A Theory of Affirmative Action in College Admissions: An All-Pay Auction Approach

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Abstract

This paper analyzes the incentive effect of affirmative action at college admissions level in an asymmetric, complete-information, all-pay auction model. One minority and one non-minority candidate compete for a seat in a college. The admissions officer may unequally weigh candidates’ academic quality to address certain policy concern. Moderately administered preferential admissions procedure, which handicaps the non-minority candidate, may correct the negative effect of discrimination in labor market on minority candidate’s incentive to make educational effort. I show there exists a positive externality between the minority and the non-minority candidates’ incentive. As a consequence, a unique optimal policy coefficient exists that maximizes individual efforts, the aggregate effort, and the expected winning academic quality simultaneously. This result then reconciles the often assumed conflicts between diversity and academic quality in the current debate. However, I also conclude that excessive preferential treatment may potentially lead to “reverse discrimination”. In addition, I find the affirmative action admissions mechanism improves the non-minority’s incentive more than the targeted minority candidate, which tends to widen the racial gap in educational attainment.

JEL classification: D78, I21, J71
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1 Introduction

Since 1960s, as a remedy for past discrimination, affirmative action programs have been extensively administered in a wide variety of activities, such as employment, education, and government procurement. In an attempt to enhance the minority’s representation in higher education, race-based preferential admissions procedures are widely administered by selective colleges or universities. The College of Arts and Science, University of Michigan, automatically assigns 20 points (150 maximum) to a minority applicant’s score in its rating system. Harvard University has a “unofficial lift” scheme targeting at minority applicants. However, controversy about affirmative action have arisen ever since its inception. The intensity of resistance even rose in recent years. In California, Texas and Florida, legislations have already terminated the use of race-based admissions in publicly funded institutions. This debate culminated in the recent Supreme Court ruling on University of Michigan’s case, while the court endorsed admissions rules that take into account race as a qualifying characteristic.

In spite of the higher profile of the current debate, neither pros nor cons based on their rhetoric on established research findings. Very few formal studies have been conducted to investigate the effect of affirmative action on education. According to Holzer and Newmark (1999), theoretical study on the efficiency of affirmative action on education is “virtually nonexistent”. It is commonly assumed that affirmative action is merely a patronage program, which awards minorities advantage by lowering the bar and naturally conflicts with meritocracy. In the debate on affirmative action practice in college admission, one major criticism accuses affirmative action of seeking diversity or equality at the cost of academic quality. However, no conclusive
evidence has been shown to either support or refute this argument. Another major source of opposition to affirmative actions maintains that affirmation action leads to reverse discrimination against non-minorities and weakens their incentives to acquire education. This argument is not supported by convincing research findings either. Supporters of this practice tend to emphasize the importance of diversity and the positive influence of diversity on the pedagogical environment. All of these views, however, do not fully recognize the incentive structure behind affirmative action policies. It is unclear whether diversity and academic quality or education incentives have to be at odds in the first place.

The process of admissions therefore can be understood from the perspective of mechanism design. College admissions largely resemble an auction mechanism. An auction mechanism allocates a limited number of objects to bidders according to their bidding messages, while bidders pay according to certain payment rule. In the context of college admissions, to compete for a limited number of seats in the incoming class, college candidates have to present their applications to the admissions officer, while the admissions officer, like an auctioneer, allocates seats to winners based on the quality of their academic credentials. To win a seat, one has to make effort to improve her academic quality. Candidates’ effort is costly and non-refundable regardless of the outcome although the winner takes all. These features then exactly correspond to an all-pay auction framework.

This study models the process of college admissions in an all-pay auction framework. Two candidates, one minority, and the other non-minority, simultaneously choose their effort levels to compete for a seat in a college. Without corrective measures, the discrimination in labor market adversely influences the minority’s incentive to acquire education. Under an affirmative action admissions rule, the minority candidate’s academic quality weighs more than the non-minority in
the admissions officer’s rating system, which improves her incentive to expend more academic effort. I show a color-blind admission rule does not generate a color-neutral outcome, while the discrimination in labor market is preserved even at college admissions level. In contrast to previous literature, there exists a positive across group externality in college candidates’ incentive. The non-minority candidate may increase her effort in response to the higher effort made by the minority candidate. As a consequence, a pro-minority preferential admissions rule may induce both candidates to make higher efforts than they do in an environment without affirmative action. This implies diversity and academic quality need not be in conflicts. I find a unique optimal rule that maximize individual academic efforts, aggregate academic efforts, and the expected winning academic quality simultaneously. Notwithstanding its positive effects, excessive preferential treatment does create “reverse discrimination” that dampens both candidates’ incentive. In addition, I show that affirmative action improves the incentive of the non-minority more than the targeted minority candidate, which may even widens the racial gap in educational attainment and raise further concerns on policy design.

A major purpose of this paper is to provide a theoretical framework to analyze how affirmative action affects college candidates’ incentive to make academic effort. Significant racial test score gap has been widely reported by numerous empirical studies. One strand of literature attempts to explain racial gap in earning as the outcome of ability differences. Neal and Johnson (1996) show that difference in premarket factor (skills) accounts for job market differences more than discrimination. Another view, however, argues that the minority has a weaker incentive to acquire skills because they face a lower return for additional education. Phelps (1972) shows if it is costly to gather information about job applicants, the employer’s statistical experience with different groups makes the test score of one group more reliable than the other group.
Then a discrimination equilibrium exists even if the employer is “racial taste free”. Lundberg and Startz (1983) shows imperfect information yields differentials in the return to training across groups. This class of theories, the so called "statistical discrimination theory", maintain that it is harder for a skilled minority to differentiate herself from a unskilled minority, which dampens the minority’s incentive to acquire skills. As a consequence, a negative bias against the minority can be sustained as a self-fulfilling prophecy, which implies corrective policy might be necessary.

Coate and Loury (1993) investigate affirmative action in labor market by extending statistical discrimination theory in a job assignment model. They show that the negative belief of an employer on one group discourage this group to acquire skills, which, in turn, confirms the negative belief. However, if an equal assignment rate across races mandated by the government may give rise to mixed result. The minority worker may or may not invest more in human capital. In the line of Coate and Loury (1993), Benoit (1998) shows discrimination can be preserved even under a color-blind assignment rule, if the test score is biased against members of the socioeconomically disadvantaged group. He also shows that a rational firm has no incentive to correct the discrimination on its own, thus affirmative action may have to be enforced. Moro and Norman (2003) considers a fully competitive economy. They show there exists a negative externality between groups. When the fraction of skilled workers in one group increases, the incentive of the other group to invest for skills will be decreasing. They conclude the affirmative may increase the minority’s incentive. All these frameworks, however, do not address the winner-take-all feature typical of an admissions mechanism. A recent study by Fryer and Loury (2003) shares substantial insights with this paper. They use a tournament model to investigate the categorical redistribution in a winner-take-all market. It is shown the optimally designed
tourneyrnent that maximizes the winner’s effort naturally involves handicapping.

This paper is organized as follows. Sections 2 set up the model. Section 3 shows the equilibrium outcome of the model. Section 4 discusses the endogenous optimal policy design and evaluate preferential admissions against several criticisms. Section 6 presents a concluding remark.

2 A Model of College Admissions

This model involves two college candidates indexed by $i = M, N$, who compete for one seat at a college. One candidate, $M$, is minority, while the other candidate, $N$, is non-minority. Candidate $i$ values her college education at $V_i \in (0, \infty)$. The value of $V_i$ represents candidate $i$’s marginal return of college education, which is common knowledge.

The role of college admissions officer is to scrutinize candidates’ application profiles, and award the seat to one of them. The admissions decision is primarily based on the quality candidate $i$’s academic credentials $q_i$, i.e. SAT score and high school GPA etc. On the other hand, to address certain policy concerns, the admissions officer may take into account characteristics other than academic qualification, e.g. race, gender, alumni relationship, and athletic expertise. She may assign a positive weight $\alpha_i \in (0, \infty)$\footnote{Some institutions adopt an alternative preferential admissions rule, which assigns additional score to certain groups of candidates. I model this rule in the Appendix A.} to candidate $i$’s academic quality. As a consequence, candidate $i$ receives a rating $\alpha_i q_i$ in the admissions officer’s assessing system. Candidate $i$ is admitted if and only if her rating is higher than her rival, i.e. $\alpha_i q_i > \alpha_j q_j$. In case $\alpha_i q_i = \alpha_j q_j$, the seat is randomly assigned to one candidate with equal chance.

I normalize the weight on candidate $N$’s academic quality $q_N$ to be unity, and the weight on candidate $M$’s academic quality $q_M$ to be $\alpha = \frac{\alpha_M}{\alpha_N}$, $\alpha \in (0, \infty)$. In the first stage of the
game, the admissions officer strategically chooses the optimal policy coefficient $\alpha$ to achieve her policy objective. A college may pursue diverse interests. In the first place, a college prefers a high quality incoming class. Secondly, it also desires a diversified student body.

Apparently, the size of the policy coefficient $\alpha$ represents the relative stand of the minority candidate in the admission contest. When $\alpha < 1$, the admissions officer’s rating system is perversely biased against the minority candidate. When $\alpha > 1$, the minority candidate’s academic quality weighs relatively more than that of her non-minority rival, which corresponds to an environment where the minority receives preferential treatment to some extent. When $\alpha = 1$, no characteristics other than academic quality makes difference in the admissions decision, which represents a “Color-blind” admission scheme.

In the second stage, upon observing the rule stipulated by the college, candidate $i$ has to choose nonrefundable academic effort $e_i \in [0, V_i]$, to improve the quality (score) of her academic credentials. I assume $q_i$ is a continuous and increasing of function of her academic effort $e_i$. I further assume there is no innate ability difference, in the sense that these two candidates have identical academic quality production technology. Without loss of generality, I assume a linear production function of $q_i$, given by $q_i = e_i$. These two candidates simultaneously choose their own academic effort level to maximize their payoffs. Higher effort creates higher academic quality, which enhances a candidate’s chance of admissions. On the other hand, academic effort
is costly and generates disutility. Their expected payoff functions are given as follows.

\[
\pi_M = \begin{cases} 
V_M - e_M & \text{if } \alpha e_M > e_N \\
\frac{1}{2}V_M - e_M & \text{if } \alpha e_M = e_N \\
-e_M & \text{if } \alpha e_M < e_N 
\end{cases} 
\]

(1)

\[
\pi_N = \begin{cases} 
V_N - e_N & \text{if } e_N > \alpha e_M \\
\frac{1}{2}V_N - e_N & \text{if } e_N = \alpha e_M \\
-e_N & \text{if } e_N < \alpha e_M 
\end{cases} 
\]

(2)

The value of \(V_i\) plays a crucial role in this model. Because of the existence, or the perception, of persistent discrimination in the labor market, I assume the minority candidate expects a lower return for higher education than her non-minority rival. Thus candidate \(M\) has a lower valuation on the seat in the college, i.e. \(V_N > V_M > 0\). Alternatively, this model reflects another aspect of inequality. The differential marginal return to higher education may also arise if the minority and the non-minority bear a higher marginal cost in learning. For instance, the past discrimination may create additional obstacle for the minority to achieve academic excellence. One may also argue the test score is biased against the minority.

3 The Equilibrium Outcome of the Admissions Contest

In this section, I characterize the equilibrium outcome of the admissions contest. The equilibrium of standard complet-Information all-pay auction has been thoroughly investigated by Hillman and Riley (1989), and Baye, Kovenock and de Vries (1996). This study adopts a line of reasoning similar to these two studies. I first establish the following preliminary results that

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2 According to Learman (1997), from 1984 to 1995, the percentage wage differential between white and black male high school graduates decreased from 24.6% to 8.3%. the percentage differential between white and black male college graduate or postgraduate decreases from 26.7% to 18.0%. This result implies that during this time period, the across group wage gap at college or above level was widened relatively to that at high school graduate level.
sketch the basic properties of this game.

**Lemma 1** There exists no pure strategy equilibrium in the admissions contest.

**Lemma 2** Both candidates must be active, in the sense that both of them make positive efforts with positive probability.

Let \( F_M = F_M(e_M) \) and \( F_N = F_N(e_N) \) be candidate \( M \) and \( N \)'s equilibrium effort distribution functions respectively in a mixed strategy equilibrium.

**Lemma 3** \( F_M \) and \( F_N \) must have zero lower supports.

**Lemma 4** \( F_M \) and \( F_N \) must be atomless above their lower supports.

**Lemma 5** \( F_M \) and \( F_N \) must be strictly increasing above the lower support.

Lemma 1 is a standard property of a complete-information all-pay auction. If contestants play pure strategies in equilibrium, the outcome is also determined. Given the loser’s strategy, the winner always wants to reduce its effort level to lower the cost, while the loser either invests nothing to avoid loss, or invests more to prevail against the winner. Because the continuity in payoff, no pure strategy equilibrium can be reached. Lemma 2 is a natural corollary of Lemma 1, since making zero effort with probability one is a pure strategy. Lemma 3 to 5, instead, sketch the outline of the mixed equilibrium of this all-pay auction.

The property of the equilibrium hinges on the inequality in candidates’ valuations and the value of policy coefficient \( \alpha \). When \( \alpha < \frac{V_N}{V_M} \), candidate \( N \) always has advantage against candidate \( M \), whereas when \( \alpha < \frac{V_N}{V_M} \), candidate \( M \) is on a better footing relative to candidate \( N \). I will show \( \frac{V_N}{V_M} \) is a critical value which connects a two-part equilibrium. To ease the notation in future analysis, I define \( \theta \) to be \( \theta = \frac{V_N}{V_M} > 1 \).

### 3.1 Equilibrium when \( \alpha \in (0, \theta] \)

To find the equilibrium, I first establish the following results.
Lemma 6 When $\alpha \in (0, \theta]$, in a mixed strategy equilibrium, $F_M$ and $F_N$ must have upper supports $V_M$, and $\alpha V_M$, respectively.

Lemma 7 When $\alpha \in (0, \theta]$, in a mixed strategy equilibrium, candidate $M$ has an expected payoff of zero, while candidate $N$ has an expected payoff of $V_N - \alpha V_M$.

Lemma 6 is intuitively straightforward. Since candidate $M$ will never make an effort higher than her own valuation $V_M$, candidate $N$ can ensure her winning by making an effort $e_N = \alpha V_M$. Any effort higher than the upper support will not be a part of equilibrium strategy. By this reasoning, it naturally follows that candidate $N$ always earns a positive expected payoff in equilibrium as long as $\alpha < \theta$, which is stated in Lemma 7. In contrast, by Lemma 7, when $\alpha$ falls in this interval, candidate $N$ always receives zero expected payoff. The differentials in payoff implies that candidate $N$ is extracts a rent from her advantage created by her higher return of additional education.

Proposition 1 When $\alpha \in (0, \theta]$, there exists a unique Nash equilibrium, where candidate $N$ continuously randomizes effort over the whole support $[0, \alpha V_M]$, while candidate $M$ continuously randomizes effort over the support $(0, V_M]$, and places probability mass at zero with a size of $\frac{V_N - \alpha V_M}{V_N}$. Distribution functions of candidate $M$ and $N$ are given by,

$$F_M(e_M) = \frac{V_N - \alpha V_M + \alpha e_M}{V_N}, \text{ and,}$$

$$F_N(e_N) = \frac{e_N}{\alpha V_M} \tag{3} \tag{4}$$

Proposition 1 states that when $\alpha \in (0, \theta)$, the minority candidate always has a nonnegative probability of staying inactive, i.e. dropping out of the schooling system.

Corollary 1 Candidate $M$’s probability of “dropout” decreases with $\alpha$ when $\alpha \in (0, \theta)$, and is reduced to zero when $\alpha = \theta$.

A larger $\alpha$ reduces candidate $N$’s relative advantage and improves candidate $M$’s relative stand, which creates an incentive for candidate $M$ to expend more effort and not to drop out of school. This result implies that preferential treatment enlarges the pool of active candidates.

The main algebraic result in this equilibrium is summarized in Table 1. Expected aggregate
effort is defined as the sum of the two candidates’ expected effort, i.e. \( E(e_1 + e_2) \). Thanks to the linear quality production function, the expected winning quality is equivalent to the expected effort made by the winner.

**Table 1: Summary of Equilibrium Results when \( \alpha \in (0, \theta] \)**

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Expected Probability of Winning</th>
<th>Expected Effort</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candidate N</td>
<td>( \frac{2V_N - \alpha V_M}{2V_M} )</td>
<td>( \frac{\alpha V_M}{V_N} )</td>
</tr>
<tr>
<td>Candidate M</td>
<td>( \frac{\alpha V_M}{V_N} )</td>
<td>( \frac{\alpha V_M^2}{2V_N} )</td>
</tr>
<tr>
<td>Aggregate Effort</td>
<td>( \frac{\alpha V_M (V_M + V_N)}{2V_N} )</td>
<td></td>
</tr>
<tr>
<td>Winning Quality</td>
<td>( \frac{V_M}{V_N} \cdot \left[ \frac{\alpha (V_N - \alpha V_M)}{2} + \frac{\alpha V_M}{3} + \frac{\alpha^2 V_M}{3} \right] )</td>
<td></td>
</tr>
</tbody>
</table>

Let \( P_M = \Pr(\alpha e_M > e_N) \) be candidate M’s expected probability of winning, and \( P_N = \Pr(e_N > \alpha e_M) \) candidate N’s expected probability. By the result summarized in Table 1, I obtain the following results.

**Proposition 2** (1) \( P_M < P_N \), when \( \alpha \in (0, \theta) \), while, \( P_M = P_N \), when \( \alpha = \theta \);
(2) when \( \alpha \in (0, \theta] \), \( P_M \) monotonically increases with \( \alpha \);
(3) when \( \alpha \in (0, \theta] \), both candidates’ expected efforts monotonically increase with \( \alpha \);
(4) when \( \alpha \in (0, \theta] \), the expected winning score monotonically increases with \( \alpha \).

When \( \alpha \in (0, \theta] \), \( \alpha V_M \leq V_N \). Candidate N possesses an advantage against her minority rival, which results in a \( P_N \in \left( \frac{1}{2}, 1 \right) \) unless \( \alpha = \theta \). In contrast, Proposition 2 (2) shows that a more level playfield does enhance the representation of the minority candidate. By Proposition 2 (3), with the increase in \( \alpha \), both candidates tend to increase their expected efforts. The increase in \( \alpha \) affects the expected winning score in two ways. A larger \( \alpha \) stimulates both candidates to expend higher efforts, which tends to raise the winning score. In contrast, a larger \( \alpha \) increases the likelihood that a lower-scored minority candidate is admitted, which
tends to reduce the score. However, in this coefficient range, the former effect dominates the latter, which leads to Proposition 2 (4).

### 3.2 Equilibrium when $\alpha \in [\theta, \infty)$

In this part, I study the equilibrium allocations when $\alpha \geq \theta$.

**Lemma 8** When $\alpha \in [\theta, \infty)$, in a mixed strategy equilibrium, $F_M$ and $F_N$ must have upper supports $\frac{V_N}{\alpha}$, and $V_N$, respectively.

**Lemma 9** When $\alpha \in [\theta, \infty)$, in a mixed strategy equilibrium, candidate $M$ has an expected payoff of $V_M - \frac{V_N}{\alpha}$, while candidate $N$ has an zero expected payoff.

In contrast to the case $\alpha \in (0, \theta]$, when $\alpha \geq \theta$, $\alpha V_M > V_N$, which gives the minority candidate an advantage against the non-minority candidate. The non-minority candidate affords no effort level higher than her own valuation $V_N$. In contrast, by choosing $\alpha V_M$, the minority candidate ensures her winning while receives a positive payoff $V_M - \frac{V_N}{\alpha}$. In this case, the minority candidate extracts positive rent from her advantage created by the excessively large policy coefficient $\alpha$.

**Proposition 3** When $\alpha \in [\theta, \infty)$, there exists a unique Nash equilibrium, where candidate $M$ continuously randomizes effort over the whole support $[0, \frac{V_N}{\alpha}]$, while candidate $N$ continuously randomizes effort over the support $[0, V_N]$, and places probability mass at zero with a size of $\frac{V_M - \frac{V_N}{\alpha}}{V_M}$. Distribution functions of candidate $M$ and $N$ are given by,

\[
F_N(e_N) = \frac{V_M - \frac{V_N}{\alpha} + \frac{e_N}{\alpha}}{V_M}, \text{ and,} \\
F_M(e_M) = \frac{\alpha e_M}{V_N}
\]

**Corollary 2** Candidate $N$’s probability of “dropout” equals zero when $\alpha = \theta$, and increases with $\alpha$ when $\alpha > \theta$.

In contrast to the preceding case, when $\alpha \in [\theta, \infty)$, which gives non-minority candidate a non-negative probability of staying inactive. This probability is positive except when $\alpha = \theta$. The size of this probability mass is also a continuous function of the policy coefficient $\alpha$, but
monotonically increases with $\alpha$. This implies that admissions policy awards exceedingly large advantage to the minority candidate, which drives the initially advantageous non-minority candidate to drop out of the schooling system. The algebraic results of this case are summarized in Table 2. By the result presented in Table 2, I obtain the following results.

**Proposition 4**

1. $P_M > P_N$, when $\alpha \in (\theta, \infty)$, while, $P_M = P_N$, when $\alpha = \theta$;
2. when $\alpha \in [\theta, \infty)$, $P_M$ monotonically increases with $\alpha$;
3. when $\alpha \in [\theta, \infty)$, both candidates’ expected efforts monotonically decrease with $\alpha$;
4. when $\alpha \in [\theta, \infty)$, the expected winning score monotonically decreases with $\alpha$.

Proposition 4 suggests that when the policy coefficient $\alpha$ exceeds the critical value $\theta$, the minority candidate is *ex ante* more likely to be admitted. This setting therefore corresponds to a "reverse discrimination". In addition, it dampens both candidates’ incentive of making effort. Because the minority candidate’s advantage has been predominantly large, the minority candidate is able to win the seat without making high effort. On the other hand, the increase in $\alpha$ lowers the chance of the non-minority candidate, which weakens her incentive to make effort. In addition, in response to the decreased effort made by the non-minority candidate, the minority candidate will also strategically lower her effort to minimize her cost. All these interactions contribute to the results of Proposition 4.

**Table 2: Summary of Equilibrium Results when $\alpha \in [\theta, \infty)$**

<table>
<thead>
<tr>
<th>Candidate $N$’s expected probability of winning</th>
<th>$\frac{V_N}{2\alpha V_M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candidate $M$’s expected probability of winning</td>
<td>$\frac{2\alpha V_M - V_N}{2\alpha V_M}$</td>
</tr>
<tr>
<td>Candidate $N$’s expected effort</td>
<td>$\frac{V_N^2}{2\alpha V_M}$</td>
</tr>
<tr>
<td>Candidate $M$’s expected effort</td>
<td>$\frac{V_N}{2\alpha}$</td>
</tr>
<tr>
<td>The expected aggregate effort</td>
<td>$\frac{V_N}{2\alpha V_M} \cdot (V_M + V_N)$</td>
</tr>
<tr>
<td>Expected winning quality</td>
<td>$\frac{V_N}{\alpha V_M} \cdot \left[ \frac{1}{2} \alpha V_M - V_N + \frac{V_N}{3} + \frac{\alpha V_N}{3} \right]$</td>
</tr>
</tbody>
</table>
3.3 A Bayesian Interpretation of mixed-strategy

Having high school student randomize her academic effort may sound controversial. According to the classical view on mixed strategy equilibrium, players in a game play a mixed strategy in order to deliberately conceal their actions. This interpretation loses its bite in the context of college admissions. A Bayesian view on mixed strategy can be a rationale that strikes the note. According to Harsanyi (1973), almost all mixed strategy equilibrium can be approximated by the equilibrium of a nearby game with perturbed payoff. The true realization of the payoff is private information of each player. Therefore, a player is uncertain about her rival’s type and her action, although she is definitely making a pure strategy choice to each realization of the perturbed payoff.

Hence, the equilibrium in this model can be simply interpreted in the light of the Harsanyi’s purification theorem. In reality, candidates may agree on the inequality in marginal return of additional education between the minority and the non-minority. However, the true value can be subject to variation. This uncertainty leads candidates to behave as if they follow some distributions in the eye of their opponents. The mixed strategy equilibrium which is solved in a complete information setting is actually the approximation of equilibria of incomplete information games where types of players are not fully revealed.

3.4 A Positive Externality

This model brings forth a different flavor than models based on conventional statistical discrimination theory. In Coate and Loury (1993), the effect of affirmative action on the minority’s incentive of acquiring human skills is mixed. The minority who benefits from the patronage program may have a weaker incentive to invest in skills. In contrast, Moro and Norman (2003)
shows affirmative action creates an cross-group externality. Moro and Norman (2003) suggests that in any group of workers, the incentive of acquiring skills is weakened if the fraction of skilled worker in the other group is increasing. Hence, they conclude that the discriminated group have a stronger incentive to invest for skills, while the initially dominant group have a weaker incentive, when an affirmative action policy is being introduced.

This model, however, suggest another type of externality, i.e. a positive “cross-group” externality. By Proposition 2, both candidates raise their average effort with the increase in $\alpha$ when $\alpha \in (0, \theta]$. This result reflects two effects of the affirmative action on candidates’ equilibrium efforts. One is a direct effect. Intuitively, a larger $\alpha$ generates a larger probability of winning for the minority candidate, which increases her marginal return of academic effort. Consequently, the minority candidate increases her equilibrium effort level. The other effect is an indirect effect. In response to the higher effort made by the minority candidate, the minority candidate has to increase her effort level as well in order to win this seat. In short, the preferential admissions rule strengthens both candidates’ incentive to make academic effort. In this sense, affirmative action should not be understood as merely a redistributational instrument, but a powerful incentive mechanism. This result then reveals an import aspect of the incentive nature of affirmative action, which has been ignored in the current debate and previous literature.

4 Discussion

The equilibrium result presented in Section 4 allows me to analyze the policy choice made by the admissions choice directed towards certain interest, and its incentive impact on educational performance. To establish a benchmark for the analysis on affirmative action program, I first
examine a case where the admissions officer makes her decision purely based on candidates’ credentials, i.e. a “color-blind” admissions rule.

4.1 A Benchmark Case: Color-Blind Admissions Rule

In this circumstance, the admissions officer does not take into account the identity of applicants, but concerns only candidates’ academic qualifications. Consequently, the applicant who makes higher effort than her rival must be admitted. This setting therefore corresponds to a standard complete-information all-pay auction with asymmetric valuations. Inserting $\alpha = 1$ to the results presented in Table 1, I have the main algebraic results of this case, which are summarized in Table 3.

<table>
<thead>
<tr>
<th>Table 3: Equilibrium Results under Color-Blind Admissions Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>candidate $N$’s expected probability of winning</td>
</tr>
<tr>
<td>$\frac{2V_M - V_N}{2V_M}$</td>
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<tr>
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<tr>
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<tr>
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</tr>
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</tr>
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</tr>
<tr>
<td>$\frac{V_N^2}{2V_N}$</td>
</tr>
<tr>
<td>the expected aggregate effort</td>
</tr>
<tr>
<td>$\frac{V_M(V_M + V_N)}{2V_N}$</td>
</tr>
<tr>
<td>expected winning quality</td>
</tr>
<tr>
<td>$\frac{V_M}{V_N} \cdot \left[ \frac{3V_N + V_M}{6} \right]$</td>
</tr>
</tbody>
</table>

By Proposition 1, in the mixed-strategy equilibrium, the two candidates, $N$ and $M$, randomly choose their effort levels according to distribution functions $F_N(e_N) = \frac{e_N}{V_N}$, and $F_M(e_M) = \frac{V_N - V_M + e_M}{V_N}$, respectively. The non-minority candidate earns a positive expected payoff, $V_N - V_M$, while the non-minority candidate earns zero on average. The non-minority candidate fully extracts the rent created by the inequality in labor market. By Proposition 1, the result shows that the discriminated minority candidate places positive probability mass at zero. In addition,
by Proposition 2, the non-minority candidate, on average, is *ex ante* more likely to be admitted than the minority counterpart.

The equilibrium outcome in this case implies that, even if the test score is not biased against the minority candidate, discrimination in the labor market can be preserved at the college admissions level without corrective policy intervention. This model echoes Coate and Loury (1993)’s concern that the absence of corrective measures may shrink the pool of qualified job candidates. The voluntary “dropout” of the minority candidate lowers the average education attainment of the minority, which, in turn, may reinforce the existing bias against the minority.

One criticism against affirmative action contends that affirmative action creates reverse discrimination in the sense it makes a lower scoring minority candidate “leapfrog” a higher scoring non-minority candidate. According to this view, candidates of comparable academic quality should have equal chance of being admitted. But nonetheless, this analysis shows that a color-blind admissions rule does not suffice this end either.

**Proposition 5 (Color-blind is not color-neutral)** Under a color-blind college admissions rule, for any effort level $e < V_M$, the minority candidate has a smaller probability of winning against her non-minority rival, i.e. $P_M(e) < P_N(e)$, $\forall e \in [0, V_M)$.

This result is fairly intuitive. Because candidate $N$’s marginal return of effort dominates candidate $M$ for any effort level $e \in [0, V_M)$, candidate $N$’s distribution function dominates candidate $M$’s distribution function. Proposition 5 states that a color-blind admissions procedure does not generates a color-neutral outcome. The minority candidate is less likely to be admitted than an equally qualified non-minority candidate. Hence, a corrective policy is worthy of consideration. Justice Harry Blackmun, in his writing on the Bakke case, contends that “in order to get beyond racism, we must first take account of race. There is no other way”. The above analysis formalizes Justice Blackmun’s argument.
4.2 The Admissions Officer’s Choice

In this subsection, I focus on the admissions officer’s policy choice. A college admissions officer may concern both the diversity and the academic quality of the entering class. At the first stage, she chooses the optimal coefficient \( \alpha \) to design the admissions mechanism which yields the most desirable outcome towards her interests. The analysis in this subsection answers the following questions: (1) what is the optimal policy coefficient to serve certain policy concerns? (2) does the goal of diversity necessarily compromise the academic quality?

The debate often centers on the tension between diversity and academic quality. These two goals are seemingly conflicting with each other. However, as shown above, a more level playfield may strengthen both candidates’ incentive to make educational efforts. These two objectives may instead be balanced in an appropriately designed policy environment. By Proposition 2, any coefficient \( \alpha \in (1, \theta] \) results in a more diversified student body and a higher winning score than otherwise.

Looking at Proposition 1 and 3, it is straightforward to see that equilibria over the two intervals are connected at the point \( \alpha = \theta = \frac{V_N}{V_M} \). When \( \alpha = \theta \),

\[
\frac{V_N - \alpha V_M + \alpha e_M}{V_N} = \frac{e_M}{V_M} = \frac{\alpha e_M}{V_N} = \frac{e_M}{V_M} = \frac{V_M - \frac{V_N}{\alpha} + \frac{e_N}{\alpha}}{V_M}
\]

When \( \alpha = \theta \), both candidates remain active with probability one.

**Proposition 6** When \( \alpha = \theta \), there exists a unique Nash equilibrium, where candidate \( N \) continuously randomizes effort over the whole support \([0, V_N]\), while candidate \( M \) continuously randomizes effort over the support \([0, V_M]\). Distribution functions of candidate \( M \) and \( N \) are given by,

\[
F_M(e_M) = \frac{e_M}{V_M}
\]

\[
F_N(e_N) = \frac{e_N}{V_N}
\]
The equilibrium results of this case are summarized in Table 4. By the results, I have the
following proposition.

**Proposition 7**  (1) $\alpha = \theta$ is the unique policy coefficient that creates equal probability of winning;
(2) when $\alpha = \theta$, both candidates’ expected efforts are maximized;
(3) when $\alpha = \theta$, the expected aggregate effort is maximized;
(4) the expected winning score is maximized.

By Proposition 6, $\theta$ is the connecting point of the equilibria over the two coefficient range.

Hence, by Proposition 3 and 5, the claim of Proposition 7 is straightforward.

<table>
<thead>
<tr>
<th>Table 4: Equilibrium Results when $\alpha = \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>candidate N’s expected probability of winning</td>
</tr>
<tr>
<td>candidate M’s expected probability of winning</td>
</tr>
<tr>
<td>candidate N’s expected effort</td>
</tr>
<tr>
<td>candidate M’s expected effort</td>
</tr>
<tr>
<td>the expected aggregate effort</td>
</tr>
<tr>
<td>expected winning quality</td>
</tr>
</tbody>
</table>

Apparently, to achieve either, or both, of these two objectives, the college admissions officer will choose $\alpha = \theta$. First of all, the equal expected probability of winning between two candidates responds to the objective of equal representation. Second, this coefficient generates the highest winning academic quality, thus it creates an entering class of the highest academic quality. As a consequence, these two objectives are achieved simultaneously. Even if the college does not concerns the racial composition of its student body, it may as well adopt a preferential admissions rule in an attempt to enhance its selectivity. This result indicates that meritocracy is still upheld and compatible with the diversity concern, which may rationalize the persistent and widespread practice of preferential admissions.
4.3 Public Education Authority’s Interest

According to this model, the public education authority may also benefit from the practice of affirmative action. The above analysis suggests that affirmative action may strengthen both candidates’ incentive to make academic effort. The expected aggregate effort level is boosted to the largest extent if \( \alpha \) is set at \( \theta \). In addition, with this policy coefficient, neither candidate places positive probability at zero effort. Put it differently, neither of them voluntarily drops out of the schooling system. Hence, this result unequivocally indicates that an appropriately designed preferential admission scheme boosts the aggregate educational output.

4.4 Overshooting

Notwithstanding its positive effect, affirmative action admissions may adversely affect candidates' incentive if the minority candidate is given an excessively large advantage. As shown in Proposition 3, when \( \alpha > \theta \), candidate \( N \) has a lower chance of being admitted than candidate \( M \). In response to the lower return of academic effort, candidate \( N \) increases the size of probability mass she places at zero effort, while the upper support of \( e_M \) is monotonically decreasing with \( \alpha \). This result indicates that affirmative action does warrant the concern of “reverse discrimination”. Consequently, to administer an affirmative action policy, the policy maker must be highly cautious of the potential overshooting, which dampens both candidates’ incentive to make academic effort.

Look at Table 2 and Table 4. I set the expected effort level in the color-blind case as a benchmark. When \( \alpha = \frac{V_N^2}{V_M^2} \), candidate \( N \)'s expected effort \( E(e_N) = \frac{V_N^2}{2\alpha V_M} = \frac{V_M}{2} \), while candidate \( M \)'s expected effort \( E(e_M) = \frac{V_N}{2\alpha} = \frac{V_M}{2V_N} \). Both candidates choose the same effort levels as they do in the Benchmark case. This result indicates that the affirmative action rule
generates a higher aggregate effort level than the color-blind case when \( \alpha \) falls in the interval between 1 and \( \frac{V_M^2}{V_N^2} \). However, whenever \( \alpha \) exceeds \( \frac{V_M^2}{V_N^2} \), the resulted effort level will be even lower than the color-blind case, where no corrective measures is implemented.

### 4.5 Does Affirmative Action Improve the Minority’s Relative Stand?

The central intent of affirmative action is to level the playfield and improve the relative stand of the minority. By the above analysis, preferential admissions does enhance the minority representation in higher education. However, it remains to investigate if this policy indeed narrows the racial gap in educational attainment. Steele (1990) contends that affirmative action offers “blacks only entitlement but no development”, and makes them “stand to lose more from it than they gain”. Another interesting criticism is wrong-fit argument. This view argues that preferential admissions rule matches students in wrong places, in the sense that a less prepared minority candidate is placed in an environment where she is unable to compete with her peers. Both of these two arguments, however, are not supported by a set of established research findings. In the following analysis, I evaluate the affirmative action admissions mechanism against these two argument.

#### 4.5.1 Entitlement vs. Development

To compare the effect of affirmative action on the minority candidate’s education attainment relative to the non-minority candidate, this part focuses on the equilibrium outcome when \( \alpha \in [1, \theta] \). Look at Table 1. Define \( \Delta e = E(e_N) - E(e_M) = \frac{\alpha V_M}{2V_N} \cdot (V_N - V_M) \) to be the difference in average academic quality between the non-minority candidate and the minority candidate.

**Remark 1** \( \Delta e \) is always positive and strictly increasing with \( \alpha \) over the interval \([1, \theta]\).
The above observation indicates that although affirmative action increases the incentive of both candidates to make effort, the gap in academic quality between the non-minority candidate and the minority candidate is even widened by affirmative action procedure. Look at the marginal effect of the policy coefficient on each candidate’s expected effort. \( \frac{\partial E(e_M)}{\partial e_M} = \frac{V_M^2}{2V_N} \), while \( \frac{\partial E(e_N)}{\partial e_N} = \frac{V_M}{2} \). Obviously, \( \frac{\partial E(e_M)}{\partial e_M} = \frac{V_M}{2} \cdot \frac{V_M}{V_N} < \frac{V_M}{2} \), which implies the preferential admissions rule has a stronger marginal effect on candidate N’s incentive. Although the minority candidate is the directed beneficiary of affirmative action, the preferential rule, nevertheless, stimulates non-minority more than the minority candidate. This analysis does not fully support Steele’s charge, because the affirmative action policy does induce the minority to develop herself. However, affirmative action does not completely defend away the concern on the widen racial gap in educational attainment. In short, affirmative action does offer “development” to the minority, but the development is inadequate.

### 4.5.2 Wrong-fit Argument

The wrong-fit argument states that preferential admissions program mismatches the minority student to an environment where she is unable to survive. To evaluate this argument, one needs to compare the academic quality of a selected minority candidate with a selected non-minority candidate. In the current framework, this argument may find its support to the extent the academic quality of a selected minority candidate is falling relative to a selected non-minority candidate.

I first find candidates’ truncated effort distribution functions conditional on being admitted, \( F_M(e_M | \alpha e_M > e_N) \), and \( F_N(e_N | e_N > \alpha e_M) \), respectively. Again, I focus on the coefficient
I have,

\[ F_M(e_M|\alpha e_M > e_N) = \frac{\Pr(e_N < \alpha e, \alpha e < \alpha e_M)}{\Pr(\alpha e_M > e_N)} = \frac{\int_0^{e_M} \int_0^e \frac{1}{\alpha e_M V_N} d\alpha d\epsilon}{\int_0^{e_M} \int_0^e \frac{1}{\alpha \epsilon M V_N} d\alpha d\epsilon} = \frac{e_M^2}{V_M^2} \]  

and

\[ F_N(e_N|e_N > \alpha e_M) = \frac{\Pr(e_N < e_N, \alpha e_M < e_N)}{\Pr(\alpha e_M < e_N)} = \frac{e_N(V_N-\alpha V_M)}{2V_N} + \int_0^{e_N} \int_0^e \frac{1}{\alpha V_M V_N} d\alpha d\epsilon de = \frac{e_N^2 + 2e_N(V_N-\alpha V_M)}{\alpha V_M (2V_N-\alpha V_M)} = \frac{2V_M}{3} \]

By (11) and (12), I can easily find the corresponding truncated expected score of a selected minority or non-minority candidate, which is presented in the following table.

**Table 5: Expected equality of a Selected Candidate when \( \alpha \in [1, \theta] \)**

<table>
<thead>
<tr>
<th></th>
<th>( \frac{2V_M}{3} )</th>
<th>( \frac{\alpha V_M (V_N-\alpha V_M)}{2V_N-\alpha V_M} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>expected quality of a selected minority candidate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>expected quality of a selected non-minority candidate</td>
<td></td>
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</tr>
</tbody>
</table>

Let \( \bar{e}_M = E(e_M|\alpha e_M > e_N) = \frac{2V_M}{3} \) be the expected score of a selected minority candidate, and \( \bar{e}_N = E(e_N|e_N > \alpha e_M) = \frac{\alpha V_M (V_N-\alpha V_M)}{2V_N-\alpha V_M} \) the expected score of a selected non-minority candidate.

**Proposition 8** When \( \alpha \in [1, \theta] \),

1. the expected score of a selected minority candidate is independent of the policy coefficient \( \alpha \);
2. the expected score of a selected non-minority candidate increases with the policy coefficient \( \alpha \).

The first part of Proposition 8 indicates that the average academic quality of the selected minority candidate does not vary with the policy coefficient \( \alpha \). Intuitively, \( \alpha \) influences the
expected score of the selected minority candidate in two ways. On one hand, a larger \( \alpha \) encourages the minority candidate to expend higher effort, which tends to shift \( \widehat{e}_M \) upward. On the other hand, with the increase of \( \alpha \), the minority candidate faces a lower standard for the college, which makes it more likely for a lower-scored minority candidate to be admitted. This effect tends to reduce the average academic quality of the selected minority candidate. These two effects function in opposite directions and exactly offset each other, which makes \( \widehat{e}_M \) a mean-preserving spread. In contrast, the expected score of a selected non-minority candidate monotonically increases with \( \alpha \). The increase in \( \widehat{e}_N \) stems from two sources. First, a larger \( \alpha \) makes a non-minority candidate expend more effort. Second, the preferential admissions rule makes it harder for a low-scored non-minority candidate to be admitted. Both of these two effects tend to lift \( \widehat{e}_N \).

Define \( \Delta \widehat{e} = \widehat{e}_N - \widehat{e}_M \) to be the difference in the expected academic quality between the selected non-minority candidate and the selected minority candidate.

**Proposition 9**

1. Under a color-blind admissions rule, \( \Delta \widehat{e} < 0 \) (re-interpretation of Proposition 5);
2. under the optimal affirmative action admissions rule, \( \Delta \widehat{e} > 0 \);
3. \( \Delta \widehat{e} \) is increasing with \( \alpha \), and is maximized under the optimal affirmative action admission rule.

Proposition 9 (1) represents the same intuition as Proposition 5. It implies that, to achieve the equivalent outcome, i.e. winning the seat in the college, the minority candidate has to expend higher effort than her non-minority counterpart. In contrast, Proposition 9 (2) shows that under the optimal rule, the academic quality of a minority college student is on average lower than a non-minority student. Proposition 9 (3) indicates the academic quality of a minority college student indeed falls against her non-minority counterpart.

By Proposition 9, it follows that there must exist a unique critical value \( \bar{\alpha} \) between 1 and \( \theta \),
which equalizes the expected academic quality of a selected minority candidate and a selected non-minority candidate. However, at \( \tilde{\alpha} \), the minority candidate still has a smaller probability of winning than the non-minority candidate. Whenever \( \alpha > \tilde{\alpha} \), a minority college student, on average, becomes less qualified in academic quality than a non-minority student, and the gap is being widened with the increase in \( \alpha \). This finding may explain the rising failure rate of black students, which underpins the “wrong-fit” argument. However, the racial gap in academic quality should not be attributed to affirmative action alone. Note that the effort level of the minority candidate is always bounded from above by the lower valuation \( V_M \). The differential return to higher education is the essential driving force of the widen gap.

5 Concluding Remark

This study provides a theoretical framework to investigate the incentive effect of affirmative action practice in college admissions. Although diversity is the most claimed goal of education policy makers, this study shows that diversity is not the sole objective affirmative action may serve. The above analysis indicates that equal opportunity, optimal education output and the highest expected winning academic quality can be achieved simultaneously under the optimal preferential admissions rule. This also provides an alternative explaination for the prevalent and persistent practice of affirmative action procedure in selective institutions.

This study reveals a novel and nontrivial aspect of the incentive structure behind preferential admissions procedure, which has been virtually ignored in the debate and literature. The main result has strong policy implication. First, the analysis shows that a totally “color blind” practice does not achieve a color-neutral outcome. Second, it suggests that a corrective policy may enhance diversity and academic quality simultaneously. Hence, it is essential for a policy
maker to understand the incentive structure underlying an affirmative action policy proposal. In addition, this analysis does indicate the possibility of undesirable “reverse discrimination”. Finally, as a redistributional instrument, a preferential admission scheme alone does not suffice to fill in the racial gap in educational attainment, but even widens it. A policy maker should concern more on “what to do” and “how to do”, rather than “whether or not to do”, in regard to affirmative action.

This study also leaves open tremendous room for future extensions. First of all, the emphasis of this study is the partial equilibrium incentive effect of affirmative action at the college admissions level. However, this framework does not consider the response of labor market. It also remains to see how affirmative action at college admissions level affects productivity and the minority’s welfare, which entails a general equilibrium approach. Second, one may concern that this study is sensitive when outside option for college candidates is available. For instance, non-minority student who is rejected by an affirmative action college may turn to other colleges. The current setting corresponds to a relatively concentrated higher education market. However, it will be compelling to investigate if this result may extend to a setting where multiple colleges compete for a fixed pool of students, and how the education market structure endogenously determines colleges’ admission policy.
References


In the main text I discuss a affirmative action admissions scheme in which the minority candidate’s academic quality weighs more than the non-minority candidate in the admissions officer’s rating system. In this scheme, the minority candidate’s academic effort is relatively more effective, which improves her incentive to make effort. However, some institutions, such as University of Michigan, automatically add a certain amount of points to the scores of the minority candidate in their rating system. The minority candidate is given a “headstart” advantage. In this section, I will explore the consequence of this scheme.

Again, I assume these two candidates, $N$ and $M$, value the seat in college at $V_N$ and $V_M$, respectively, with $V_N > V_M > 0$, and $V_N - V_M < V_M$. Candidate $i$ chooses her effort $e_i$ to obtain her academic score $q_i = e_i$. The admissions officer automatically adds a fixed amount of points $k$ to candidate $M$’s score, with $0 < k \leq V_N - V_M$. As a consequence, the minority candidate is admitted whenever $e_M + k > e_N$, while the non-minority candidate is able to win with certainty if and only if $e_N > e_M + k$. Again, if they tie, the seat is randomly assigned to one of them. These two candidates have the following payoff functions.

$$\pi_M = \begin{cases} V_M - e_M & \text{if } e_M + k > e_N \\ \frac{1}{2}V_M - e_M & \text{if } \alpha e_M + k = e_N \\ -e_M & \text{if } e_M + k < e_N \end{cases}$$

(13)

$$\pi_N = \begin{cases} V_N - e_N & \text{if } e_N > e_M + k \\ \frac{1}{2}V_N - e_N & \text{if } e_N = e_M + k \\ -e_N & \text{if } e_N < e_M + k \end{cases}$$

(14)

By the standard technique, it can be shown that Lemma 1 and 2 still hold in this all-pay auction. It is also straightforward to see that candidate $M$’s effort distribution function has a
lower support zero, In contrast, any $e \in [0, k)$ will never be in the support of $F_N$, because she will lose with certainty. Lemma 4 and 5 still hold, because $F_N$ in this case is exactly a parallel upward shift of the distribution function $F'_N$ in the color-blind case, with $e_N = e'_N + k$. I have the following Lemmas.

**Lemma 10** In a mixed strategy equilibrium, $F_N$ and $F_M$ have lower supports $k$ and zero, respectively.

**Lemma 11** In a mixed strategy equilibrium, $F_N$ and $F_M$ have upper supports $V_M + k$, and $V_M$, respectively.

**Lemma 12** In a mixed strategy equilibrium, candidate $N$ and $M$ have expected payoffs $V_N - V_M - k$, and zero, respectively.

Obviously, candidate $M$ will never make any effort higher than her own valuation. On the other hand, candidate $V_N$ will not make effort higher than $V_N - V_M - k$, because an effort $e_N = V_N - V_M - k$ guarantees she wins the seat with probability one.

**Proposition 10** When $k \in (0, V_N - V_M]$, there exists a unique Nash equilibrium, where candidate $N$ continuously randomizes effort over the whole support $[0, V_M + k]$, while candidate $N$ continuously randomizes effort over the support $(0, V_M]$, and places probability mass at zero with a size of $\frac{V_N - V_M}{V_N}$. Distribution functions of candidate $M$ and $N$ are given by,

$$F_N(e_N) = \frac{V_N - V_M + e_M}{V_N}, \quad \text{and,}$$

$$F_M(e_M) = \frac{e_N - k}{V_M}$$

It is interesting to note that the size of the probability mass candidate $M$ places at zero is independent of $k$. In addition, candidate $M$’s equilibrium distribution function, as given by (25) does not vary with the degree of preferential treatment either, which takes exactly the same form as the color-blind case. However, the lower support of $F_N$ increases with $k$. Under this rule, the distribution of candidate $N$’s effort is shifted upward by $k$. The main result is summarized in Table 6.
Table 6: Equilibrium Results of the Alternative Admissions Scheme

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>candidate $N$’s expected probability of winning</td>
<td>$\frac{2V_N - V_M}{2V_N}$</td>
</tr>
<tr>
<td>candidate $M$’s expected probability of winning</td>
<td>$\frac{V_M}{2V_N}$</td>
</tr>
<tr>
<td>candidate $N$’s expected effort</td>
<td>$\frac{V_M}{2} + k$</td>
</tr>
<tr>
<td>candidate $M$’s expected effort</td>
<td>$\frac{V_M^2}{2V_N}$</td>
</tr>
<tr>
<td>the expected aggregate effort</td>
<td>$\frac{V_M}{2V_N} \cdot (V_N + V_M) + K$</td>
</tr>
<tr>
<td>expected winning quality</td>
<td>$\frac{3V_N V_M + V_M^2 + 3(2V_N - V_M)K}{6V_N}$</td>
</tr>
</tbody>
</table>

By the result presented in Table 6, it is apparent that this admission scheme does not enhance the chance of the minority candidate. Candidates have the same expected probabilities of winning as under the color-blind admissions rule. Although the minority is given a “head-start” advantage against the non-minority candidate. However, this policy does not influence candidate $M$’s incentive to make effort. On the other hand, this policy does affect candidate $N$’s choice. She has to increase her effort by $k$. Given candidate $N$’s increased effort, which exactly offsets candidate $M$’s *ex ante* advantage.

By the result in Table 6, it is straightforward to observe that the winning score is increasing with $k$. This policy only encourages candidate $N$ to make an higher effort, which also lifts the expected winning score. However, the increase in the winning score stems purely from the upward shift of $F_N$. Consequently, an academic quality oriented college still has an incentive to adopt this admissions rule. However, this mechanism does not respond to the goal of diversity.
7 Appendix B: Proof

7.1 Proof of Lemma 1

Proof. Consider the case $\alpha \leq \theta$. Suppose there exists a pure strategy equilibrium, in which, candidate $N$ chooses effort $e_N$, while candidate $M$ chooses $e_M$. If $e_N \geq \alpha V_M$, then the best response of candidate $M$ is to choose $e_M = 0$. Given candidate $M$’s best response $e_M = 0$, candidate $N$ should choose an infinitely small positive effort. However, candidate $M$ should choose a positive effort too, which breaks the equilibrium. If candidate $N$ chooses any effort $e_N \in (0, \alpha V_M)$, candidate $M$ can get a positive payoff if she chooses $e_M = \frac{e_N}{\alpha} + \varepsilon$, where $\varepsilon$ is a infinitely small positive number. Given $e_M = \frac{e_N}{\alpha} + \varepsilon$, candidate $N$’s best response is to choose $e_N = \alpha e_M + \varsigma$, where $\varsigma$ is an infinitely small positive number, and gets positive payoff, which breaks the equilibrium. Hence, no pure strategy equilibrium can be reached. The same reasoning applies to the case $\alpha \geq \theta$. Q.E.D.

7.2 Proof of Lemma 2

Proof. By Lemma 1, neither candidate plays pure strategy with probability one, which implies that neither will choose $e_N = e_M = 0$ with probability one. Q.E.D.

7.3 Proof of Lemma 3

Proof. Consider the case $\alpha \leq \theta$. Define $e_N$ as the lower support of $F_N$, and $e_M$ the lower support of $F_M$. This proof contains two steps.

Step 1: In equilibrium, $e_N = \alpha e_M$. Suppose $e_N > \alpha e_M$. Then it always pays for candidate $N$ to move the probability mass downward to a $\varepsilon$-neighbor above $\alpha e_M$. If $e_N < \alpha e_M$, then it always pays for candidate $M$ to move the probability mass upward.

Step 2: In equilibrium, $e_N = \alpha e_M = e_M = 0$. Suppose $e_N = \alpha e_M > 0$. By the property of
a mixed strategy equilibrium, \( \pi_M = \pi_M(e_M) = -e_M \), and \( \pi_N = \pi_N(e_N) = -e_N \). Then both of them will be strictly better by moving the probability mass downward to zero, which breaks the equilibrium.

The same reasoning applies to the case \( \alpha \geq \theta \). Q.E.D.

7.4 Proof of Lemma 4

Proof. Consider the case \( \alpha \leq \theta \). Suppose the contrary, candidate \( N \) places probability mass at \( e > 0 \). Now look at candidate \( M \). Let \( F_N^+(e) \) denote \( \lim_{\varepsilon \downarrow e} F_N(e) \). It is straightforward to see that \( F_N^+(e) > F_N(e) \). Hence, the payoff function of candidate \( M \), \( F_N(e) - \frac{\alpha}{\alpha} \), has an upward jump at \( \frac{\alpha}{\alpha} \). Candidate \( M \) will therefore transfer probability mass from a \( \varepsilon \)-neighborhood below \( \frac{\alpha}{\alpha} \) to \( \frac{\alpha}{\alpha} \). Since candidate \( M \) does not make effort in this neighborhood, candidate \( N \) will not place probability mass at \( e \). This line of reasoning also holds for candidate \( M \). Hence, \( F_N \) and \( F_M \) must be atomless above zero.

The same reasoning applies to the case \( \alpha \geq \theta \). Q.E.D.

7.5 Proof of Lemma 5

Proof. Consider the case \( \alpha \leq \theta \). Suppose the contrary. If the density of \( F_N \) in an open interval \((e_1, e_2)\) is zero, it follows the density of \( F_M \) is also zero in an open interval \((\frac{e_1}{\alpha}, \frac{e_2}{\alpha})\). Candidate \( M \) will be strictly better off by choosing \( e_M = \frac{e_1}{\alpha} \) than \( \frac{e_2}{\alpha} \), which contracts with the property of a mixed strategy equilibrium. Then \( F_N \) and \( F_M \) must be strictly increasing above the lower support.

The same reasoning applies to the case \( \alpha \geq \theta \). Q.E.D.

7.6 Proof of Lemma 6

Proof. Because \( \alpha V_M \leq V_N \), \( e_N = \alpha V_M \) is feasible. Suppose candidate \( N \) puts an upper support
strictly smaller than $\alpha V_M$. Then it always pays for candidate $M$ to transfer probability mass to a $\epsilon$-neighborhood above $\frac{e_N}{\alpha}$. In turn, the best response of candidate $N$ is to move probability mass upward. There is no equilibrium unless they have upper supports $\alpha V_M$, and $V_M$, respectively. Q.E.D. ■

### 7.7 Proof of Lemma 7

**Proof.** By lemma 1 to 6, in a mixed strategy equilibrium, $\pi_M = \pi_M(e_M) = F_N(\alpha V_M)V_M - V_M = 0$, while $\pi_N = \pi_N(e_N) = F_M(V_M)V_N - \alpha V_M = V_N - \alpha V_M$. Q.E.D. ■

### 7.8 Proof of Proposition 1

**Proof.** By Lemma 1 to 7, equilibrium distribution functions are uniquely determined by the following set of equations:

\[
F_M\left(\frac{e_N}{\alpha}\right)V_N - e_N = V_N - \alpha V_M \quad (17)
\]

\[
F_N(\alpha e_M)V_M - e_M = 0 \quad (18)
\]

\[
F_M(0)V_N = V_N - \alpha V_M \quad (19)
\]

By solving (17), (18) and (19), I have the results stated in Proposition 1. Q.E.D. ■

### 7.9 Proof of Proposition 2

**Proof.** (1) When $\alpha \in (0, \theta]$, $P_M - P_N = \frac{2V_N - \alpha V_M}{2V_M} - \frac{\alpha V_M}{V_N} = \frac{\alpha V_M - V_N}{V_N} \leq 0$. This equality is always strict unless $\alpha = \theta$.

(2) and (3) are straightforward.

(4) Define $\tilde{\pi}_I$ as the expected winning score when $\alpha \in (0, \theta]$. Then take the first order
derivative of $\pi$ with respect to $\alpha$.

\[
\frac{\partial e_1}{\partial \alpha} = \frac{V^2_M}{V_N} \cdot \left[ \frac{\alpha(V_N - aV_M)}{2} + \frac{\alpha V_M}{3} + \frac{\alpha^2 V_M}{3} \right]
\]

\[
= \frac{V^2_M}{V_N} \cdot \left( \frac{3V_N + 2V_M - 2\alpha V_M}{6} \right)
\]  

(20)

Because $\alpha \in (0, \theta]$, (8) $\geq \frac{V^2_M}{V_N} \cdot \left( \frac{2V_N + 2V_M - 2\alpha V_M}{6} \right) = \frac{V^2_M}{V_N} \cdot \left( \frac{V_N + 2V_M}{6} \right) > 0$. Q.E.D.

7.10 Proof of Lemma 8

Proof. Because $V_M \geq \frac{V_N}{\alpha}$, $e_M = \frac{V_N}{\alpha}$ is feasible. Suppose candidate $M$ puts an upper support $e_M$ strictly smaller than $\frac{V_N}{\alpha}$. Then it always pays for candidate $N$ to transfer probability mass to a $\varepsilon$-neighborhood above $\alpha e_M$. In turn, the best response of candidate $M$ is to move probability mass upward. There is no equilibrium unless they have upper supports $V_N$, and $\frac{V_N}{\alpha}$, respectively. Q.E.D.

7.11 Proof of Lemma 9

Proof. By lemma 1 to 5, and 8, in a mixed strategy equilibrium, $\pi_M = \pi_M(e_M) = F_N(V_N)V_M - \frac{V_N}{\alpha} = V_M - \frac{V_N}{\alpha}$, while $\pi_N = \pi_N(e_N) = F_M(\frac{V_N}{\alpha})V_N - V_N = 0$. Q.E.D.

7.12 Proof of Proposition 3

Proof. By Lemma 1 to 5, 8 and 9, candidate $M$ and $N$’s equilibrium effort distribution functions are uniquely determined by the following set of equations.

\[
F_N(\alpha e_M)V_B - e_M = V_M - \frac{V_N}{\alpha}
\]  

(21)

\[
F_M(\frac{e_N}{\alpha})V_N - e_N = 0
\]  

(22)

\[
F_N(0)V_M = V_M - \frac{V_N}{\alpha}
\]  

(23)

By solving (11), (12) and (13), I can obtain the result stated in the proposition. Q.E.D.
7.13 Proof of Proposition 4

**Proof.** (1) \( P_M - P_N = \frac{aV_M - V_N}{aV_N} \geq 0 \). This inequality is strict whenever \( \alpha \in (\theta, \infty) \).

(2) and (3) are straightforward.

(4) To prove (4), let \( \bar{e}_2 = \frac{V_N}{\alpha^2 V_M} \cdot \left[ \frac{1}{2} (\alpha V_M - V_N) + \frac{V_N}{3} + \frac{\alpha V_N}{3} \right] \) be the expected winning score when \( \alpha \in [\theta, \infty) \). Take the first order derivative of \( \bar{e}_2 \) with respect to \( \alpha \), I have

\[
\frac{\partial \bar{e}_2}{\partial \alpha} = \frac{\partial \left\{ \frac{V_N}{\alpha^2 V_M} \cdot \left[ \frac{1}{2} (\alpha V_M - V_N) + \frac{V_N}{3} + \frac{\alpha V_N}{3} \right] \right\}}{\partial \alpha}
= \frac{V_N}{\alpha^2 V_M} \cdot \left( \frac{V_N}{3 \alpha} - \frac{V_N}{3} - \frac{V_M}{2} \right) .
\] (24)

Since \( \alpha \geq \theta \), (14) \( \leq \frac{V_N}{\alpha^2 V_M} \cdot \left( \frac{V_M}{3} - \frac{V_N}{3} - \frac{V_M}{2} \right) < 0 \). Q.E.D.

7.14 Proof of Proposition 5

**Proof.** Suppose both candidate choose an effort \( e \). Let \( P_M = P_M(e) \) be the minority candidate’s probability of winning, and \( P_N = P_N(e) \) the non-minority candidate’s winning. By Proposition 1, when \( \alpha = 1 \), \( F_M = \frac{V_N - V_M + e}{V_N} \), and \( F_N = \frac{e}{V_M} \). Their equilibrium distribution functions have common upper and lower supports. \( P_M = \Pr(e_N < e) = \frac{e}{V_M} \), and \( P_N = \Pr(e_M < e) = \frac{V_N - V_M + e}{V_N} \). \( P_M - P_N = \frac{e}{V_M} - \frac{V_N - V_M + e}{V_N} = \frac{(V_N - V_M)(e - V_M)}{V_M V_N} \). Apparently, for any \( e < V_M \), \( P_M - P_N < 0 \). Q.E.D.

7.15 Proof of Proposition 8

**Proof.** Proposition 8 (1) is straightforward, since \( \widehat{\epsilon}_M \) does not contain \( \alpha \). It remains to show

(2). Take the first order derivative of \( \widehat{\epsilon}_N \), with respect to \( \alpha \), I have

\[
\frac{\partial \widehat{\epsilon}_N}{\partial \alpha} = \frac{[V_M \cdot (V_N - \frac{\alpha V_M}{3}) - \alpha V_M - \frac{V_M}{3}] \cdot (2V_N - \alpha V_M) + \alpha V_M^2 \cdot (V_N - \frac{\alpha V_M}{3})}{(2V_N - \alpha V_M)^2}
= \frac{V_M \cdot (2V_N^2 + \frac{\alpha V_M^2}{3} - \frac{4 \alpha V_N V_M}{3})}{(2V_N - \alpha V_M)^2} .
\] (25)
Because \( \alpha \in [1, \theta] \),

\[
(21) \geq \frac{V_M \cdot \left(2V_N^2 + \frac{\alpha V_M^2}{3} - \frac{4V_M^2}{3} \right)}{(2V_N - \alpha V_M)^2} > 0
\]

Q.E.D. ■

7.16 Proof of Proposition 9

Proof. (1) When \( \alpha = 1 \), \( \Delta \hat{\epsilon} = \hat{e}_N - \hat{e}_M = \frac{V_M - \frac{V_M}{2} - \frac{2}{3} V_M}{V_N - V_M} = -\frac{V_M(V_N - V_M)}{3(2V_N - V_M)} < 0. \)

(2) When \( \alpha = \theta \), \( \Delta \hat{\epsilon} = \hat{e}_N - \hat{e}_M = \frac{\theta V_M - \frac{\theta V_M}{2} - \frac{2}{3} V_M}{2V_N - \theta V_M} = \frac{2}{3} V_N - V_M^2 V_M = \frac{2}{3}(V_N - V_M) > 0. \)

(3) This proof is straightforward. \( \frac{\partial \Delta \hat{\epsilon}}{\partial \alpha} = \frac{\partial \hat{e}_N}{\partial \alpha} - \frac{\partial \hat{e}_M}{\partial \alpha} \). By Proposition 8, \( \frac{\partial \hat{e}_N}{\partial \alpha} > 0 \) and \( \frac{\partial \hat{e}_M}{\partial \alpha} = 0 \). Hence \( \frac{\partial \Delta \hat{\epsilon}}{\partial \alpha} = \frac{\partial \hat{e}_N}{\partial \alpha} - \frac{\partial \hat{e}_M}{\partial \alpha} > 0 \). Q.E.D. ■

7.17 Proof of Lemma 10

Proof. Define \( e_N \) as the lower support of \( F_N \), and \( e_M \) the lower support of \( F_M \). This proof contains four steps.

**Step 1:** In equilibrium, \( e_N > 0 \). Candidate \( N \) can get a payoff \( \pi_N = V_N - V_M - k \geq 0 \) by choosing \( e_N = V_M + k \). Hence, candidate \( N \) places no probability mass at zero.

**Step 2:** The density of \( F_N \) over the range \((0, k)\) must be zero. Any \( e_N \in (0, k) \) brings only negative payoff with probability one. Thus candidate \( N \) must transfer probability mass toward \( k \).

**Step 3:** In equilibrium, \( e_N = e_M + k \). Suppose \( e_N > e_M + k \). Then it always pays for candidate \( N \) to move the probability mass downward to a \( \varepsilon \)-neighbor above \( e_M + k \). If \( e_N < e_M + k \), then it always pays for candidate \( M \) to move the probability mass upward.

**Step 4:** In equilibrium, \( e_N - k = e_M = 0 \). Suppose \( e_N - k = e_M > 0 \). By the property of a mixed strategy equilibrium, \( \pi_M = \pi_M(e_M) = -e_M \), and \( \pi_N = \pi_N(e_N) = -e_N \). Then both of them will be strictly better by moving the probability mass downward to zero, which breaks
7.18 Proof of Lemma 11

**Proof.** Because $V_M + k \leq V_N$, $e_N = V_M + k$ is feasible. Suppose candidate $N$ puts an upper support $e_N$ strictly smaller than $V_M + k$. Then it always pays for candidate $M$ to transfer probability mass to a $\varepsilon$-neighborhood above $e_N - k$. In turn, the best response of candidate $N$ is to move probability mass upward. There is no equilibrium unless they have upper supports $V_M + k$, and $V_M$, respectively. ■

7.19 Proof of Lemma 12

**Proof.** By Lemma 1, 2, 4, 5, and 10, in a mixed strategy equilibrium, $\pi_M = \pi_M(e_M) = F_N(V_M + k)V_M - V_M = 0$, while $\pi_N = \pi_N(e_N) = F_M(V_M)V_N - V_M - k = V_N - V_M - k$. ■

7.20 Proof of Proposition 10

**Proof.** By Lemma 1, 2, 4, 5, and 10, 11 and 12, equilibrium distribution functions are uniquely determined by the following set of equations:

\begin{align*}
F_M(e_N - k)V_N - e_N &= V_N - V_M - k \quad (27) \\
F_N(e_M + k)V_M - e_M &= 0 \quad (28) \\
F_M(0)V_N - k &= V_N - V_M - k \quad (29)
\end{align*}

Q.E.D. ■