. Conversely, if $\lambda$, represents the extent of optimism of industrial sectors, is above its critical value, then all sectors will choose industrialization as their equilibrium strategies, and giving rise to “big push”.

4.1 Example of The “Big Push”

We start with this simple binary game. Each sector is deciding to choose either constant return to scale technology (CRS), which mean non-industrialization or increasing return to scale (IRS), representing “industrialization” which gives a highest payoff if the other sector choose IRS. Payoffs are given by the following matrix, where $M > m > 0, l > 0$.

<table>
<thead>
<tr>
<th></th>
<th>Industrialization</th>
<th>Non-Industrialization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Industrialization</td>
<td>$M, M$</td>
<td>$-l, 0$</td>
</tr>
<tr>
<td>Non-Industrialization</td>
<td>$0, -l$</td>
<td>$m, m$</td>
</tr>
</tbody>
</table>

This is a coordination game with strategic complements. Now we will show if both sectors are sufficiently optimistic (considering $\gamma \to 0$) then industrialization is the only equilibrium. Suppose, if possible, a non-industrialization equilibrium exists, the payoff for sector 1 is,

$$P_1(\bar{e}, v_1) = M \cdot \lambda - l \cdot (1 - \lambda) = (M + l) \cdot \lambda - l$$

$$P_1(e, v_1) = m \cdot \lambda + (1 - \lambda) \cdot m = m$$

It is easily to see when $\lambda > \frac{m + l}{M + l}$, non-industrialization is not an optimal choice for sector 1. So when sectors are getting more optimism, they will choose industrialization as their equilibrium strategy. Conversely, only when $\lambda < \frac{m + l}{M + l}$, the non-
industrialization is the equilibrium. Inequality $\lambda > \frac{m+l}{M+l}$ also says that industrialization is more likely if the higher is the payoff from industrialization $M$ or the lower is the profit from non-industrialization. Thus it is more plausible to say, unlike Nash Equilibrium, all these factors will affect the equilibrium resulted. This result is consistent with the one from Eichberger and Kelsey (1999), where they use simple capacity to model player’s beliefs.

Conversely, suppose sectors are pessimistic, then $\lambda = 0$, non-industrialization is naturally the only equilibrium. Similarly assuming if industrialization is equilibrium, then,

$$P_1 (In, v_1) = -l \cdot \gamma + M \cdot (1 - \gamma) = M - (M + l) \cdot \gamma$$

$$P_1 (Non, v_1) = 0$$

So if $\gamma > \frac{M}{M+l}$, industrialization will not be an equilibrium, sector 1 will choose non-industrialization as his equilibrium choice. Similarly, in equation $\gamma > \frac{M}{M+l}$ says that the lower is the payoff from industrialization, the more likely the sector will choose non-industrialization as equilibrium.

Finally, it is easy to derive from the results above that, when $0 < \lambda < \frac{m+l}{M+l}$ and $0 < \gamma < \frac{M}{M+l}$, there are the indeterminate, multiple equilibria. In this case, the level of ambiguity is less than $\frac{M+m+l}{M+l}$. Because the game is symmetric, sector 2 is similarly applied.

4.2 Model

The model is restricted to one-period, multiple continuum sectors which are indexed by $i$ on the unit interval $[0, 1]$. There are two types of firms in each sector, one is a
competitive fringe of firms with a constant returns to scale technology which convert
one unit of input into one unit of output. The other is referred to as a monopolist who
is able to access to two types of technologies, low and high productivity technologies.
The low technology is free CRS technology but superior to the competitive fringe
with marginal cost of 1, which is:  \( q_i = n_i \). The adoption of low technology represents
the non-industrialization state of economy. The high technology, representing the
industrialization, will incur a fixed cost which is the same in all industrializing sectors.
In the model we use superscript “l” and “h” separately representing low technology
adopted and high technology accepted. The only cost of production is labour, denoted
by \( N \), which is inelastically supplied.

Define \( y_i \) as the expenditure of a monopolist \( i \) in each sector, then the budget
constraint is equal to \( \int_0^1 y_i = Y = \Pi + W \) where \( Y \) is aggregate income (expenditure),
\( \Pi \) represents aggregate profits by all firms and \( W \) represents total wages. We assume
the consumption is allocated identical share along sectors, expenditure in each sector,\( y_i \) will equal to \( Y \) by normalizing on the unit interval.

The production function for low technology is :  \( q_i = \alpha_l n_i \) where \( \alpha_l > 1 \). Assume the demand is unit-elastic, the price in each sector will be set as 1 and the
monopoly firms capture all of the market. The profits of each (monopoly) firm using
low technology are:

\[
\pi^l_i = q_i - \frac{q_i}{\alpha_L} = a_l q_i. \quad (1)
\]

where \( a_l = (\alpha_l - 1) / \alpha_l \)

Next, we allow firms to pay a cost and adopt the “high” technology, which im-
proves production function as, \( q_i = \alpha_h n_i \) where \( \alpha_h > \alpha_l \). The cost incurred is denoted as \( I \). So the profit of the firm with high technology will be,

\[
\pi^h_i = q_i - \frac{q_i}{\alpha_h} - I = a_h q_i - I. \tag{2}
\]

where \( a_h = \frac{\alpha_h - 1}{\alpha_h} \).

Again, our normalization gives that gross production, \( Q \) is equal to production in each sector, \( q_i \). The incurrence of cost can also be thought as the expenditure on intermediate goods, and drives a wedge between income \( Y \) and gross production \( Q \). Thus the industrialized firm contributes to the demand for other firms’ goods by raising gross production not aggregate income. This creates the channel of spillover other than profits. It is important because complementarity arises in the economy through this expenditure-demand relationship across sectors.

Demand here is endogenous and depends on the production of other sectors. Thus firms are interested in the productive potentials of other sectors of the economy. That means, if some sectors in the economy industrialize, then they will be spending more on the products in the remaining sectors by at least incurring industrialization cost, no matter whether this industrialization increases aggregate income. These increased demand will induce the output expansion and make industrialization in other sectors profitable. In particular, this expenditure-demand linkages can create multipliers and, when combined with nonconvexities in technology, can lead to multiple, Pareto-ranked equilibria. In the case of big push, this means, at a low aggregate level of industrialization, the equilibrium strategy played by firms will be non-industrialization because it is individually unprofitable to industrialize. Conversely, as long as a sufficient num-
ber of other sectors industrialize, the increased demand will make industrialization individually profitable to become a Pareto-preferred equilibrium action. We illustrate this as below.

Let $\mu$ represents the fraction of monopoly firms using the high technology. Formally we have

$$Q = Y + \mu I = \Pi + N + \mu I.$$  \hfill (3)

By substituting equations (1) and (2) into Eq.(3), we have,

$$Q = \mu (a_h Q - I) + (1 - \mu) a_l Q + N + \mu I \Rightarrow Q = \frac{N}{1 - \mu a_h - (1 - \mu) a_l}.$$  

We can see that $Q$ is an increasing function of $\mu$, which works as a multiplier.

For firm $i$, the profit for industrialization is,

$$\pi_i^h = a_h \cdot \frac{N}{1 - \mu a_h - (1 - \mu) a_l} - I.$$  

The profit of non-industrialization is,

$$\pi_i^l = a_l \cdot \frac{N}{1 - \mu a_h - (1 - \mu) a_l}.$$  

Decision rule is,

$$\pi_i^h - \pi_i^l = (a_h - a_l) \cdot \frac{N}{1 - \mu a_h - (1 - \mu) a_l} - I > 0.$$  \hfill (4)

In this inequality, the entire first term is a strictly increasing function of $\mu$, the second term is fixed and constant. It is clear that there is a marginal rate of $\frac{\hat{\mu}}{\mu}$ which makes,

$$\pi_i^h - \pi_i^l = (a_h - a_l) \cdot \frac{N}{1 - \mu a_h - (1 - \mu) a_l} - I = 0.$$
The intuition is the increased profit from increased demand which is created by $\hat{\mu}$ industrialized firms will exactly compensate the cost of industrialization $I$. Thus there is a possibility of multiple equilibria. When more firms industrialized and $\mu > \hat{\mu}$, Eq. (4) holds, so equilibrium will be “every firm industrializes” and $\mu = 1$. Conversely, when $\mu < \hat{\mu}$, Eq. (4) does not hold, then economy will stuck in the inefficient equilibrium, $\mu = 0$ and “no firm industrializes”. This multiplicity of equilibrium is an very simple kind of coordination failure: if every firm industrializes then demand is high and industrialization is optimal; if no firm accept high technology then demand is low and non-industrialization is optimal.

Next, we will show that ambiguity can play a role in equilibrium selection. Ambiguity here refers to one sector concerning industrialization of others. The above analysis shows, with the presence of strategic complementarity, the scale of industrialization in the whole economy will influence the return of an industrialized sector. Expecting a low aggregate level of industrialization in which industrialization is individually unprofitable, a sector won’t industrialize; conversely, if a sector are quite optimistic to expect a sufficient number of sectors industrialized then it is individually profitable to industrialize by itself. We will show that sufficient optimism will have enough power to create “big push” in economy, but pessimism will make it difficult and economy is stuck in an inefficient state.

With presence of ambiguity, the Choquet expected value of demand faced by a firm is
\[ CEU(Q) = \gamma \cdot \frac{N}{1 - a_l} + \lambda \cdot \frac{N}{1 - a_h} + (1 - \gamma - \lambda) \frac{N}{1 - \mu a_h - (1 - \mu) a_l}. \]  

(5)

where, the highest aggregate production is \( Q^h = \frac{N}{1-a_h} \) when all firms are industrialized, \( \mu = 1 \), and lowest aggregate production is \( Q^l = \frac{N}{1-a_l} \), when all firms are non-industrialized, \( \mu = 0 \). The expected value of demand is \( \frac{N}{1-\mu a_h - (1-\mu) a_l} \).

Then, the firm’s profit when it industrialized is, \( \pi^h_i = a_h CEU(Q) - I \); when it is non-industrialized, the profit is \( \pi^l_i = a_l CEU(Q) \).

The decision rule is,

\[ \pi^h_i - \pi^l_i = (a_h - a_l) CEU(Q) - I > 0. \]  

(6)

Substitute Eq. (5) into Eq. (6) and rearrange it, the decision rule is,

\[
\begin{align*}
\pi^h_i - \pi^l_i &= (a_h - a_l) \left[ \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu) a_l} \right) + \lambda \left( \frac{N}{1-a_h} - \frac{N}{1-\mu a_h - (1-\mu) a_l} \right) + \frac{N}{1-\mu a_h - (1-\mu) a_l} - I \right] \\
&> 0.
\end{align*}
\]

The first term in bracket is negative and the second term in bracket is positive.

The other variables in the equation are given as fixed. Now considering \( \mu \) is at marginal level of \( \tilde{\mu} \), if firms are more pessimistic and \( \gamma \) is sufficiently larger than \( \lambda \), then the negative term will dominate the positive one, we will have,

\[
\begin{align*}
\pi^h_i - \pi^l_i &= (a_h - a_l) \left[ \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu) a_l} \right) + \lambda \left( \frac{N}{1-a_h} - \frac{N}{1-\mu a_h - (1-\mu) a_l} \right) + \frac{N}{1-\mu a_h - (1-\mu) a_l} - I \right] \\
&< 0.
\end{align*}
\]
Thus, firm i won’t industrialize. Conversely, if optimism prevails and \( \lambda \) is sufficiently large, the positive term will dominate the negative one, then,

\[
\pi_i^h - \pi_i^l = (a_h - a_l) \cdot \left[ \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu)a_l} \right) + \lambda \left( \frac{N}{1-a_h} - \frac{N}{1-\mu a_h - (1-\mu)a_l} \right) + \frac{N}{1-\mu a_h - (1-\mu)a_l} - I \right] > 0.
\]

and firm i will industrialize.

The intuition is that a firm is more likely to industrialize when it is more optimistic and less likely to do so when it is more pessimistic. Why is this so? Since pessimism/optimism makes anticipated scale of industrialization smaller/larger than it actually is, which discourage firms to industrialize. This can be clearly shown as diagram 1. Curve \( \Delta \pi \) depict the value of \( \pi^h - \pi^l \) responding to value of \( \mu \). When firms are more optimism, curve \( \Delta \pi \) shift leftward to \( \Delta \pi^\lambda \), the anticipated scale of industrialization now is \( \hat{\mu}_2 \) instead of \( \hat{\mu} \); when pessimism prevails, curve \( \Delta \pi \) shift rightward to \( \Delta \pi^\gamma \), the anticipated scale of industrialization now is \( \hat{\mu}_1 \) which is clearly less than \( \hat{\mu} \).

Now we could show separately two pure equilibria in the economy, given the corresponding beliefs held by firms.

1. Suppose firm i is extremely pessimistic, which mean, \( \lambda = 0 \), then the profit of industrialization or non-industrialization of it own will be,

\[
\pi_i^h = a_h \cdot \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu)a_l} \right) + \frac{N}{1-\mu a_h - (1-\mu)a_l} - I.
\]

Similarly, \( \pi_i^l = a_l \cdot \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu)a_l} \right) + \frac{N}{1-\mu a_h - (1-\mu)a_l} \).

Thus, \( \pi_i^h - \pi_i^l = (a_h - a_l) \cdot \gamma \left( \frac{N}{1-a_l} - \frac{N}{1-\mu a_h - (1-\mu)a_l} \right) + (a_h - a_l) \cdot \frac{N}{1-\mu a_h - (1-\mu)a_l} - I \).

Therefore, when firm i is sufficient pessimistic, let us say \( \gamma \to 1 \), we get \( \pi_i^h - \).
\( \pi_i^l = (a_h - a_l) \cdot \frac{N}{1 - \mu_i} - I \) strictly less than 0. This imply that firm \( i \) will undoubtly adopt non-industrialization. If the pessimism prevails in the economy, every firm will do so and the equilibrium is “no firm industrialize”, \( \mu = 0 \). It is private optimal for single firm but the economy will have lowest aggregate production.

2. Suppose firm \( i \) is extremely optimistic and \( \gamma = 0 \). The possible profit for firm \( i \) will be,

\[
\pi^h_i = a_h \cdot \lambda \left( \frac{N}{1 - \mu_h} - \frac{N}{1 - \mu_h - (1 - \mu)a_i} \right) + \frac{N}{1 - \mu_h - (1 - \mu)a_i} - I.
\]

\[
\pi^l_i = a_l \cdot \lambda \left( \frac{N}{1 - \mu_l} - \frac{N}{1 - \mu_l - (1 - \mu)a_i} \right) + \frac{N}{1 - \mu_l - (1 - \mu)a_i}.
\]

\[
\pi^h_i - \pi^l_i = (a_h - a_l) \cdot \lambda \left( \frac{N}{1 - \mu_h} - \frac{N}{1 - \mu_h - (1 - \mu)a_i} \right) + (a_h - a_l) \cdot \frac{N}{1 - \mu_h - (1 - \mu)a_i} - I.
\]

The same analysis apply and the equilibrium in this case is that all firms invest with prevailing optimism and we attain both private optimality and social optimality. The economy has the highest aggregate production.
5 A General Model for Coordination Games With Keynesian Features

In this section we take one step to make the argument more general. We change the problem to the choice of output level for the purpose of constructing continuous strategy space. For the reason of being readable, we present model of two-sector economy. However the results are hold more generally as demonstrated in Appendix B.

5.1 Uniqueness and Determinacy of Equilibrium

It is an imperfect competitive economy with 2 industrial sectors \( i, i = 1, 2 \). Each sector produces a unique product and chooses output level \( e_i \) with a certain technology in the interval of possible level \([e, e]\) to maximise profit. When facing high demands and sales, it will be profitable for sectors to utilize efficient technology and get high level of output. The aggregate output of economy is \( f(e_1, e_2) \). Each sector incur cost \( c(e_i) \), which is the increasing function of \( e_i \). Considering sectors consume equally total output of whole economy, the payoff function for sector \( i \) is defined as following.

Definition 5.1 \( U_i (e_i, e_{-i}, \theta_i) = \frac{f(e_i, e_{-i})}{2} - c(e_i) \), where \( \theta_i \) parameterizes the payoff of sector \( i = 1, 2 \).

Assumption 5.1 The utility function has the following properties:

1. the payoff function is strictly concave, \( \frac{\partial^2 u}{\partial e_i^2} < 0 \).

2. there is strategic complementarity \( \frac{\partial^2 u}{\partial e_i \partial e_j} > 0 \) for \( i \neq j \).
3. \( \frac{\partial^2 u}{\partial e_i \partial e_j} > 0, \) for \( i = 1, 2 \)

Condition (2) says that an increase in effort by sector \( j \), increases sector \( i \)'s marginal benefit of increasing his/her effort; condition (3) says that an increase in \( \theta_i \) raises the marginal benefit of \( i \)'s effort. Effectively this defines a way of ordering the set of possible \( \theta_i \).

It is clear that there are multiple equilibria in this coordination game. Denote sector \( i \)'s expected payoff by \( u_i(e_i, e, \theta_i) \) when the other sector other than \( i \) choose output level \( e \) while sector \( i \) choose \( e_i \), in this paper we show that a sufficient ambiguity will lead to the uniqueness of equilibrium.

**Proposition 5.1** Consider a game, which satisfies Assumption 5.1 when there is sufficient ambiguity, the equilibrium is unique.

If player \( i \) has degree of ambiguity aversion \( \gamma_i \), ambiguity preference \( \lambda_i \), then his/her (Choquet) expected utility will be:

\[
\gamma u(e_i, \bar{e}, \bar{\theta}) + \lambda u(e_i, \bar{e}, \bar{\theta}) + (1 - \gamma - \lambda) u(e_i, e, \theta).
\]

Let \( R_i(e) \) denote individual \( i \)'s best response to this profile, defined by:

\[
\gamma \frac{\partial u}{\partial e_i}(R_i(e), \bar{e}, \bar{\theta}) + \lambda \frac{\partial u}{\partial e_i}(R_i(e), \bar{e}, \bar{\theta}) + (1 - \gamma - \lambda) \frac{\partial u}{\partial e_i}(R_i(e), e, \theta) = 0.
\]

Differentiating with respect to \( e \),

\[
\gamma \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), \bar{e}, \bar{\theta}) \frac{\partial R_i(e)}{\partial e_i} + \lambda \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), \bar{e}, \bar{\theta}) \frac{\partial R_i(e)}{\partial e_i} + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), e, \theta)
\]

\[
+ (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), e, \theta) \frac{\partial R_i(e)}{\partial e_i} = 0.
\]

Solving,

\[
\frac{\partial R_i(e)}{\partial e_i} = \frac{\lambda \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), \bar{e}, \bar{\theta}) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), e, \theta)}{\gamma \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), \bar{e}, \bar{\theta}) + \lambda \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), \bar{e}, \bar{\theta}) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j}(R_i(e), e, \theta)}.
\] (7)
Equation (5.1) is the slope of sector $i$’s best reaction function curve. With presence of strategic complements, we know the reaction curve of player $i$ always slope upwards. As shown in Cooper (1999), p21-22, a necessary condition for multiple equilibria, is that there exists a Nash equilibrium with $R_i'(e) \geq 1$. Therefore, by proving $R_i'(e) < 1$ when there is sufficient ambiguity, we can establish uniqueness of equilibrium. Proof is in Appendix A.

Next we will show that the determinacy of equilibrium depends on the relative level of pessimism or optimism. Although there are multiple Nash equilibria, but a given sector will wish to play one of the extreme strategies, provided that it believes that the other act similarly. We further the analysis and proved that only the Pareto inferior equilibrium survives if there are sufficiently high level of pessimism, and conversely, the only equilibrium will be the Pareto superior one if sectors are sufficiently optimistic.

**Proposition 5.2** If there is sufficient much ambiguity, there exists $\gamma$ (resp. $\bar{\lambda}$) such that if the degree of pessimism (resp. optimism) is $\gamma \geq \bar{\gamma}$ (resp. $\lambda \geq \bar{\lambda}$), the equilibrium strategies are unique and all players play strategy $e_-$ (resp. $\bar{e}$).

### 5.2 Comparative statics

#### 5.2.1 Effect of Ambiguity on Strategy Level

In the presence of strategic complementarity and unique equilibrium, the following proposition shows that more pessimism or more optimism have respectively different effects on the effort levels in the highest and lowest equilibria. Simply to ex-
plain, more optimistic/pessimistic, higher/lower weight player \( i \) will place on high/low strategy played by the opponent. With presence of strategic complementarity, this gives \( i \) incentive to increase/decrease his strategy. Therefore, an increase in optimism/pessimism will increase/decrease equilibrium strategies of both players. The set of equilibria increases/decreases, in the sense that the strategies played in the highest and lowest equilibria increase/decrease.

**Proposition 5.3** An increase in pessimism decreases the strategies played in the highest and lowest symmetric equilibria. Conversely, an increase in optimism increases the strategies played in the highest and lowest symmetric equilibria.

The proposition 5.3 says, \( \frac{d e^*}{d \gamma} < 0 \) and \( \frac{d e^*}{d \lambda} > 0 \)

The results make very usefully economic intuition. Firstly, we argue that this offers theoretical rationale as to why bad fundamentals are somehow “more likely” to trigger a financial crisis, or to tip the economy into recession. When there is more ambiguity about other’s activities, weak fundamentals will more likely make pessimism prevail. Therefore more ambiguity will induce lower economic activities. Conversely, facing high level of ambiguity, the prevailing optimism will create a favorable circumstances for economy developing: the IT boom in middle of 90s can be a case to name. Even how big the role and how important of developing IT in economy is not clear, investors, technologiests behave highly optimistic about the future prospects which lead to the boom in this area.
5.2.2 Effect of Ambiguity on the Multiplier

With presence of strategic complements, multiplier is the other result arose. In this section, we will show how ambiguity affects multiplier.

The first order condition for the model is,

\[ \gamma \frac{\partial u}{\partial e_i} (e_i, \bar{e}, \theta) + \lambda \frac{\partial u}{\partial e_i} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial u}{\partial e_i} (e_i, e, \theta) = 0, \text{ for } i = 1, 2. \]

Let \( \frac{de}{d\theta} \) denote the partial equilibrium response to a change in \( \theta \). This is defined implicitly by:

\[ \gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) \frac{de}{d\theta} + \gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) \frac{de}{d\theta} + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta) \frac{de}{d\theta} = 0 \]

Solving,

\[ \frac{de}{d\theta} = \frac{\gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta) \frac{de}{d\theta}}{\gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta)} \]

Supposing both players have the same marginal response to the change of \( \theta \), the equilibrium responses to an increase in \( \theta \), \( \frac{de}{d\theta} \), is:

\[ \gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) \frac{de}{d\theta} + \gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) \frac{de}{d\theta} + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta) \frac{de}{d\theta} = 0 \]

Solving,

\[ \frac{de}{d\theta} = \frac{\gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta)}{\gamma \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial \theta} (e_i, e, \theta)} \]

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Define the multiplier $\mu$ by:

$$\frac{de}{d\theta} = \mu \frac{de}{d\theta}$$

then

$$\mu = \frac{\gamma \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, e, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i^2} (e_i, e, \theta)}{\gamma \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, e, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i^2} (e_i, e, \theta)}$$

(5.2)

This may be rewritten as:

$$\mu = \left(1 + \frac{(1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, e, \theta)}{\gamma \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, e, \theta) + \lambda \frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i^2} (e_i, e, \theta)}\right)^{-1}$$

By the assumption of strategic complementarities and equation 7, $\mu > 1$. Hence $\mu$ is indeed a multiplier.

**Proposition 5.4** For any $\delta > 0$, there exists $\hat{k}$, such that if $1 > \gamma + \lambda \geq \hat{k}$, the multiplier is less than $1 + \delta$.

**Proof.** From equation (5.2), when $\lambda + \gamma \to 1$, $(1 - \gamma - \lambda) \to 0$, then $\mu \to 1$. ■

The intuition of $\lambda + \gamma$ is the measure of ambiguity. Therefore we could say term $1 - \gamma - \lambda$ represents the confidence. The economic explanation here is that as ambiguity increases, people become less confident in the actions of others. As a result, they respond less to changes in others’ behavior. Hence the multiplier is reduced by ambiguity.

If $\frac{\partial^2 u}{\partial e_i \partial e_\theta} (e_i, e, \theta) \geq \frac{\partial^2 u}{\partial e^2} (e, e, \theta)$, then the relationship between ambiguity and the multiplier is monotonically decreasing. However, in general, the relationship between the amount of ambiguity and the multiplier is not necessarily monotonic.
6 Conclusion

In this paper, by representing decision-maker’s beliefs as a Neo-additive capacity and modelling excessive pessimism or optimism, we applied CEU decision theory to macroeconomic problems with presence of strategic complementarities. The equilibrium analysis show clearly the consistent results with Keynesian views about economic fluctuation, in both “big push” model and macroeconomic coordination model: under ambiguity, equilibrium economic activities, in terms of either level of output or investment, are subject to changes in expectation, or "animal spirits". Pessimism will cause a collapse in investment leading to a depression and optimism will lead to higher equilibrium strategy level and booming economy.

Furthermore, the comparative statics present a interesting results about Keynes’ multiplier. We show that when there is enough ambiguity, economy will converge to a unique equilibrium, either stuck in a low-level trap with sufficient pessimism, or in economic boom with high level of optimism. In both case, multiplier effects disappear. We shall argue that despite appearances this is consistent with Keynesian views about business cycle. In Keynesian views, assume there is an increase in aggregate demand (from an increase in consumer or business optimism), the multiplier magnifies the increased demand which leads to income growth and output expansion. Unemployment declines and the economy experiences a boom. Eventually the economy reaches full capacity (here it is our highest level of output) which constrains additional growth. Then the income-expenditure link become very weak and multiplier effect from business optimism gradually disappear. However, as growth slows, opti-
mism becomes less and business may begin to cut back expenditures. The multiplier will be generated from this reduced optimism and magnifies the decreased expenditure. Pessimism gradually build up and economy begin to experience the recession eventually reach the lowest level of equilibrium, where multiplier from pessimism will dissipate eventually and a new business cycle starts.

As stated before, Keynesian multiplier is commonly known as psychologically based effect. The intuition behind our results suggests that multiplier only work when ambiguity level is very low and business are very confidence. In other words, multiplier doesn’t exist in extreme economic equilibria where either optimism or pessimim are sufficiently high enough to reach economic constraint. It only exists during the macroeconomic adjustment process. Therefore, considering traditional Keynesian policy suggestions of increasing government expenditure to cure a recession (pyramid building) which is naturally related to sufficient pessimism, we suggest that it may be only helpful when this policy trigger confidence and optimism to build among the general pessimistic enviroment. Then optimism multiplier begin to work, business become gradually confident and economy are pushed out of recession.

We think the result in the paper may further contribute to the debate about Alan Greenspan’s policy to 98’s economic bubbles. In our opinion, Mr Greenspan obviously took a right action. According to our analysis, policies only work during the process of forming equilibrium. Governments can stop bubble only during it forms but, this implies the risk of curbing the possible economic push rather than the bubble. Therefore we would like to agree that no response to the prevailing optimism. However, once the economics start turning the way or the worst, bubble bursts,
government should respond fast and far enough to stop the loss the confidence and cushion the consequences. Recession is anyway never welcomed for any economics.

Finally, considering the potential criticism about modelling "big push" as one-period process, we would like to say that we seek to develop the dynamic model but rely upon the static framework. Dynamic enters through the strategic complementarities which will enforce the belief to be consistent. However, it is still natural direction to extend the paper into the dynamic framework.

Appendix

A Proof of Lemma and Proposition

Proposition 5.1: Consider a game, which satisfies Assumption 5.1, when there is sufficient ambiguity, the equilibrium is unique.

Proof. As stated in Cooper (1999), a necessary condition for multiple equilibria is the slope of the reaction function is greater than or equal to 1. Hence, the sufficient condition for unique equilibrium is $R'_i(e)$ less than 1. When there is sufficient ambiguity, $\gamma + \lambda \rightarrow 1$. If we want equation 7 less than 1, it is equal to require the following inequality hold.

$$ - (1 - \gamma - \lambda) \frac{\partial^2 u_{i}}{\partial e_{i} \partial e_{j}}(R_{i}(e), e, \theta) + \gamma \frac{\partial^2 u_{i}}{\partial e_{i}}(R_{i}(e), e, \theta) $$

$$ + \lambda \frac{\partial^2 u_{i}}{\partial e_{i}}(R_{i}(e), \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u_{i}}{\partial e_{i}}(R_{i}(e), e, \theta) $$

It is easily to see the inequality above is true when $\gamma + \lambda \rightarrow 1$ with strict concave
utility function. Rearrange it, we get,

\[
R'_i(e) = -\frac{(1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta)}{\gamma \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) + \lambda \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) + (1 - \gamma - \lambda) \frac{\partial u}{\partial e_i} (R_i(e), e, \theta)} < 1
\]

Therefore, we proved that when there is sufficient ambiguity, the slope of reaction curve of sector i is always less than 1, which imply the existence of unique equilibrium.

**Proposition 5.2:** If there are sufficiently much ambiguity, there exist \( \tilde{\gamma} \) (resp. \( \tilde{\lambda} \)) such that if the degree of pessimism (resp. optimism) is \( \gamma \geq \tilde{\gamma} \) (resp. \( \lambda \geq \tilde{\lambda} \)), the equilibrium strategies are unique and all players play strategy \( \bar{e} \) (resp. \( \check{e} \)).

**Proof.** (1) Pessimism

Consider a given sector \( i \), if (s)he plays strategy \( e^* > \bar{e} \), then the payoff will be:

\[
U_i^* = \gamma \left( \frac{f(e^*; e)}{I} - c(e^*_i) \right) + (1 - \gamma) \left( \frac{f(e^*; e)}{I} - c(e^*_i) \right)
\]

If (s)he play strategy \( \bar{e} \), the payoff will be:

\[
U_i = \gamma \left( \frac{f(e; e)}{I} - c(\bar{e}) \right) + (1 - \gamma) \left( \frac{f(e; e)}{I} - c(\bar{e}) \right)
\]

Hence the extra payoff from producing \( e^* \) rather than \( \bar{e} \) is:

\[
U_i^* - U = \gamma \left[ (u(e^*_i, \bar{e}) - u(\bar{e}, \bar{e})) \right] + (1 - \gamma) \left[ u(e^*_i, \bar{e}) - u(\bar{e}, \bar{e}) \right]
\]

By assumption \( \frac{\partial u}{\partial e_i} (\bar{e}, \bar{e}, \theta) \leq 0 \), which together with concavity of payoff function \( u \) implies \( (u(e^*, \bar{e}) - u(\bar{e}, \bar{e})) < 0 \). Therefore, for \( \gamma \) sufficiently large, \( \gamma \left[ (u(e^*, \bar{e}) - u(\bar{e}, \bar{e})) \right] + (1 - \gamma) \left[ u(e^*, e) - u(\bar{e}, e) \right] < 0 \). Hence the strategy \( e^*_i \) is not a best response. It follows that for large degrees of pessimism, the only equilibrium is where all individuals play strategy \( \bar{e} \).

(2) Optimism

Consider a given sector \( i \), if (s)he plays strategy \( e^* < \check{e} \), then the payoff will be:

\[
U^* = \lambda \left( \frac{f(e^*; e)}{I} - c(e^*_i) \right) + (1 - \lambda) \left( \frac{f(e^*; e)}{I} - c(e^*_i) \right)
\]
If (s)he play strategy $e$, the payoff will be:

$$
\hat{U} = \lambda \left( \frac{f(e,e)}{I} - c(e) \right) + (1 - \lambda) \left( \frac{f(e,e)}{I} - c(e) \right)
$$

Hence the extra payoff from producing $e^*$ rather than $e$ is:

$$
\hat{U} - U^* = \lambda \left( u(\bar{e}, \bar{e}) - u(e^*, \bar{e}) \right) + (1 - \lambda) \left( u(\bar{e}, e) - u(e^*, e) \right),
$$

By assumption $\frac{\partial u_i}{\partial e_i} (e, \bar{e}, \theta) \geq 0$, which together with concavity of $p$ implies $u(\bar{e}, \bar{e}) - u(e^*, \bar{e}) > 0$. Therefore, for $\lambda$ sufficiently large, $\lambda \left( u(\bar{e}, \bar{e}) - u(e^*, \bar{e}) \right) + (1 - \lambda) \left( u(\bar{e}, e) - u(e^*, e) \right) > 0$. Hence the strategy $e^*_i$ is not the best response. It follows that for large degrees of optimism, the only equilibrium is where all individuals play strategy $\bar{e}$.

**Proposition A.1** 5.3 An increase in pessimism decreases the strategies played in the highest and lowest symmetric equilibria. Conversely, an increase in optimism increases the strategies played in the highest and lowest symmetric equilibria.

**Proof.** We have known first order condition is 5.1,

$$
\gamma \frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta) + \lambda \frac{\partial u_i}{\partial e_i} (R_i(e), \bar{e}, \theta) + (1 - \gamma - \lambda) \frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta) = 0.
$$

Now differentiated to $\gamma$, we have,

$$
\frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta) + \gamma \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), e, \theta) R'_i(e) \frac{de^*}{d\gamma}
$$

$$
+ \lambda \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), \bar{e}, \theta) R'_i(e) \frac{de^*}{d\gamma} - \frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta)
$$

$$
+ (1 - \gamma - \lambda) \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), e, \theta) R'_i(e) \frac{de^*}{d\gamma}
$$

$$
+ (1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u_i}{\partial e_i \partial e_j} (R_i(e), e, \theta) \frac{de^*}{d\gamma} = 0
$$

Rearranging,

$$
R'_i(e) \left( \gamma \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), e, \theta) \frac{de^*}{d\gamma} + \lambda \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), \bar{e}, \theta) \frac{de^*}{d\gamma} + (1 - \gamma - \lambda) \frac{\partial^2 u_i}{\partial e_i^2} (R_i(e), e, \theta) \frac{de^*}{d\gamma} \right)
$$

$$
= - \frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta) + \frac{\partial u_i}{\partial e_i} (R_i(e), e, \theta) - (1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u_i}{\partial e_i \partial e_j} (R_i(e), e, \theta) \frac{de^*}{d\gamma}
$$

then, as explained before, at the highest and lowest equilibrium, we hold the follow inequality,
\[ R'_i (e) = \]
\[
- \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) + \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) - (1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta) \frac{\partial e^*}{\partial e_i} < 1
\]

Denote \( \epsilon = \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) + \lambda \frac{\partial^2 u}{\partial e_i^2} (R_i(e), e, \theta) + (1 - \gamma - \lambda) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta) \), it is seen that \( \epsilon < 0 \) by concavity.

Thus,
\[
- \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) + \frac{\partial u}{\partial e_i} (R_i(e), e, \theta) \frac{d \epsilon}{d e^*} < 1 + \frac{(1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta)}{\epsilon}
\]

Rearranging,
\[
\frac{d e^*}{d \gamma} \cdot \frac{\epsilon}{\frac{\partial u}{\partial e_i} (R_i(e), e, \theta) - \frac{\partial u}{\partial e_i} (R_i(e), e, \theta)} > \frac{\epsilon}{\epsilon + (1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta)}
\]

Because of strategic complementarities in the game,
\[
\frac{\partial u}{\partial e_i} (R_i(e), e, \theta) > \frac{\partial u}{\partial e_i} (R_i(e), e, \theta),
\]

numerator is positive. From Eq (5.1), we know denominator
\[
\epsilon + (1 - \gamma - \lambda) (I - 1) \frac{\partial^2 u}{\partial e_i \partial e_j} (R_i(e), e, \theta)
\]
is negative.

Therefore we get, \( \frac{d e^*}{d \gamma} < 0 \)

This result indicates that facing high level of ambiguity, the effort level is decreased if people are more pessimistic;

The same proof applied to \( \lambda \), we get \( \frac{d e^*}{d \lambda} > 0 \), which means when the degree of ambiguity increase, the effort level is increased if people are more optimistic.
B Macroeconomic Coordination Games with Multi-player

This appendix contains proofs of our results in a multi-sector economy, i.e., \( I \subset R \), the aggregate output of economy is \( f(\sum e_i) \), the payoff function for sector \( i \) is \( u_i(e_i, e_{-i}, \theta_i) = \frac{f(\sum e_i)}{I} - c(e_i) \). Except the assumption that players’ beliefs are independent, all the assumptions are held as same as two-sector model.

**Proof of Proposition 5.1**

Sector \( i \)’s Choquet expected utility now is,

\[
CEU(e_i) = \gamma_i \cdot \left[ \frac{F((n-1)\tilde{e} + e_i, \theta)}{n} - c(e_i) \right] + \lambda_i \left[ \frac{F((n-1)\tilde{e} + e_i, \theta)}{n} - c(e_i) \right] \\
+ (1 - \gamma_i - \lambda_i) \cdot \\
\left[ \sum_{x1=0}^{n-2} \sum_{x2=0}^{n-2-x1} (x1, x2, n-1-x1-x2) \cdot \lambda_{j1}^{x2} \cdot \gamma_{j1}^{x1} (1 - \lambda_j - \gamma_j)^{n-1-x1-x2} \cdot \\
\frac{F(x_1 \tilde{e} + x_2 \tilde{e} + (n-1-x1-x2)e_{j1} + e_i, \theta)}{n} \right] \\
- (1 - \gamma_i - \lambda_i) \cdot c(e_i)
\]

Denote \( Z_{x1,x2}^{n-1} = (x1, x2, n-1-x1-x2) \cdot \lambda_{j1}^{x2} \cdot \gamma_{j1}^{x1} (1 - \lambda_j - \gamma_j)^{n-1-x1-x2} \)

\[
\bar{F} = \frac{F((n-1)\tilde{e} + e_i, \theta)}{n}, \quad \bar{F} = \frac{F((n-1)\tilde{e} + e_i, \theta)}{n}, \quad \bar{F} = \frac{F(x_1 \tilde{e} + x_2 \tilde{e} + (n-1-x1-x2)e_{j1} + e_i, \theta)}{n}
\]

Then

\[
CEU(e_i) = \gamma_i \cdot \bar{F} + \lambda_i \cdot \bar{F} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x1=0}^{n-1} \sum_{x2=0}^{n-1-x1} Z_{x1,x2}^{n-1} \cdot \bar{F} - c(e_i)
\]
We have first order condition:

\[ 0 = \gamma_i \cdot \frac{\partial F}{\partial e_i} + \lambda_i \cdot \frac{\partial F}{\partial e_i} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial F}{\partial e_i} - c'(e_i) \]

The same, we differentiate this FOC to \( e_j \) to get the slope of reaction function as,

\[
R'(e_j) = - \frac{(1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}{\gamma_i \cdot \frac{\partial^2 F}{\partial e_i^2} + \lambda_i \cdot \frac{\partial^2 F}{\partial e_i^2} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} - \epsilon''(e_i)}
\]

As in the case of two sectors, \( \gamma + \lambda \to 1 \), \( R'(e_j) < 1 \), which implies the existence of unique equilibrium.

**Proof of Proposition 5.2**

Differentiating first order condition with respect to \( \gamma \), (note here we assume the symmetric games, \( e_i = e_j = e \in (e, \bar{e}) \))

\[
\frac{\partial F}{\partial e_i} + \gamma_i \cdot \frac{\partial^2 F}{\partial e_i^2} R'(e) \frac{de}{d\gamma} + \lambda_i \cdot \frac{\partial^2 F}{\partial e_i^2} R'(e) \frac{de}{d\gamma} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \frac{de}{d\gamma} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2) R'(e) \frac{de}{d\gamma}
\]

\[
- \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial F}{\partial e_i} - c''(R(e_j)) R'(e_j) \frac{de}{d\gamma} = 0
\]

Rearranging,
\[
\gamma_i \cdot \frac{\partial^2 F}{\partial e_i^2} R'(e) \frac{de}{d\gamma} + \lambda_j \cdot \frac{\partial^2 F}{\partial e_i^2} R'(e) \frac{de}{d\gamma} - c''(R(e_j)) R'(e_j) \frac{de_j}{d\gamma}
\]  
\[(8)\]
\[
+ (1 - \gamma_i - \lambda_j) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2) R'(e) \cdot \frac{de}{d\gamma}
\]
\[
= \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \frac{\partial F}{\partial e_i} - \frac{\partial F}{\partial e_i} - (1 - \gamma_i - \lambda_j) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \frac{\partial^2 F}{\partial e_i^2} \frac{de}{d\gamma}
\]

We know that, at the highest and lowest equilibrium, \( R'(e_j) < 1 \), (Propostion 5.2)

and denote

\[
\epsilon = \gamma_i \cdot \frac{\partial^2 F}{\partial e_i^2} + \lambda_j \cdot \frac{\partial^2 F}{\partial e_i^2} + (1 - \gamma_i - \lambda_j) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2) - c''(R(e_j))
\]

and

\[
\delta = \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \cdot \frac{\partial F}{\partial e_i} - \frac{\partial F}{\partial e_i}
\]

Because of assumption of strategic complementarities in the game,

\[
\sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \cdot \frac{\partial F}{\partial e_i} > \frac{\partial F}{\partial e_i}
\]

we have,

\[
\delta > 0
\]

Rearrange B,

\[
R'(e_j) = \frac{\delta - (1 - \gamma_i - \lambda_j) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{-1} \cdot \frac{\partial^2 F}{\partial e_i^2} \cdot \frac{de_i}{d\gamma}}{\epsilon \cdot \frac{de_i}{d\gamma}} < 1
\]

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Hence,

\[
\frac{\delta}{\epsilon} \cdot \frac{d\gamma}{de_j} < 1 + \frac{(1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}{\epsilon}
\]

Rearranging,

\[
\frac{de_j}{d\gamma} < \frac{\delta}{\epsilon + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}
\]

we know \(\delta > 0\), and denominator is negative with denotation of \(\epsilon\).

Therefore we get, \(\frac{de_j}{d\gamma} < 0\). The same proof applied to \(\lambda\), we get \(\frac{de_j}{d\lambda} > 0\).

**Proof of Proposition 5.4**

Solve partial equilibrium response \(\frac{de}{d\theta}\) now as,

\[
\frac{de}{d\theta} = -\gamma_i \cdot \frac{\partial^2 F}{\partial e_i \partial \theta} + \lambda_i \cdot \frac{\partial^2 F}{\partial e_i \partial \theta} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial F}{\partial e_i \partial \theta}
\]

Denote

\[
\alpha = \gamma_i \cdot \frac{\partial^2 F}{\partial e_i \partial \theta} + \lambda_i \cdot \frac{\partial^2 F}{\partial e_i \partial \theta} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1-x_1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial F}{\partial e_i \partial \theta}
\]

\[
\beta = \gamma_i \cdot \frac{\partial^2 F}{\partial e_i^2} + \lambda_i \cdot \frac{\partial^2 F}{\partial e_i^2} + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1-x_1} \sum_{x_2=0}^{n-1-x_1} Z_{x_1,x_2}^{n-1} \cdot \frac{\partial^2 F}{\partial e_i^2}
\]

The equilibrium responses to an increase in \(\theta\), \(\frac{de}{d\theta}\), is:
\[
\frac{de}{d\theta} = - \frac{\beta + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} \alpha Z_{x_1,x_2} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}{eta + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} \alpha Z_{x_1,x_2} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}
\]

Then multiplier \( \mu \) now is,

\[
\mu = \frac{\beta}{\beta + (1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} \alpha Z_{x_1,x_2} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}
\]

This may be rewritten as

\[
\mu = \left(1 + \frac{(1 - \gamma_i - \lambda_i) \cdot \sum_{x_1=0}^{n-1} \sum_{x_2=0}^{n-1-x_1} \alpha Z_{x_1,x_2} \cdot \frac{\partial^2 F}{\partial e_i \partial e_j} \cdot (n - 1 - x_1 - x_2)}{\beta} \right)^{-1}
\]

It is easy to see we hold \( \mu \to 1 \) when \( \lambda + \gamma \to 1 \), \( (1 - \gamma - \lambda) \to 0 \).

**References**


