Technical Appendix for "Bootstrapping GMM Estimators for Time Series"

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This technical appendix contains proofs of the lemmas in "Bootstrapping GMM Estimators for Time Series" by A. Inoue and M. Shintani.

Notation

$\otimes$ denotes the Kronecker product operator. If α is an n-dimensional nonnegative integral, $|\alpha|$ denotes its length, i.e., $|\alpha| = \sum_{i=1}^n |\alpha_i|$. $\|\cdot\|$ denotes the Euclidean norm i.e., $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$, where x is an n-dimensional vector. We will write $\omega(j/\ell)$ as ω_j for notational simplicity. $\kappa_j(x)$ denotes the jth cumulant of a random variable x. vec$(\cdot)$ is the column-by-column vectorization function. vech$(\cdot)$ denotes the column stacking operator that stacks the elements on and below the leading diagonal. For a nonnegative integral vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, let

$$D^\alpha \;=\; \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

ℓ and l are treated differently: ℓ denotes the lag truncation parameter and l denotes an integer.

Let $u_t = y_t - \beta_0' x_t$, $\hat{u}_t = y_t - \tilde{\beta}_T' x_t$, $v_t = z_t u_t$, $\hat{v}_t = z_t \hat{u}_t$, $w_t = z_t x_t'$,

$$\hat{\Gamma}_j = \begin{cases} \frac{1}{T}\sum_{t=1}^T \hat{v}_{t+j}\hat{v}_t' & j \geq 0 \\ \frac{1}{T}\sum_{t=1}^T \hat{v}_t\hat{v}_{t-j}' & j < 0 \end{cases}, \quad \nabla\tilde{\Gamma}_j\delta = \begin{cases} \frac{1}{T}\sum_{t=1}^T (v_{t+j}z_t'x_t' + z_{t+j}v_t'x_{t+j}')\delta & j \geq 0 \\ \frac{1}{T}\sum_{t=1}^T (v_t z_{t-j}'x_{t-j}' + z_t v_{t-j}'x_t')\delta & j < 0 \end{cases},$$

$$\tilde{\Gamma}_j = \begin{cases} \frac{1}{T}\sum_{t=1}^T v_{t+j}v_t' & j \geq 0 \\ \frac{1}{T}\sum_{t=1}^T v_t v_{t-j}' & j < 0 \end{cases}, \quad \nabla\Gamma_j\delta = \begin{cases} E(v_{t+j}z_t'x_t' + z_{t+j}v_t'x_{t+j}')\delta & j \geq 0 \\ E(v_t z_{t-j}'x_{t-j}' + z_t v_{t-j}'x_t')\delta & j < 0 \end{cases},$$

$$\Gamma_j = \begin{cases} E(v_{t+j}v_t') & j \geq 0 \\ E(v_t v_{t-j}') & j < 0 \end{cases}, \qquad \delta'\nabla^2\tilde{\Gamma}_j\delta = \begin{cases} \frac{1}{T}\sum_{t=1}^T \delta' x_{t+j} z_{t+j} z_t' x_t' \delta & j \geq 0 \\ \frac{1}{T}\sum_{t=1}^T \delta' x_t z_t z_{t-j}' x_{t-j}' \delta & j < 0 \end{cases},$$

$$\begin{array}{lll} \hat{S}_T = \sum_{j=-\ell}^{\ell} \omega_j \hat{\Gamma}_j, & \tilde{S}_T = \sum_{j=-\ell}^{\ell} \omega_j \tilde{\Gamma}_j, & \bar{S}_T = \sum_{j=-\ell}^{\ell} \omega_j \Gamma_j, \\ S_T = \sum_{j=-T+1}^{T-1} (1 - \frac{|j|}{T})\Gamma_j, & \nabla\tilde{S}_T\delta = \sum_{j=-\ell}^{\ell} \omega_j \nabla\tilde{\Gamma}_j\delta, & \nabla\bar{S}_T\delta = \sum_{j=-\ell}^{\ell} \omega_j \nabla\Gamma_j\delta, \\ \nabla S\delta = \sum_{j=-\infty}^{\infty} \nabla\Gamma_j\delta, & \delta'\nabla^2\tilde{S}_T\delta = \sum_{j=-\ell}^{\ell} \omega_j \delta'\nabla^2\tilde{\Gamma}_j\delta, & \end{array}$$

where δ is a p-dimensional vector. Let $G_T = (1/T)\sum_{t=1}^T z_t x_t'$ and $m_T = T^{-1/2}\sum_{t=1}^T v_t$.

Then the studentized statistic can be written as

$$f_T \;=\; \sqrt{T}(c'\hat{\Sigma}c)^{-1/2}c'(\hat{\beta}_T - \beta_0) \;=\; (c'(G_T'\hat{S}_T^{-1}G_T)^{-1}c)^{-1/2}c'(G_T'\hat{S}_T^{-1}G_T)^{-1}G_T'\hat{S}_T^{-1}m_T.$$

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We use the following notation for the bootstrap. Let

$$\begin{aligned}
m^*_T &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(z^*_t u^*_t - \mu^*_T\right) = \frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k}, \\
B_{N_k} &= \frac{1}{\sqrt{\ell}}\sum_{i=1}^{\ell}\left(z_{N_k+i}\hat{u}_{N_k+i} - \mu^*_T\right) = \frac{1}{\sqrt{\ell}}\sum_{i=1}^{\ell}\left(\hat{v}^*_{N_k+i} - \mu^*_T\right), \\
\widehat{B}_{N_k} &= \frac{1}{\sqrt{\ell}}\sum_{i=1}^{\ell}\left(z_{Nk+i}\hat{u}^*_{Nk+i} - \mu^*_T\right) \ ,\hat{u}^*_i = y_i - \widetilde{\beta}^{*\prime} x_i, \\
G^*_T &= \frac{1}{T}\sum_{t=1}^{T} z^*_t x^{*\prime}_t = \frac{1}{b}\sum_{k=1}^{b} F_{N_k}, \\
F_{N_k} &= \frac{1}{\ell}\sum_{i=1}^{\ell} z_{N_k+i} x^{\prime}_{N_k+i} = \frac{1}{\ell}\sum_{i=1}^{\ell} w_{N_k+i}.
\end{aligned}$$

$$\hat{S}^*_T = \frac{1}{b}\sum_{k=1}^{b}\widehat{B}_{N_k}\widehat{B}^{\prime}_{N_k}, \quad \widetilde{S}^*_T = \frac{1}{b}\sum_{k=1}^{b} B_{N_k}B^{\prime}_{N_k}, \quad S^*_T = \mathrm{Var}^*\left(m^*_T\right).$$

Then the bootstrap version of the first-step and the second-step GMM estimators can be written as

$$\begin{aligned}
\widetilde{\beta}^* &= \hat{\beta} + \left[\frac{1}{b}\sum_{k=1}^{b} F^{\prime}_{N_k} V_T \frac{1}{b}\sum_{k=1}^{b} F_{N_k}\right]^{-1} \frac{1}{b}\sum_{k=1}^{b} F^{\prime}_{N_k} V_T \frac{1}{\sqrt{Tb}}\sum_{k=1}^{b} B_{N_k} \\
&= \hat{\beta} + \left[G^{*\prime}_T V_T G^*_T\right]^{-1} G^{*\prime}_T V_T \frac{1}{\sqrt{T}} m^*_T, \\
\hat{\beta}^* &= \hat{\beta} + \left[\frac{1}{b}\sum_{k=1}^{b} F^{\prime}_{N_k} \hat{S}^{*-1}_T \frac{1}{b}\sum_{k=1}^{b} F_{N_k}\right]^{-1} \frac{1}{b}\sum_{k=1}^{b} F^{\prime}_{N_k} \hat{S}^{*-1}_T \frac{1}{\sqrt{Tb}}\sum_{k=1}^{b} B_{N_k} \\
&= \hat{\beta} + \left[G^{*\prime}_T \hat{S}^{*-1}_T G^*_T\right]^{-1} G^{*\prime}_T \hat{S}^{*-1}_T \frac{1}{\sqrt{T}} m^*_T,
\end{aligned}$$

respectively.

Proofs of Lemmas

Next, we will present the lemmas used in the proofs of the theorems. Lemma A.1 produces a Taylor series expansion of the studentized statistic f_T. Lemma A.2 provides bounds on the moments and will be used in the proofs of Lemmas A.3–A.6. Lemma A.3 shows the limits and the convergence rates of the first three cumulants of g_T in (A.1), that will be used to derive the formal Edgeworth expansion. Lemmas A.5 and A.6 provide bounds on the approximation error. For convenience, we present Lemma A.8 that will be used in the proofs of Lemmas A.9 and A.10. Lemma A.9 shows the consistency and convergence rate of the bootstrap version of the moments. Lemma A.10 shows the limits and the convergence rates of the first three cumulants of the bootstrap version.

Lemma A.1:

$$\begin{aligned}
& f_T \\
= \; & \mathbf{a}^{\prime} m_T + \mathbf{b}^{\prime}[\mathrm{vec}(G_T - G_0)\otimes m_T] + \mathbf{c}^{\prime}[\mathrm{vech}(\hat{S}_T - S_0)\otimes m_T] \\
& +\mathbf{d}^{\prime}[\mathrm{vec}(G_T - G_0)\otimes \mathrm{vech}(\hat{S}_T - S_0)\otimes m_T] + \mathbf{e}^{\prime}[\mathrm{vech}(\hat{S}_T - S_0)\otimes \mathrm{vech}(\hat{S}_T - S_0)\otimes m_T] \\
& +O_p((\ell/T)^{3/2}) \\
= \; & \mathbf{a}^{\prime} m_T + \mathbf{b}^{\prime}[\mathrm{vec}(G_T - G_0)\otimes m_T] + \mathbf{c}^{\prime}[\mathrm{vech}(\hat{S}_T - \bar{S}_T)\otimes m_T] + \mathbf{c}^{\prime}[\mathrm{vech}(\bar{S}_T - S_0)\otimes m_T] \\
& +\mathbf{d}^{\prime}[\mathrm{vec}(G_T - G_0)\otimes \mathrm{vech}(\hat{S}_T - \bar{S}_T)\otimes m_T] + \mathbf{e}^{\prime}[\mathrm{vech}(\hat{S}_T - \bar{S}_T)\otimes \mathrm{vech}(\hat{S}_T - \bar{S}_T)\otimes m_T]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{d}'[\mathrm{vec}(G_T-G_0)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]+\mathbf{e}'[\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]\\
&+\mathbf{e}'[\mathrm{vech}(\bar{S}_T-S_0)\otimes\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes m_T]+\mathbf{e}'[\mathrm{vech}(\bar{S}_T-S_0)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]\\
&+O_p((\ell/T)^{3/2})\\
\equiv\ & g_T+\mathbf{c}'[\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]+\mathbf{d}'[\mathrm{vec}(G_T-G_0)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]\\
&+\mathbf{e}'[\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]+\mathbf{e}'[\mathrm{vech}(\bar{S}_T-S_0)\otimes\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes m_T]\\
&+\mathbf{e}'[\mathrm{vech}(\bar{S}_T-S_0)\otimes\mathrm{vech}(\bar{S}_T-S_0)\otimes m_T]+O_p((\ell/T)^{3/2}), \qquad \text{(A.1)}
\end{aligned}
$$

where $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ and $\mathbf{e}$ are k, k^2p, $k(k^2+k)/2$, $k^2(k^2+k)p/2$ and $k((k^2+k)/2)^2$-dimensional vectors of smooth functions of G_0 and S_0, respectively.

Proof of Lemma A.1:

(A.1) immediately follows from a Taylor series expansion of f_T around

$$(m_T',\mathrm{vec}(G_T)',\mathrm{vech}(\hat{S}_T)')' \ = \ (0_{1\times q},\mathrm{vec}(G_0)',\mathrm{vech}(S_0)')'$$

and from Theorem 1 of Andrews (1991). *Q.E.D.*

Lemma A.2:

$$
\begin{aligned}
E\|m_T\|^{r+\eta} &= O(1), &\text{(A.2)}\\
E\|T^{1/2}\mathrm{vec}(G_T-G_0)\|^{r+\eta} &= O(1), &\text{(A.3)}\\
E\|(T/\ell)^{1/2}\mathrm{vech}(\tilde{S}_T-\bar{S}_T)\|^{r/2} &= O(1), &\text{(A.4)}\\
E\|(T/\ell)^{1/2}\mathrm{vech}(\nabla\tilde{S}_T-\nabla\bar{S}_T)\|^{r/2} &= O(1), &\text{(A.5)}\\
E\|T^{1/2}\mathrm{vech}(\hat{S}_T-\tilde{S}_T)\|^{r/2} &= O(1). &\text{(A.6)}
\end{aligned}
$$

Proof of Lemma A.2:

First, (A.2) and (A.3) immediately follow from the moment inequality of Yokoyama (1980). Second, we will show (A.4). Note that

$$
\begin{aligned}
(T/\ell)^{1/2}(\tilde{S}_T-\bar{S}_T) &= (T/\ell)^{1/2}\sum_{j=-\ell}^{\ell}\omega_j(\tilde{\Gamma}_j-\Gamma_j) \ = \ (\ell/T)^{1/2}\sum_{i=1}^{[T/\ell]}W_i\\
&= (\ell/T)^{1/2}(\sum_{i=0mod3}W_i+\sum_{i=1mod3}W_i+\sum_{i=2mod3}W_i), \qquad \text{(A.7)}
\end{aligned}
$$

where

$$W_i \ = \ \frac{1}{\ell}\sum_{t=(i-1)\ell+1}^{i\ell}\{v_tv_t'-E(v_tv_t')+\sum_{j=1}^{\ell}\omega_j[v_{t+j}v_t'-E(v_{t+j}v_t')+v_tv_{t+j}'-E(v_tv_{t+j}')]\}.$$

Note that the summands in each sum on the RHS of (A.7), (e.g., $W_3,W_6,W_9,\ldots$), form strong mixing sequence with mixing coefficient $\{\alpha_{\ell m}\}_{m=1}^{\infty}$. Let

$$W_2(1)=\ell^{-1}\sum_{t=\ell+1}^{2\ell}\sum_{j=0}^{\ell-1}\omega_jv_{t+j}v_t',\ W_2(2)=\ell^{-1}\sum_{t=\ell+1}^{2\ell}\sum_{j=-\ell+1}^{-1}\omega_jv_tv_{t-j}',\ W_2(3)=\sum_{j=-\ell+1}^{\ell-1}E(v_0v_{-j}')$$

If

$$
\begin{aligned}
E|W_2(1)^{(i,j)}|^{\frac{r}{2}} &= O(1), &\text{(A.8)}\\
E|W_2(2)^{(i,j)}|^{\frac{r}{2}} &= O(1), &\text{(A.9)}\\
E|W_2(3)^{(i,j)}|^{\frac{r}{2}} &= O(1), &\text{(A.10)}
\end{aligned}
$$

where $W_2(\cdot)^{(i,j)}$ denotes the (i,j)th element of $W_2(\cdot)$ for $i,j=1,2,\ldots,q$, then it follows from the moment inequality that

$$E\left\|(T/\ell)^{1/2}\mathrm{vech}(\tilde{S}_T-\bar{S}_T)\right\|^{\frac{r}{2}} \ = \ O(1). \qquad \text{(A.11)}$$

It remains to show (A.8), (A.9) and (A.10). By Assumptions 1(a) and 1(f), it follows that

$$E|W_2(1)^{(i,j)}|^{r/2} \;=\; O(\ell^{r/2} \sum_{t_1 \le t_2 \le \cdots \le t_r} E|v_{t_1}^{(k_1)} v_{t_2}^{(k_2)} \cdots v_{t_r}^{(k_r)}|), \tag{A.12}$$

where $0 \le t_l \le 2\ell$ and $k_l = i, j$ for $l = 1, 2, ..., r$. Then the standard arguments used in proofs of the moment inequality complete the proof of (A.8). The proof of (A.9) is analogous to that of(A.8) and thus is omitted. By the mixing inequality of Hall and Heyde (1980, Corollary A.2), it follows that for some $d' > 0$

$$E|W_2(3)^{(i,j)}|^{\frac{r}{2}} = (\sum_{j=-\ell+1}^{\ell-1} E(v_0 v_{-j}'))^{\frac{r}{2}} = (\sum_{j=-\ell+1}^{\ell-1} \alpha_j^{d'})^{\frac{r}{2}} = O(1), \tag{A.13}$$

and thus (A.10) holds. Therefore, (A.4) immediately follows from (A.7)–(A.10). The proof of (A.5) is analogous to that of (A.4) and thus is omitted.

Lastly, we will prove (A.6). Note that

$$T^{1/2}(\hat{S}_T - \tilde{S}_T) \;=\; T^{1/2}\nabla\tilde{S}_T(\tilde{\beta}_T - \beta_0) + T^{1/2}(\tilde{\beta}_T - \beta_0)'\nabla^2\tilde{S}_T(\tilde{\beta}_T - \beta_0). \tag{A.14}$$

Thus it follows from (A.5) and Minkowski's inequality that, for an arbitrary p-dimensional vector δ,

$$[E\|\nabla\tilde{S}_T\delta\|^r]^{1/r} \;\le\; [E\|(\nabla\tilde{S}_T - \nabla\bar{S}_T)\delta\|^r]^{1/r} + [E\|\nabla\bar{S}_T\delta\|^r]^{1/r} \;=\; O(\ell^{1/2}T^{-1/2}) + O(1), \tag{A.15}$$

$$\begin{aligned} [E\|\delta'\nabla^2\tilde{S}_T\delta\|^r]^{1/r} \;&\le\; [E\|\sum_{j=-\ell}^{\ell} \omega_j \delta'(\nabla^2\Gamma_j - E(\nabla^2\Gamma_j)\delta)\|^r]^{1/r} + [E\|\sum_{j=-\ell}^{\ell} \omega_j [\delta' E(\nabla^2\Gamma_j)\delta\|^r]^{1/r} \\ &=\; O(\ell T^{-1/2}) + O(\ell). \end{aligned} \tag{A.16}$$

Therefore (A.6) follows from (A.14), (A.15), (A.16), Assumption 1(i) and Hölder's inequality. *Q.E.D.*

Lemma A.3:

$$\begin{aligned} T^{1/2}\kappa_1(g_T) \;&=\; \alpha_\infty + O(\ell^{-q}) + o(\ell T^{-1/2}), && \text{(A.17)} \\ (T/\ell)(\kappa_2(g_T) - 1) \;&=\; \gamma_\infty + O(\ell^{-1/2}), && \text{(A.18)} \\ T^{1/2}\kappa_3(g_T) \;&=\; \kappa_\infty - 3\alpha_\infty + O(\ell^{-q}) + o(\ell T^{-1/2}), && \text{(A.19)} \\ (T/\ell)(\kappa_4(g_T) - 3) \;&=\; \zeta_\infty + O(\ell^{-1/2}), && \text{(A.20)} \end{aligned}$$

where

$$\begin{aligned} \alpha_\infty \;&=\; \mathbf{b}' \sum_{i=-\infty}^{\infty} E[w_0 \otimes v_i] + \mathbf{c}' \sum_{i,j=-\infty}^{\infty} E[\mathrm{vech}(v_0 v_i') \otimes v_j] \\ &\quad + \mathbf{c}' \sum_{i=-\infty}^{\infty} E\{\mathrm{vech}[\nabla\bar{S}(E(w_0)'VE(w_0))^{-1}E(w_0)'Vv_0] \otimes v_i\} \\ \gamma_\infty \;&=\; 2\lim_{T\to\infty} \frac{1}{\ell} \sum_{j=-\ell}^{\ell} \sum_{i,k=-T}^{T} E\{\mathbf{a}'v_0\mathbf{c}'[\mathrm{vech}(v_i v_{i-j}' - \Gamma_j) \otimes v_k]\} \\ &\quad + 2\lim_{T\to\infty} \frac{1}{\ell T} \sum_{i,l=-\ell}^{\ell} \sum_{i,k,m=-T}^{T} E\{\mathbf{a}'v_0\mathbf{e}'[\mathrm{vech}(v_i v_{i-j}' - \Gamma_j) \otimes \mathrm{vech}(v_k v_{k-l}' - \Gamma_l) \otimes v_m]\} \\ &\quad + \lim_{T\to\infty} \frac{1}{\ell T} \sum_{j,k,m=-T}^{T} \sum_{i,l=-\ell}^{\ell} E\{\mathbf{c}'[\mathrm{vech}(v_0 v_{-i}' - \Gamma_i) \otimes v_j]\mathbf{c}'[\mathrm{vech}(v_k v_{k-l}' - \Gamma_k) \otimes v_m]\}, \\ \kappa_\infty \;&=\; \sum_{i,j=-\infty}^{\infty} E(\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{a}'v_j) + 3\lim_{T\to\infty} \frac{1}{T} \sum_{i,j,k=-T+1}^{T-1} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{b}'[\mathrm{vech}(w_j - E(w_j)) \otimes v_k]\} \end{aligned}$$

$$
\begin{aligned}
&+3\lim_{T\to\infty}\frac{1}{T}\sum_{i,j,k,l=-T}^{T} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{c}'[\mathrm{vech}(v_jv_{j-k}'-\Gamma_k)\otimes v_l\} \\
&+3\lim_{T\to\infty}\frac{1}{T^2}\sum_{i,j,k=-T}^{T} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{c}'\mathrm{vech}[\nabla\bar{S}(E(w_0)'VE(w_0))^{-1}E(w_0)'Vv_j]\otimes v_k\},
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_\infty \\
= \;&\frac{4}{\ell T}\sum_{i,j,k,m=-T}^{T}\sum_{l=-\ell}^{\ell} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{a}'v_j\mathbf{c}'[\mathrm{vech}(v_kv_{k-l}'-\Gamma_l)\otimes v_m]\} \\
&+\lim\frac{4}{\ell T^2}\sum_{i,j,k,m,o=-T}^{T}\sum_{l,n=-\ell}^{\ell} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{a}'v_j\mathbf{e}'[\mathrm{vech}(v_kv_{k-l}'-\Gamma_l)\otimes\mathrm{vech}(v_mv_{m-n}'-\Gamma_n)\otimes v_o]\} \\
&+\lim\frac{6}{\ell T^2}\sum_{i,j,l,m,o=-T}^{T}\sum_{k,n=-\ell}^{\ell} E\{\mathbf{a}'v_0\mathbf{a}'v_i\mathbf{c}'[\mathrm{vech}(v_jv_{j-k}'-\Gamma_k)\otimes v_l]\mathbf{c}'[\mathrm{vech}(v_mv_{m-n}'-\Gamma_n)\otimes v_o]\} \\
&-12\lim\frac{1}{\ell}\sum_{j,l=-T}^{T}\sum_{k=-\ell}^{\ell} E\{\mathbf{a}'v_0\mathbf{c}'[\mathrm{vech}(v_jv_{j-k}'-\Gamma_k)\otimes v_l]\} \\
&-12\lim\frac{1}{\ell T}\sum_{j,l,n=-T}^{T}\sum_{k,m=-\ell}^{\ell} E\{\mathbf{a}'v_0\mathbf{e}'[\mathrm{vech}(v_jv_{j-k}'-\Gamma_k)\otimes(v_lv_{l-m}'-\Gamma_m)\otimes v_n]\} \\
&-6\lim\frac{1}{\ell T^2}\sum_{j,k,m=-T}^{T}\sum_{i,l=-\ell}^{\ell} E\{\mathbf{c}'[(v_0v_{-i}'-\Gamma_i)\otimes v_j]\mathbf{c}'[(v_kv_{k-l}'-\Gamma_l)\otimes v_m]\}.
\end{aligned}
$$

Proof of Lemma A.3: First, we will prove (A.17). By Hölder's inequality and Lemma A.2, it suffices to show that

$$
\begin{aligned}
T^{1/2}E[\mathrm{vec}(G_T-G_0)\otimes m_T] &= \sum_{i=-\infty}^{\infty}E[w_0\otimes v_i]+O(T^{-1}), && \text{(A.21)} \\
T^{1/2}E[\mathrm{vech}(\tilde{S}_T-\bar{S}_T)\otimes m_T] &= \sum_{i,j=-\infty}^{\infty}E[\mathrm{vech}(v_0v_i')\otimes v_j]+O(\ell^{-q})+O(\ell T^{-1}), && \text{(A.22)} \\
T^{1/2}E[\mathrm{vech}(\hat{S}_T-\tilde{S}_T)\otimes m_T] &= \sum_{i=-\infty}^{\infty}E\{\mathrm{vech}[\nabla\bar{S}(E(w_0)'VE(w_0))^{-1}E(w_0)'Vv_0]\otimes v_i\} \\
&\quad +O(\ell^{1/2}T^{-1/2}), && \text{(A.23)} \\
(T/\ell)E[\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes\mathrm{vech}(\hat{S}_T-\bar{S}_T)\otimes m_T] &= o(1). && \text{(A.24)}
\end{aligned}
$$

First, (A.21) follows from several applications of the mixing inequality. Second, we will show (A.22). We have

$$
\begin{aligned}
&T^{\frac{1}{2}}E[\sum_{j=0}^{\ell}\omega_j\mathrm{vech}(\tilde{\Gamma}_j-\Gamma_j)\otimes m_T] \\
= \;&\sum_{i=0}^{\ell}\omega_i\sum_{j=-\ell-T+1}^{T-1}\frac{T-i1(j>i)-|j|1(j>0\text{ or }j\le -i)}{T}E[\mathrm{vech}(v_0v_{-i}')\otimes v_j] \\
= \;&\sum_{i=0}^{\ell}\omega_i\sum_{j=-\ell-T+1}^{T-1}E[\mathrm{vech}(v_0v_{-i}')\otimes v_j]+O(\ell T^{-1}) \\
= \;&\sum_{i=0}^{\ell}\sum_{j=-\ell-T+1}^{T-1}E[\mathrm{vech}(v_0v_{-i}')\otimes v_j]+O(\ell^{-q})+O(\ell T^{-1})
\end{aligned}
$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} E[\mathrm{vech}(v_0 v_{-i}') \otimes v_j] + O(\ell^{-q}) + O(\ell T^{-1}). \tag{A.25}$$

The first equality follows from strict stationarity. Repeated applications of the moment inequality of Yokoyama (1980) produce

$$\begin{aligned}
& \sum_{i=0}^{\ell} \omega_i \sum_{j=-\ell-T+1}^{T-j} \frac{T - i1(j > i) - |j|1(j > 0 \text{ or } j \leq -i)}{T} E[\mathrm{vech}(v_0 v_{-i}') \otimes v_j] \\
= \; & O\left(T^{-1} \sum_{i=0}^{\ell} \omega_i \left[\sum_{j=-\ell-T}^{-2j-1} |j| \alpha_{-i-j}^{r'} + \sum_{j=-2j}^{-j} |j| \alpha_i^{r'} + \sum_{j=-i}^{-(1/2)i} i \alpha_{-j} \right.\right. \\
& \left.\left. + \sum_{j=-(1/2)i+1}^{-1} i \alpha_{i+j}^{r'} + \sum_{j=0}^{i} (i+j) \alpha_i^{r'} + \sum_{j=i+1}^{T-1} (i+j) \alpha_j^{r'} \right] \right) \\
= \; & O(\ell T^{-1}).
\end{aligned} \tag{A.26}$$

for some $r' \in (0, 1)$, from which the second equality follows. Arguments analogous to the proof of Theorem 10 of Hannan (1970, pp.283-284) yield the last two equalities. By symmetric arguments, it follows that

$$\begin{aligned}
& T^{\frac{1}{2}} E[\sum_{j=-\ell}^{-1} \omega_j \mathrm{vech}(\tilde{\Gamma}_j - \Gamma_j) \otimes m_T] \\
= \; & \sum_{i=-\infty}^{-1} \sum_{j=-\infty}^{\infty} E[\mathrm{vech}(v_0 v_{-i}') \otimes v_j] + O(\ell^{-q}) + O(\ell T^{-1}).
\end{aligned} \tag{A.27}$$

Hence, (A.23) follows from (A.25) and (A.27). Third, we will show (A.23). It follows from (A.14), Assumption 1(a) and Lemma A.2 that

$$\begin{aligned}
& T^{\frac{1}{2}} E[\mathrm{vech}(\hat{S}_T - \tilde{S}_T) \otimes m_T] \\
= \; & T^{\frac{1}{2}} E[\mathrm{vech}(\nabla \tilde{S}_T (\tilde{\beta}_T - \beta_0) + (\tilde{\beta}_T - \beta_0)' \nabla^2 \tilde{S}_T (\tilde{\beta}_T - \beta_0)) \otimes m_T] \\
= \; & T^{\frac{1}{2}} E[\mathrm{vech}((\nabla \tilde{S}_T - \nabla \bar{S}_T)(\tilde{\beta}_T - \beta_0) \otimes m_T)] + T^{\frac{1}{2}} E[\mathrm{vech}(\nabla \bar{S}_T (\tilde{\beta}_T - \beta_0) \otimes m_T)] \\
& + T^{\frac{1}{2}} E[\mathrm{vech}(((\tilde{\beta}_T - \beta_0)' \nabla^2 \tilde{S}_T - \nabla^2 \bar{S}_T)(\tilde{\beta}_T - \beta_0)) \otimes m_T] \\
& + T^{\frac{1}{2}} E[\mathrm{vech}(((\tilde{\beta}_T - \beta_0)' \nabla^2 \bar{S}_T (\tilde{\beta}_T - \beta_0)) \otimes m_T)] \\
= \; & \sum_{i=-\infty}^{\infty} E\{\mathrm{vech}[\nabla \bar{S} (E(x_0 z_0')' V E(z_0 x_0'))^{-1} E(x_0 z_0')' V v_0 \otimes v_i]\} + O(\ell^{1/2} T^{-1/2}),
\end{aligned} \tag{A.28}$$

which completes the proof of (A.23). Lastly, we will show (A.24). We have

$$\begin{aligned}
& (T/\ell) E[\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes \mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T] \\
= \; & (T/\ell) E[\mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes \mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T] + o(1) \\
= \; & \ell^{-1} T^{-3/2} \sum_{t,s,u=1}^{T} E[\sum_{i=-\ell}^{\ell} \mathrm{vech}(v_{t+i} v_t' - \Gamma_i) \otimes \sum_{j=-\ell}^{\ell} \mathrm{vech}(v_{s+j} v_s - \Gamma_j) \otimes v_u] + o(1).
\end{aligned} \tag{A.29}$$

Consider two sets of s, t and u: In Set (i) the maximum of $|t-s|$, $|s-t|$ and $|u-t|$ is greater than ℓ and in Set (ii) the maximum of $|t-s|$, $|s-t|$ and $|u-t|$ is smaller than or equal to ℓ. For t, s, u that belong to Set (i), it follows from repeated applications of the moment inequality as in Yokoyama (1980) that the sum of

$$E[\sum_{i=-\ell}^{\ell} \mathrm{vech}(v_{t+i} v_t' - \Gamma_i) \otimes \sum_{j=-\ell}^{\ell} \mathrm{vech}(v_{s+j} v_s - \Gamma_j) \otimes v_u]$$

over Set (i) is of order $O(\ell^2 T)$ where ℓ^2 comes from the variances of $\sum_{i=-\ell}^{\ell} \text{vech}(v_{t+i}v_t' - \Gamma_i)$ and $\sum_{j=-\ell}^{\ell} \text{vech}(v_{s+j}v_s - \Gamma_j)$. For t, s, u that belong to Set (ii), the sum over (ii) is the sum of

$$E[\text{vech}(v_{t+i}v_t') \otimes \text{vech}(v_{s+j}v_s) \otimes v_u],$$

$$E[\text{vech}(\Gamma_i) \otimes \text{vech}(v_{s+j}v_s) \otimes v_u],$$

and

$$E[\text{vech}(v_{t+i}v_t') \otimes \text{vech}(\Gamma_j) \otimes v_u].$$

By repeated applications of the moment inequality again, the sum of each of the three terms is of order $O(\ell^2 T)$ because i, j, $|t-s|$, $|s-u|$ and $|u-t|$ vary from 1 to ℓ. Thus, the first term on the R.H.S. of (A.29) is of order $O(\ell T^{-1/2})$ which is $o(1)$. Therefore, (A.17) follows from (A.21)–(A.24).

Next, we will prove (A.18). It follows from (A.17), Hölder's inequality and Lemma A.2 that

$$\begin{aligned}
\kappa_2(g_T) - 1 &= E(g_T^2) - [E(g_T)]^2 - 1 \\
&= 2E\{\mathbf{a}' m_T \mathbf{b}'[\text{vec}(G_T - G_0) \otimes m_T]\} + 2E\{\mathbf{a}' m_T \mathbf{c}'[\text{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
&\quad + 2E\{\mathbf{a}' m_T \mathbf{e}'[\text{vech}(\tilde{S}_T - \bar{S}_T) \otimes \text{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
&\quad + E\{\mathbf{c}'[\text{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2 + O(\ell^{1/2} T^{-1}). \qquad (A.30)
\end{aligned}$$

Thus, we only need to analyze the first four terms on the RHS of (A.30). First, by repeated applications of the mixing inequality as in the proof of moment inequalities (e.g, the proof of Lemma 4 of Billingsley, 1968, pp.172–174), one can show that

$$TE\{\mathbf{a}' m_T \mathbf{b}'[\text{vec}(G_T - G_0) \otimes m_T]\} = O(1). \qquad (A.31)$$

Second, it follows from arguments similar to the one used in the proof of (A.17) that

$$\begin{aligned}
&(T/\ell)E\{\mathbf{a}' m_T \mathbf{c}'[\text{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
= &\ (\ell T)^{-1} \sum_{j=-\ell}^{\ell} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{u=1}^{T} \omega_j E\{\mathbf{a}' v_t \mathbf{c}'[\text{vech}(v_s v_{s-j}' - \Gamma_j) \otimes v_u]\} \\
= &\ \ell^{-1} \sum_{j=-\ell}^{\ell} \sum_{i,k=-T+1}^{T-1} \omega_j (1 - \tau_{i,k}) E\{\mathbf{a}' v_0 \mathbf{c}'[\text{vech}(v_i v_{i-j}' - \Gamma_j) \otimes v_k]\} \\
= &\ \ell^{-1} \sum_{j=-\ell}^{\ell} \sum_{i,k=-T+1}^{T-1} \omega_j E\{\mathbf{a}' v_0 \mathbf{c}'[\text{vech}(v_i v_{i-j}' - \Gamma_j) \otimes v_k]\} + O(\ell T^{-1}) \\
= &\ \ell^{-1} \sum_{j=-\ell}^{\ell} \sum_{i,k=-T+1}^{T-1} E\{\mathbf{a}' v_0 \mathbf{c}'[\text{vech}(v_i v_{i-j}' - \Gamma_j) \otimes v_k]\} + O(\ell^{-q_w}) + O(\ell T^{-1}) \\
= &\ \lim_{T\to\infty} \ell^{-1} \sum_{j=-\ell}^{\ell} \sum_{t=-T+1}^{T-1} \sum_{s=-T+1}^{T-1} E\{\mathbf{a}' v_0 \mathbf{c}'[\text{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_s]\} + O(\ell^{-1}), \qquad (A.32)
\end{aligned}$$

$$\begin{aligned}
&(T/\ell)E\{\mathbf{a}' m_T \mathbf{e}'[\text{vech}(\tilde{S}_T - \bar{S}_T) \otimes \text{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
= &\ \frac{1}{\ell T^2} \sum_{i,j=-\ell}^{\ell} \sum_{r,s,t,u=1}^{T} \omega_i \omega_j E\{\mathbf{a}' v_r \mathbf{e}'[\text{vech}(v_s v_{s-i}' - \Gamma_i) \otimes \text{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_u]\} \\
= &\ \frac{1}{\ell T} \sum_{i,j=-\ell}^{\ell} \sum_{s,t,u=-T}^{T} \omega_i \omega_j (1 - \tau_{s,t,u}) E\{\mathbf{a}' v_0 \mathbf{e}'[\text{vech}(v_s v_{s-i}' - \Gamma_i) \otimes \text{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_u]\} \\
= &\ \frac{1}{\ell T} \sum_{i,j=-\ell}^{\ell} \sum_{s,t,u=-T}^{T} \omega_i \omega_j E\{\mathbf{a}' v_0 \mathbf{e}'[\text{vech}(v_s v_{s-i}' - \Gamma_i) \otimes \text{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_u]\}
\end{aligned}$$

$$
\begin{aligned}
& +O(\ell^2 T^{-1}) \\
= \; & \frac{1}{\ell T} \sum_{i,j=-\ell}^{\ell} \sum_{s,t,u=-T}^{T} E\{\mathbf{a}' v_0 \mathbf{e}' [\mathrm{vech}(v_s v_{s-i}' - \Gamma_i) \otimes \mathrm{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_u]\} \\
& +O(\ell^{-q}) + O(\ell^2 T^{-1}) \\
= \; & \lim_{T\to\infty} \frac{1}{\ell T} \sum_{i,j=-\ell}^{\ell} \sum_{s,t,u=-T}^{T} E\{\mathbf{a}' v_0 \mathbf{e}' [\mathrm{vech}(v_s v_{s-i}' - \Gamma_i) \otimes \mathrm{vech}(v_t v_{t-j}' - \Gamma_j) \otimes v_u]\} \\
& +O(\ell^{-1}), \qquad \text{(A.33)}
\end{aligned}
$$

and

$$
\begin{aligned}
& (T/\ell) E\{\mathbf{c}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2 \\
= \; & \ell^{-1} T^{-2} \sum_{t,s,u,v=1}^{T} \sum_{i,j=-\ell}^{\ell} \omega_i \omega_j E\{\mathbf{c}' [\mathrm{vech}(v_s v_{s-i}' - \Gamma_i) \otimes v_t] \mathbf{c}' [\mathrm{vech}(v_u v_{u-j}' - \Gamma_j) \otimes v_v]\} \\
= \; & (\ell T)^{-1} \sum_{j,k,m=-T}^{T} \sum_{i,l=-\ell}^{\ell} \omega_i \omega_j (1 - \tau_{j,k,m}) E\{\mathbf{c}' [\mathrm{vech}(v_0 v_{-i}' - \Gamma_i) \otimes v_j] \\
& \times \mathbf{c}' [\mathrm{vech}(v_k v_{k-l}' - \Gamma_k) \otimes v_m]\} \\
= \; & (\ell T)^{-1} \sum_{j,k,m=-T}^{T} \sum_{i,l=-\ell}^{\ell} \omega_i \omega_j E\{\mathbf{c}' [\mathrm{vech}(v_0 v_{-i}' - \Gamma_i) \otimes v_j] \mathbf{c}' [\mathrm{vech}(v_k v_{k-l}' - \Gamma_k) \otimes v_m]\} \\
& +O(\ell T^{-1}) \\
= \; & (\ell T)^{-1} \sum_{j,k,m=-T}^{T} \sum_{i,l=-\ell}^{\ell} E\{\mathbf{c}' [\mathrm{vech}(v_0 v_{-i}' - \Gamma_i) \otimes v_j] \mathbf{c}' [\mathrm{vech}(v_k v_{k-l}' - \Gamma_k) \otimes v_m]\} \\
& +O(\ell^{-q}) + O(\ell T^{-1}) \\
= \; & \lim_{T\to\infty} \ell^{-1} T^{-1} \sum_{j,k,m=-T}^{T} \sum_{i,l=-\ell}^{\ell} E\{\mathbf{c}' [\mathrm{vech}(v_0 v_{-i}' - \Gamma_i) \otimes v_j] \mathbf{c}' [\mathrm{vech}(v_k v_{k-l}' - \Gamma_k) \otimes v_m]\} \\
& +O(\ell^{-1}), \qquad \text{(A.34)}
\end{aligned}
$$

where $\tau_{i,k} = (1/T)\min(\max(|i|, |k|, |i-k|), T)$ and $\tau_{s,t,u} = (1/T)\min(\max(|s|, |t|, |u|, |s-t|, |t-u|, |u-s|), T)$. The proofs of (A.32), (A.33) and (A.34) are similar to that of (A.17) and thus details are omitted. Therefore, (A.18) follows from (A.30)–(A.33).

Third, we will prove (A.19). By (A.17), (A.18) and

$$
\kappa_3(g_T) \; = \; E(g_T^3) - 3E(g_T^2)E(g_T) + 2(E(g_T))^3, \qquad \text{(A.35)}
$$

it suffices to show that

$$
T^{1/2} E(g_T^3) \; = \; \kappa_\infty + O(\ell^{-q}) + o(\ell T^{-1/2}). \qquad \text{(A.36)}
$$

It follows from Assumption 1(a), Hölder's inequality and Lemma A.2 that

$$
\begin{aligned}
E(g_T^3) \; = \; & E[(\mathbf{a}' m_T)^3] + 3E\{(\mathbf{a}' m_T)^2 \mathbf{b}' [\mathrm{vec}(G_T - G_0)' \otimes m_T]\} \\
& +3E\{(\mathbf{a}' m_T)^2 \mathbf{c}' [\mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
& +3E\{(\mathbf{a}' m_T)^2 \mathbf{c}' [\mathrm{vech}(\hat{S}_T - \tilde{S}_T) \otimes m_T]\} + o(\ell T^{-1}). \qquad \text{(A.37)}
\end{aligned}
$$

The rest of the proof is similar to that of (A.17), and thus we will only show that

$$
\begin{aligned}
& T^{\frac{1}{2}} E\{(\mathbf{a}' m_T)^2 \mathbf{c}' [\sum_{j=-\ell}^{\ell} \mathrm{vech}(\tilde{\Gamma}_j - \Gamma_j) \otimes m_T]\} \\
= \; & \lim_{T\to\infty} (1/T) \sum_{\tau,t,s,k=-T+1}^{T-1} E\{\mathbf{a}' v_0 \mathbf{a}' v_\tau \mathbf{c}' [\mathrm{vech}(v_t v_{t-k}' - \Gamma_k) \otimes v_s]\}. \qquad \text{(A.38)}
\end{aligned}
$$

It follows from arguments similar to the proof of (A.21) that

$$\begin{aligned}
& T^{\frac{1}{2}} E\{(\mathbf{a}' m_T)^2 \mathbf{c}' [\mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} \\
= \; & (1/T) \sum_{s,t,u=-T+1}^{T-1} \sum_{j=-\ell}^{\ell} \omega_j (1 - \tau_{s,t,u}) E\{\mathbf{a}' v_0 \mathbf{a}' v_s \mathbf{c}' [\mathrm{vech}(v_t v_{t-j} - \Gamma_j) \otimes v_u]\} \\
= \; & (1/T) \sum_{s,t,u=-T+1}^{T-1} \sum_{j=-\ell}^{\ell} \omega_j E\{\mathbf{a}' v_0 \mathbf{a}' v_s \mathbf{c}' [\mathrm{vech}(v_t v_{t-j} - \Gamma_j) \otimes v_u]\} + O(T^{-1}) \\
= \; & (1/T) \sum_{s,t,u=-T+1}^{T-1} \sum_{j=-\ell}^{\ell} E\{\mathbf{a}' v_0 \mathbf{a}' v_s \mathbf{c}' [\mathrm{vech}(v_t v_{t-j} - \Gamma_j) \otimes v_u]\} + O(\ell^{-q}) \\
= \; & \lim_{T \to \infty} T^{-1} \sum_{\tau,t,s=-T+1}^{T-1} \sum_{j=-\ell}^{\ell} E\{\mathbf{a}' v_0 \mathbf{a}' v_\tau \mathbf{c}' [\mathrm{vech}(v_t v_{t-j}' - \Gamma_j)' \otimes v_s]\} + O(\ell^{-q}). \qquad \text{(A.39)}
\end{aligned}$$

By arguments similar to the proof of Lemma 1 of Andrews (1991, pp.850–851), one can show that the RHS of (A.39) equals the infinite sum of the product of two expectations plus some finite number. By the mixing inequality, it follows that the infinite sum of the product of two expectations is finite. Therefore, the RHS of (A.39) is well defined.

Lastly, we will show (A.20).

$$\begin{aligned}
\kappa_4(g_T) - 3 \; = \; & 4E\{(\mathbf{a}' m_T)^3 \mathbf{c}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\} \\
& + 4E\{(\mathbf{a}' m_T)^3 \mathbf{e}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes \mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\} \\
& + 6E\left((\mathbf{a}' m_T)^2 \{\mathbf{c}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2\right) \\
& - 12E\{\mathbf{a}' m_T \mathbf{c}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\} \\
& - 12E\{\mathbf{a}' m_T \mathbf{e}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes \mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\} \\
& - 6E\{\mathbf{c}' [\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2 + O(\ell^{1/2} T^{-1}), \qquad \text{(A.40)}
\end{aligned}$$

from which the desired result follows by similar arguments. *Q.E.D.*

Lemma A.4: Let $\psi_{g,T}(x)$ denote the characteristic function of g_T. Then

$$\begin{aligned}
& \psi_{g,T}(x) \\
= \; & \exp(-\frac{\theta^2}{2}) \left[T^{-\frac{1}{2}} (\alpha_\infty (i\theta) - \frac{i\theta^3}{6} (\kappa_\infty - 3\alpha_\infty)) - \frac{\ell}{T} (\frac{\theta^2}{2} \gamma_\infty + \frac{\theta^4}{24} \zeta_\infty) + o(\frac{\ell}{T}) \right], \qquad \text{(A.41)}
\end{aligned}$$

$$P(g_T \leq x) \; = \; \Psi(x) + T^{-1/2} p_1(x) + (\ell/T) p_2(x) + o(\ell/T). \qquad \text{(A.42)}$$

Proof of Lemma A.4: The proof of (A.41) follows from the series expansion argument. (A.42) can be obtained by inverting (A.41). *Q.E.D.*

Lemma A.5: Following Götze and Künsch (1996), define a truncation function by

$$\tau(x) \; = \; T^\gamma x f(T^{-\gamma} \|x\|) / \|x\|$$

where $\gamma \in (2/r, 1/2)$ and $f \in C^\infty(0, \infty)$ satisfies (i) $f(x) = x$ for $x \leq 1$; (ii) f is increasing; and (iii) $f(x) = 2$ for $x \geq 2$. Let $f_T^\dagger$ denote f_T with $\bar{R}_t \equiv (v_t', \tilde{v}_t, \mathrm{vec}(w_t)')$ replaced by

$$\bar{R}_t^\dagger = (v_t^{\dagger\prime}, \tilde{v}_t^{\dagger\prime}, \mathrm{vec}(w_t^\dagger)')' \; = \; \tau\left((v_t', \tilde{v}_t', \mathrm{vec}(w_t)')'\right).$$

Let $\Psi_T^\dagger$ and $\Psi_{g,T}^\dagger$ denote the Edgeworth expansions of $f_T^\dagger$ and $g_T^\dagger$, respectively. Let $\psi_{g,T}^\dagger(x)$ and $\tilde{\psi}_{g,T}^\dagger(x)$ denote the characteristic functions of $g_T^\dagger$ and $\Psi_{g,T}^\dagger$, respectively. Then

$$\sup_x |P(f_T \leq x) - \Psi_T(x)| \leq C \int_{|\theta| < T^{1-2/r}} |\psi_{g,T}^\dagger(\theta) - \tilde{\psi}_{g,T}^\dagger(\theta)| |\theta|^{-1} d\theta + O(\ell^{-q}) + o(\ell T^{-1}). \qquad \text{(A.43)}$$

Proof of Lemma A.5: First, we will show that

$$\sup_{-\infty<x<\infty} |P(f_T \le x) - \Psi_T(x)| \;=\; \sup_{-\infty<x<\infty} |P(f_T^\dagger \le x) - \Psi_T^\dagger(x)| + o(\ell T^{-1}). \tag{A.44}$$

Since

$$P\left(\max_{1\le t\le T} \|\bar{R}_t\| > T^\gamma\right) \;\le\; \sum_{t=1}^{T} P(\|\bar{R}_t\| > T^\gamma) \;=\; O(T^{1-\gamma r}), \tag{A.45}$$

it follows that

$$\sup_{-\infty<x<\infty} \left|P(f_T \le x) - P(f_T^\dagger \le x)\right| \;=\; O(T^{1-\gamma r}) \;=\; O(T^{-1}). \tag{A.46}$$

Then it follows from Lemma A.2 and (A.45) that

$$\begin{aligned} E\|m_T^\dagger - m_T\|^j &\le 2^j E[\|m_T\|^j I(\max_{1\le t\le T}\|R_t\| > T^\gamma)] \\ &\le 2^j (E\|m_T\|^{2j})^{1/2} P(\max_{1\le t\le T}\|R_t\| > T^\gamma)^{1/2} \\ &= o(T^{-1/2}) \end{aligned} \tag{A.47}$$

for $j \le r/2$. Similarly, we obtain that

$$E\|T^{1/2}\mathrm{vec}[(G_T^\dagger - G_0^\dagger) - (G_T - G_0)]\|^j \;=\; o(T^{-1/2}), \tag{A.48}$$

$$E\|(T/\ell)^{1/2}[\mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) - \mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger)]\delta\|^j \;=\; o(T^{-1/2}), \tag{A.49}$$

$$E\|(T/\ell)^{1/2}[\mathrm{vech}(\nabla\tilde{S}_T - \nabla\bar{S}_T) - \mathrm{vech}(\nabla\tilde{S}_T - \nabla\bar{S}_T)]\|^j \;=\; o(T^{-1/2}), \tag{A.50}$$

$$E\|T^{1/2}[\mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) - \mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger)]\|^j \;=\; o(T^{-1/2}), \tag{A.51}$$

for $j \le r/2$. Thus it follows from Lemma A.2, (A.45), (A.47)-(A.51) that

$$\sup_{-\infty<x<\infty} \left|\Psi_T(x) - \Psi_T^\dagger(x)\right| \;=\; o(\ell T^{-1}). \tag{A.52}$$

Therefore (A.44) follows from (A.46) and (A.52).

Next, we will show that

$$\sup_x |P(f_T^\dagger \le x) - \Psi_T^\dagger(x)| = \sup_x |P(g_T^\dagger \le x) - \Psi_{g,T}^\dagger(x)| + O(\ell^{-q}) + O(\ell^{3/2}T^{-3/2}). \tag{A.53}$$

Let

$$\begin{aligned} h_T^\dagger \;=\; & g_T^\dagger + \mathbf{c}'[\mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes m_T^\dagger] + \mathbf{d}'[\mathrm{vec}(G_T - G_0) \otimes \mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes m_T^\dagger] \\ & + \mathbf{e}'[\mathrm{vech}(\hat{S}_T^\dagger - \bar{S}_T^\dagger) \otimes \mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes m_T^\dagger] \\ & + \mathbf{e}'[\mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes \mathrm{vech}(\hat{S}_T^\dagger - \bar{S}_T^\dagger) \otimes m_T^\dagger] \\ & + \mathbf{e}'[\mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes \mathrm{vech}(\bar{S}_T^\dagger - S_0^\dagger) \otimes m_T^\dagger], \end{aligned}$$

and let $\Psi_{h,T}(x)$ denote its Edgeworth expansion. Using the definition of Taylor series expansions, Lemma A.2 and Markov's inequality, $P(|f_T^\dagger - h_T^\dagger| > \ell^{3/2}T^{-3/2})$ can be made arbitrarily small. Thus we have

$$\sup_x |P(f_T^\dagger \le x) - \Psi_T^\dagger(x)| \;=\; \sup_x |P(h_T^\dagger \le x) - \Psi_{h,T}^\dagger(x)| + O(\ell^{3/2}T^{-3/2}). \tag{A.54}$$

Since the difference between the Edgeworth expansions of $g_T^\dagger$ and of $h_T^\dagger$ is $O(\bar{S}_T^\dagger - S_T^\dagger)$, it follows that

$$\sup_x |P(h_T^\dagger \le x) - \Psi_{h,T}^\dagger(x)| \;=\; \sup_x |P(g_T^\dagger \le x) - \Psi_{g,T}^\dagger(x)| + O(\ell^{-q}). \tag{A.55}$$

Therefore, (A.53) follows from (A.54) and (A.55).

Lastly, it follows from the so-called smoothing lemma (e.g., Proposition C1 of Fan and Linton, 1997) that

$$\sup_x |P(g_T^\dagger \leq x) - \Psi_{g,T}^\dagger(x)| \leq C \int_{|\theta|<T^{1-2/r}} |\psi_{g,T}^\dagger(\theta) - \tilde{\psi}_{g,T}^\dagger(\theta)||\theta|^{-1} d\theta + O(T^{-1+2/r}). \tag{A.56}$$

Therefore, Lemma A.5 follows from (A.44), (A.53) and (A.56) as $r > 12$. *Q.E.D.*

Lemma A.6: For $0 < \varepsilon < 1/6$,

$$\int_{|\theta|\leq T^\varepsilon} |\psi_{g,T}^\dagger(\theta) - \tilde{\psi}_{g,T}^\dagger(\theta)||\theta|^{-1} d\theta \ = \ o(\ell T^{-1}). \tag{A.57}$$

Proof of Lemma A.6: Write $g_T^\dagger$ as

$$\begin{aligned} g_T^\dagger \ &= \ \mathbf{a}' m_T^\dagger + \mathbf{b}'[\mathrm{vec}(G_T^\dagger - G_0^\dagger) \otimes m_T^\dagger] + \mathbf{c}'[\mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T) \otimes m_T^\dagger] \\ &\quad + \mathbf{c}'[\mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T) \otimes m_T^\dagger] + \mathbf{d}'[\mathrm{vec}(G_T^\dagger - G_0^\dagger) \otimes \mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) \otimes m_T^\dagger] \\ &\quad + \mathbf{d}'[\mathrm{vec}(G_T^\dagger - G_0^\dagger) \otimes \mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) \otimes m_T^\dagger] \\ &\quad + \mathbf{e}'[\mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) \otimes \mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) \otimes m_T^\dagger] \\ &\quad + \mathbf{e}'[\mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) \otimes \mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) \otimes m_T^\dagger] \\ &\quad + \mathbf{e}'[\mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) \otimes \mathrm{vech}(\tilde{S}_T^\dagger - \bar{S}_T^\dagger) \otimes m_T^\dagger] \\ &\quad + \mathbf{e}'[\mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) \otimes \mathrm{vech}(\hat{S}_T^\dagger - \tilde{S}_T^\dagger) \otimes m_T^\dagger] \\ &\equiv \ g_{T,1}^\dagger + g_{T,2}^\dagger + ... + g_{T,10}^\dagger. \end{aligned}$$

Then a Taylor series expansion of $E(\exp(i\theta g_T^\dagger))$ around $g_{T,2}^\dagger + g_{T,3}^\dagger + ... + g_{T,10}^\dagger = 0$ yields

$$\begin{aligned} E(\exp(i\theta g_T^\dagger)) \ &= \ E(\exp(i\theta g_{T,1}^\dagger) + i\theta E[\exp(i\theta g_{T,1}^\dagger)(g_{T,2}^\dagger + g_{T,3}^\dagger + g_{T,4}^\dagger)] \\ &\quad + \frac{(i\theta)^2}{2} E[\exp(i\theta g_{T,1}^\dagger)(2g_{T,1}^\dagger g_{T,3}^\dagger + 2g_{T,1}^\dagger g_{T,7}^\dagger + g_{T,3}^{\dagger 2})] \\ &\quad + \frac{(i\theta)^3}{6} E[\exp(i\theta g_{T,1}^\dagger)(3g_{T,1}^{\dagger 2} g_{T,2}^\dagger + 3g_{T,1}^{\dagger 2} g_{T,3}^\dagger + 3g_{T,1}^{\dagger 2} g_{T,4}^\dagger)] \\ &\quad + \frac{(i\theta)^4}{24} E[\exp(i\theta g_{T,1}^\dagger)(4g_{T,1}^{\dagger 3} g_{T,3}^\dagger + 4g_{T,1}^{\dagger 3} g_{T,7}^\dagger + 6g_{T,1}^{\dagger 2} g_{T,3}^{\dagger 2})] \\ &\quad + O(\theta^4[E(g_{T,2}^{\dagger 4}) + E(g_{T,3}^{\dagger 4}) + ... + E(g_{T,10}^{\dagger 4})]). \end{aligned} \tag{A.58}$$

We will analyze each term on the RHS of (A.58) in turn. First, it follows from Lemma 3.33 of Götze and Hipp (1983) that

$$\begin{aligned} & E\left\{\exp(i\theta g_{T,1}^\dagger) - \left[1 + \frac{(i\theta)^3}{6} E(\mathbf{a}' m_T^\dagger)^3 + \frac{\theta^4}{24}(E(\mathbf{a}' m_T^\dagger)^4 - 3) - \frac{\theta^6}{72}(E(\mathbf{a}' m_T^\dagger)^3)^2\right] \exp(-\frac{\theta^2}{2})\right\} \\ = \ & O((1 + |\theta|^9) \exp(-\theta^2) T^{-1-\varepsilon}). \end{aligned} \tag{A.59}$$

Second, let $\tilde{\psi}_X$ denote the multivariate expansion of $E(\exp(ic' T^{-1/2} \sum_{t=1}^T X_t^\dagger))$ where $X_t^\dagger = (\mathbf{a}' v_t^\dagger, v_t^{\dagger\prime}, (w_t^\dagger - \mathrm{vec} G_0^\dagger)')'$. Then an application of Lemma 3.33 of Götze and Hipp (1983) with $\vartheta = (\theta, 0, ..., 0)'$ yields

$$\begin{aligned} & |E\{\exp(i\theta g_{T,1}^\dagger)[i\theta g_{T,2}^\dagger + \frac{(i\theta)^3}{2} g_{T,1}^{\dagger 2} g_{T,2}^\dagger]\} \\ & - \left((i\theta - \frac{(i\theta)^3}{2}) E\{\mathbf{b}'[\mathrm{vec}(G_T^\dagger - G_0^\dagger) \otimes m_T^\dagger]\} + \frac{(i\theta)^3}{2} E\{(\mathbf{a}' m_T^\dagger)^2 \mathbf{b}'[\mathrm{vec}(G_T^\dagger - G_0^\dagger) \otimes m_T^\dagger]\}\right) \exp(-\frac{\theta^2}{2})| \\ \leq \ & T^{-1/2} \sum_\alpha |c_\alpha||D^\alpha[E(\exp(i\vartheta' T^{-1/2} \sum_{t=1}^T X_{2t}^\dagger)) - \tilde{\psi}_X]| \\ = \ & O((1 + |\theta|^8 + |\theta|^{10}) \exp(-\theta^2) T^{-1-\varepsilon}), \end{aligned} \tag{A.60}$$

where c_α are the corresponding elements of a, b and G_0.

Third, we will show that

$$\begin{aligned}
& |i\theta E[\exp(i\theta g^\dagger_{T,1})[i\theta g^\dagger_{T,3} + (i\theta)^2 g^\dagger_{T,1} g^\dagger_{T,3} + \frac{(i\theta)^3}{2} g^{\dagger 2}_{T,1} g^\dagger_{T,3} + \frac{(i\theta)^4}{6} g^{\dagger 3}_{T,1} g^\dagger_{T,3}]] \\
- & \Big((i\theta - \frac{1}{2}(i\theta)^3) E\{\mathbf{c}'[\mathrm{vech}(\tilde{S}^\dagger_T - \bar{S}^\dagger_T) \otimes m^\dagger_T]\} + (i\theta)^2 E\{\mathbf{a}' m^\dagger_T \mathbf{c}'[\mathrm{vech}(\tilde{S}^\dagger_T - \bar{S}^\dagger_T) \otimes m^\dagger_T]\} \\
& + \frac{(i\theta)^3}{2} E\{(\mathbf{a}' m^\dagger_T)^2 \mathbf{c}'[\mathrm{vech}(\tilde{S}^\dagger_T - \bar{S}^\dagger_T) \otimes m^\dagger_T]\} \\
& + \frac{(i\theta)^4}{6} E\{(\mathbf{a}' m^\dagger_T)^3 \mathbf{c}'[\mathrm{vech}(\tilde{S}^\dagger_T - \bar{S}^\dagger_T) \otimes m^\dagger_T]\}\Big) \exp(-\frac{1}{2}\theta^2)| \\
= & O((1 + |\theta|^6) \exp(-\theta^2) \ell T^{-1-\varepsilon}). \qquad \text{(A.61)}
\end{aligned}$$

Note that the first term of (A.61) can be written as a weighted sum of

$$E\{\exp(i\theta g^\dagger_{T,1})[i\theta + (i\theta)^2 \mathbf{a}' m^\dagger_T + \frac{(i\theta)^3}{2}(\mathbf{a}' m^\dagger_T)^2 + \frac{(i\theta)^4}{6}(\mathbf{a}' m^\dagger_T)^3] \mathbf{c}'[\mathrm{vech}(\tilde{\Gamma}^\dagger_j - \Gamma^\dagger_j) \otimes m^\dagger_T]\} \qquad \text{(A.62)}$$

and that the rest of the terms can be written as a weighted sum of

$$E\{[i\theta - \frac{1}{2}(i\theta)^3 + (i\theta)^2 \mathbf{a}' m^\dagger_T + \frac{(i\theta)^3}{2}(\mathbf{a}' m^\dagger_T)^2 + \frac{(i\theta)^4}{6}(\mathbf{a}' m^\dagger_T)^3] \mathbf{c}'[\mathrm{vech}(\tilde{\Gamma}^\dagger_j - \Gamma^\dagger_j) \otimes m^\dagger_T]\} \exp(-\frac{\theta^2}{2}) \qquad \text{(A.63)}$$

We will apply Lemma 3.33 of Götze and Hipp (1983) to (A.62) and (A.63). Let $\tilde{\psi}_Y$ denote the multivariate expansion of $E(\exp(i\vartheta' T^{-1/2} \sum_{t=1}^T Y^\dagger_t))$ where $\vartheta = (\theta, 0, ..., 0)$ and

$$Y^\dagger_t = (\mathbf{a}' m^\dagger_T, m^{\dagger\prime}_T, T^{-1/2} \sum_{t=1}^T \mathrm{vech}[v_t v'_{t-j} - E(v_t v'_{t-j})]')'.$$

Then the difference between (A.61) and (A.62) are bounded by

$$T^{-1/2} \sum_\alpha |c_\alpha| D^\alpha \left| E(\exp(i\vartheta' T^{-1/2} \sum_{t=1}^T Y^\dagger_t)) - \tilde{\psi}_Y) \right| = O((1 + |\theta|^6) \exp(-\theta^2) T^{-1-\varepsilon}), \qquad \text{(A.64)}$$

where c_α are the corresponding linear combinations of a and c. Thus (A.61) follows.

Fourth, by arguments analogous to the proof of (A.61), one can show that

$$\begin{aligned}
& |E[\exp(i\theta g^\dagger_{T,1})(i\theta g^\dagger_{T,4} + \frac{(i\theta)^3}{2} g^{\dagger 2}_{T,1} g^\dagger_{T,4})] \\
- & ((i\theta - \frac{1}{2}(i\theta)^3) E\{\mathbf{c}'[\mathrm{vech}(\hat{S}^\dagger_T - \tilde{S}^\dagger_T) \otimes m^\dagger_T]\} + (i\theta)^3 E\{(\mathbf{a}' m^\dagger_T)^2 \mathbf{c}'[\mathrm{vech}(\hat{S}^\dagger_T - \tilde{S}^\dagger_T) \otimes m^\dagger_T]\}) \\
& \times \exp(-\frac{\theta^2}{2})| \\
= & O((1 + |\theta|^6) \exp(-\theta^2) \ell T^{-1-\varepsilon}), \qquad \text{(A.65)}
\end{aligned}$$

and

$$\begin{aligned}
& |E[\exp(i\theta g^\dagger_{T,1})[\frac{(i\theta)^2}{2}(2 g^\dagger_{T,1} g^\dagger_{T,7} + g^2_{T,3}) + \frac{(i\theta)^4}{6} g^{\dagger 3}_{T,1} g^\dagger_{T,7} + \frac{(i\theta)^4}{4} g^{\dagger 2}_{T,1} g^{\dagger 2}_{T,3}] \\
- & (\frac{(i\theta)^2}{2}(2E\{\mathbf{a}' m_T \mathbf{c}'[\mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T]\} + E\{\mathbf{c}'[\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2) \\
& + \frac{(i\theta)^4}{6} E\{\mathbf{a}' m^\dagger_T \mathbf{e}'[\mathrm{vech}(\hat{S}^\dagger_T - \bar{S}^\dagger_T) \otimes \mathrm{vech}(\hat{S}^\dagger_T - \bar{S}^\dagger_T) \otimes m^\dagger_T]\} \\
& + \frac{(i\theta)^4}{6} E\{\mathbf{c}'[\mathrm{vech}(\hat{S}_T - \bar{S}_T) \otimes m_T]\}^2) \exp(-\frac{\theta^2}{2})| \\
= & O((1 + |\theta|^6) \exp(-\theta^2) \ell^2 T^{-3/2-\varepsilon}). \qquad \text{(A.66)}
\end{aligned}$$

Lastly, it follows from Lemma A.2 that

$$\theta^4[E(g_{T,2}^{\dagger 4}) + E(g_{T,3}^{\dagger 4}) + \ldots + E(g_{T,4}^{\dagger 4})] \;=\; O(\theta^4\ell^2T^{-2}). \tag{A.67}$$

Combining and integrating (A.59), (A.60), (A.61), (A.65), (A.66) and (A.67) produces the desired result. *Q.E.D.*

Lemma A.7:

$$\int_{T^\varepsilon<|\theta|<T^{1-2/r}} |\psi_{g,T}^{\dagger}(\theta) - \tilde{\psi}_{g,T}^{\dagger}(\theta)||\theta|^{-1}d\theta = o(\ell T^{-1}). \tag{A.68}$$

Proof of Lemma A.7: We closely follow the proof of Götze and Künsch (1996, pp.1927–1930). To simplify the notation, we will omit the superscript †. Let $m = M\log T$ for some $M > 0$. Let $N = [(T/\theta^2+1)m^2]$ for $T^\varepsilon < |\theta| < T^{1-2/r}$. Then $m \le N \le T$ for sufficiently large T. Define

$$\begin{aligned}
m_N = T^{-1/2}\sum_{t=1}^{N} v_t, &\qquad m_{T-N} = T^{-1/2}\sum_{t=N+1}^{T} v_t,\\
G_N - E(G_N) = (1/T)\sum_{t=1}^{N}(w_t - E(w_t)), &\qquad G_{T-N} - E(G_{T-N}) = (1/T)\sum_{t=N+1}^{T}(w_t - E(w_t)),\\
\tilde{S}_N - \bar{S}_N = \sum_{j=-\ell}^{\ell}\omega_j(\tilde{\Gamma}_{j,N} - \Gamma_j), &\qquad \tilde{S}_{T-N} - \bar{S}_{T-N} = \sum_{j=-\ell}^{\ell}\omega_j(\tilde{\Gamma}_{j,T-N} - \Gamma_j),\\
\hat{S}_N - \tilde{S}_N = \sum_{j=-\ell}^{\ell}\omega_j(\hat{\Gamma}_{j,N} - \tilde{\Gamma}_{j,N}), &\qquad \hat{S}_{T-N} - \tilde{S}_{T-N} = \sum_{j=-\ell}^{\ell}\omega_j(\hat{\Gamma}_{j,T-N} - \tilde{\Gamma}_{j,T-N})
\end{aligned}$$

so that

$$\begin{aligned}
m_T &= m_N + m_{T-N},\\
G_T - G_0 &= G_N - E(G_N) + G_{T-N} - E(G_{T-N}),\\
\tilde{S}_T - \bar{S}_T &= \tilde{S}_N - \bar{S}_N + \tilde{S}_{T-N} - \bar{S}_{T-N},\\
\hat{S}_T - \tilde{S}_T &= \hat{S}_N - \tilde{S}_N + \hat{S}_{T-N} - \tilde{S}_{T-N}.
\end{aligned}$$

Write

$$g_T = \mathbf{a}'m_T + Q(m_T, G_T, \hat{S}_T, \tilde{S}_T, \bar{S}_T).$$

Then a Taylor series expansion of Q around $v_t = 0$ and $w_t = 0$ for $t = 1, 2, ..., N$ yields

$$\begin{aligned}
&E\exp(i\theta g_T)\\
=\;& E[\exp(i\theta\mathbf{a}'m_T + i\theta Q(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N})\\
&\times\sum_{\alpha,\beta} v^{\mu}w^{\nu}Q_{\mu\nu}(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N})]\\
&+O(|\theta|^r E|Q(m_T, G_T, \hat{S}_T, \tilde{S}_T, \bar{S}_T) - Q(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N})|^r)
\end{aligned} \tag{A.69}$$

where the power is element-by-element and the indices satisfy

$$\mu \;=\; (\mu_1, ..., \mu_{N+\ell-1}, 0, ..., 0),\;\; \nu \;=\; (\nu_1, ..., \nu_N, 0, ..., 0),\;\; |\mu| + |\nu| \;\le\; 5(r-1).$$

First, we will consider the expansion terms in (A.69). Let

$$\begin{aligned}
\{j_1^0, ..., j_{5(r-1)}^0\} &= \{j : \mu_j \text{ or } \nu_j > 0\},\\
I &= \{j \in \{1, ..., N-m\} : |j - j_k^0| \ge 3m, k = 1, ..., 5(r-1)\},\\
j_{k+1} &= \inf\{j \in I : j \ge j_k + 7m\}
\end{aligned}$$

and $j_1 = \inf I$. Let s denote the smallest integer for which the inf is undefined. Let

$$\begin{aligned}
A_k &= \prod\{\exp(i\theta T^{-1/2}\mathbf{a}'v_t : j \in I, |j - j_k| \leq m\},\ k = 1, ..., s, \\
B_k &= \prod\{\exp(i\theta T^{-1/2}\mathbf{a}'v_t : j \in I, j_k + m + 1 \leq j \leq j_{k+1} - m - 1\}, k = 1, ..., s - 1, \\
B_s &= \prod\{\exp(i\theta T^{-1/2}\mathbf{a}'v_t : j \in I, j \geq j_s - m - 1\}, \\
R &= \prod_{j \notin I} \exp(i\theta T^{-1/2}\mathbf{a}'v_t) \exp(i\theta Q(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N}))v^{\mu}w^{\nu}Q_{\mu\nu}.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&E[\exp(i\theta\mathbf{a}'m_T + i\theta Q(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N}) \\
&\quad \times \sum_{\alpha,\beta} v^{\mu}w^{\nu}Q_{\mu\nu}(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N})] \;=\; \prod_{k=1}^{s} A_k B_k R. \qquad \text{(A.70)}
\end{aligned}$$

Note that $|A_k| \leq 1$, $|B_k| \leq 1$, $|R| \leq T^{\gamma(s-1)r}$, and that A_k, B_k and R are measurable with respect to $\mathcal{F}_{j_k-2m}^{j_k+2m}$, $\mathcal{F}_{j_k-1}^{j_k+1}$, $\{\mathcal{F}_l : \exists j \notin I,\ |l - j| \leq m\}$, respectively. By Assumption 1(d), it follows that

$$\begin{aligned}
&|E[\prod_{k=1}^{s} A_k B_k R] - E[\prod_{k=1}^{s} E(A_k|\mathcal{F}_j : |j - j_k| \leq 3m)B_k R]| \\
\leq\ & \sum_{j=1}^{s} |E[\prod_{k=1}^{j-1} A_k B_k (A_j - E(A_j|\mathcal{F}_j : |j - j_k| \leq 3m)) \prod_{l=j+1}^{s} E(A_l|\mathcal{F}_j, |j - j_l| \leq 3m)B_l| \\
=\ & \sum_{j=1}^{s} |E[\prod_{k=1}^{j-1} A_k B_k (E(A_j|\mathcal{F}_{-\infty}^{j_k-1} \cup \mathcal{F}_{j_k+1}^{\infty}) - E(A_j|\mathcal{F}_j : |j - j_k| \leq 3m)) \\
& \times \prod_{l=j+1}^{s} E(A_l|\mathcal{F}_j : |j - j_l| \leq 3m)B_l| \\
=\ & O(T^{c_1}\exp(-dm)) \;=\; o(T^{-c_2}) \qquad \text{(A.71)}
\end{aligned}$$

for any arbitrary $c_2 > 0$ by choosing sufficiently large M. By the mixing inequality of Hall and Heyde (1980), we obtain

$$\begin{aligned}
&|E[R\prod_{k=1}^{s} E(A_k|\mathcal{F}_j : |j - j_k| \leq 3m)B_k]| \\
\leq\ & T^{c_3} E \prod_{j=1}^{s} |E(A_k|\mathcal{F}_k : 0 < |j - j_k| \leq 3m)| \\
& + T^{c_3} \prod_{j=1}^{s} E|E(A_k|\mathcal{F}_j : 0 < |j - j_k| \leq 3m)| + 4T^{c_3}(q/d)\exp(-dm) \qquad \text{(A.72)}
\end{aligned}$$

for some $c_3 > 0$. For $|\theta| \geq d$, we have $E|E(A_k|\mathcal{F}_j, j \neq j_k)| \leq \exp(-d)$. Thus by Lemma 3.2 of Götze and Hipp (1983) and Assumption 1(d), it follows that

$$\begin{aligned}
E|E(A_k|\mathcal{F}_j, |j - j_k| \leq 3m)| &\leq E|E(A_k|\mathcal{F}_j : |j - j_k| \neq 0)| + O(T^{c}\exp(-dm)) \\
&\leq \max(\exp(-d\theta^2/T), \exp(-d)) + O(T^{c_3}\exp(-dm)) \qquad \text{(A.73)}
\end{aligned}$$

$$E[\prod_{k=1}^{s} A_k B_k R] = O(T^{-c}) \qquad \text{(A.74)}$$

for arbitrary $c > 0$ by choosing sufficiently large M.

Next, consider the remainder term in (A.69). It follows from Lemma A.2 that

$$\begin{aligned} E|m_N|^r &= O((N/T)^r), && \text{(A.75)} \\ E|T^{1/2}\text{vec}(G_N - G_0)|^r &= O((N/T)^r), && \text{(A.76)} \\ E|(T/\ell)^{1/2}\text{vech}(\tilde{S}_N - \bar{S}_N)|^r &= O((N/T)^{r/2}), && \text{(A.77)} \\ E|(T/\ell)^{1/2}\text{vech}(\nabla\tilde{S}_N - \nabla\bar{S}_N)|^r &= O((N/T)^{r/2}), && \text{(A.78)} \\ E|T^{1/2}\text{vech}(\hat{S}_N - \tilde{S}_N)|^r &= O((N/T)^{r/2}). && \text{(A.79)} \end{aligned}$$

Using the definition of N and $\varepsilon r > 2$, we obtain that

$$\begin{aligned} & |\theta|^r E|Q(m_T, G_T, \hat{S}_T, \tilde{S}_T, \bar{S}_T) - Q(m_{T-N}, G_{T-N}, \hat{S}_{T-N}, \tilde{S}_{T-N}, \bar{S}_{T-N})|^r \\ = \; & O(\ell^{r/2}|\theta|^r N^{r/2} T^{-r}) \\ = \; & \begin{cases} O(\ell^{r/2} m^r T^{-r/2}) & \text{for } |\theta| \leq T^{1/2} \\ O(|\theta|^r \ell^{r/2} m^r T^{-r}) & \text{for } T^{1/2} < |\theta| \leq \ell^{-1/2} T^{1-\varepsilon} \end{cases} \\ = \; & o(\ell T^{-1}). \qquad \text{(A.80)} \end{aligned}$$

Lastly, it follows from (A.69), (A.71)-(A.73) and (A.80) that

$$\begin{aligned} E\exp(i\theta g_T) &= T^c \max(\exp(-d\theta^2/T), \exp(-d))^{N/M} + O(T^c \exp(-dm)) + o(\ell T^{-1}) \\ &= o(\ell T^{-1}) \qquad \text{(A.81)} \end{aligned}$$

for $s \geq N/M$ and sufficiently large M, which completes the proof. *Q.E.D.*

Lemma A.8: For $1 \leq s \leq r/2$,

$$\begin{aligned} E^*[\|\text{vec}(F_{N_j})\|^s] - E\left\{E^*[\|\text{vec}(F_{N_j})\|^s]\right\} &= O_p(b^{-1/2}), && \text{(A.82)} \\ E^*[\|B_{N_j}\|^s] - E\left\{E^*[\|B_{N_j}\|^s]]\right\} &= O_p(b^{-1/2}). && \text{(A.83)} \end{aligned}$$

Proof of Lemma A.8: First we will prove (A.82). We can write the LHS of (A.82) as

$$(1/(T-\ell+1)) \sum_{t=0}^{T-\ell} \|\text{vec}(F_j)\|^s - E[\|\text{vec}(F_j)\|^s] = (1/(T-\ell+1))(1/\ell) \sum_{\nu=1}^{\ell} f_{s,\nu}, \qquad \text{(A.84)}$$

where

$$f_{s,\nu} = (1/b) \sum_{\mu=0}^{b-1} \left(\|\text{vec}(F_{\mu\ell+\nu})\|^s - E(\|\text{vec}(F_{\mu\ell+\nu})\|^s)\right).$$

Note that $\{\text{vec}(F_{\mu\ell+\nu})\}_{\mu=0}^{b-1}$ is a triangular array of strong mixing sequence with mixing coefficients given by $\{\alpha_{\mu\ell}\}$ where α_m is the mixing coefficient of the original variables. So is $\|\text{vec}(F_{\mu\ell+\nu})\|^s$. Thus it follows that

$$f_{s,\nu} = O_p(b^{-1/2}). \qquad \text{(A.85)}$$

Since the decay rate of the mixing coefficients is uniform in ν, (A.85) also holds uniformly in ν. Hence (A.82) follows from (A.84) and (A.85).

Next we will prove (A.83). Note that the LHS of (A.83) is bounded by

$$\begin{aligned} & O\left((1/(T-\ell+1)) \sum_{t=0}^{T-\ell} \|\tilde{B}_t\|^s - E\|\tilde{B}_t\|^s\right) && \text{(A.86)} \\ & +O\left((1/(T-\ell+1)) \sum_{t=0}^{T-\ell} \|\hat{B}_t\|^s - \|\tilde{B}_t\|^s - E(\|\hat{B}_t\|^s - \|\tilde{B}_t\|^s)\right) && \text{(A.87)} \\ & +O\left(\|\mu_T^*\|^s - E(\|\mu_T^*\|^s)\right), && \text{(A.88)} \end{aligned}$$

where $\hat{B}_t = \ell^{-1/2}\sum_{j=1}^{\ell}\tilde{v}_{t+j}$ and $\tilde{B}_t = \ell^{-1/2}\sum_{j=1}^{\ell} v_{t+j}$. First, the proof that (A.86) is $O_p(b^{-1/2})$ is analogous to the proof of (A.82) and thus is omitted. Second, we will prove that (A.87) is $O_p(b^{-1/2})$. A Taylor series expansion yields

$$\|\hat{B}_t\| - \|\tilde{B}_t\| \;=\; s\|\tilde{B}_t\|^{s-2}F_t\ell^{\frac{1}{2}}(\tilde{\beta}_T - \beta_0). \tag{A.89}$$

Thus we have

$$(1/(T-\ell+1))\sum_{t=0}^{T-\ell}\|\hat{B}_t\|^s - \|\tilde{B}_t\|^s \;=\; (1/(T-\ell+1))\sum_{t=0}^{T-\ell} s\|\tilde{B}_t\|^{s-2}F_t\ell^{\frac{1}{2}}(\tilde{\beta}_T - \beta_0). \tag{A.90}$$

By using arguments analogous to the one used in the proof of (A.82), it follows from the ergodic theorem that

$$(1/(T-\ell+1))\sum_{t=0}^{T-\ell} s\|\tilde{B}_t\|^{s-2}F_t \;=\; O_{as}(1). \tag{A.91}$$

Thus it follows from Assumption 1(i) that

$$(1/(T-\ell+1))\sum_{t=0}^{T-\ell}(\|\hat{B}_t\|^s - \|\tilde{B}_t\|^s) = O_p(b^{-1/2}). \tag{A.92}$$

Similarly we obtain

$$(1/(T-\ell+1))\sum_{t=0}^{T-\ell}E(\|\hat{B}_t\|^s - \|\tilde{B}_t\|^s) = O(b^{-1/2}). \tag{A.93}$$

Hence it follows from (A.92) and (A.93) that (A.87) is $O_p(b^{-1/2})$. Third, we will prove that (A.88) is $O_p(b^{-1/2})$. We can write μ_T^* as

$$\begin{aligned}
\mu_T^* &= (1/(T-\ell+1))\sum_{t=0}^{T-\ell}\hat{B}_t \\
&= (1/(T-\ell+1))\sum_{t=0}^{T-\ell}\tilde{B}_t + (1/(T-\ell+1))\sum_{t=0}^{T-\ell}F_t\ell^{1/2}(\tilde{\beta}_T - \beta_0) \\
&\quad + (1/(T-\ell+1))\sum_{t=0}^{T-\ell}H_t\ell^{1/2}(\tilde{\beta}_T - \beta_0)^2.
\end{aligned} \tag{A.94}$$

Thus we obtain

$$\begin{aligned}
&\|\mu_T^*\|^s - E\|\mu_T^*\|^s \\
=\; & O((1/(T-\ell+1))\sum_{t=0}^{T-\ell}\|\tilde{B}_t\|^s - E\|\tilde{B}_t\|^s) \\
& + O((1/(T-\ell+1))\sum_{t=0}^{T-\ell}\|F_t\ell^{1/2}(\tilde{\beta}_T - \beta_0)\|^s - E\|F_t\ell^{1/2}(\tilde{\beta}_T - \beta_0)\|^s).
\end{aligned} \tag{A.95}$$

The rest of the proof is analogous to the proofs of (A.86) and (A.87). Therefore (A.83) follows from (A.86), (A.87) and (A.88). *Q.E.D.*

Lemma A.9: Let $G_0^* = E^*(G_T^*)$ and B_T^* and C_T^* denote the bootstrap version of B and C in Lemma A.1 with S_0 replaced by S_T^*, respectively. Then

$$G_0^* \;=\; G_0 + O_p(T^{-1/2}), \tag{A.96}$$

$$S_T^* \;=\; S + O(\ell^{-1}) + O_p(b^{-1/2}). \tag{A.97}$$

Proof of Lemma A.9: First, we will prove (A.96).

$$\begin{aligned}
G_0^* &= E^*[G_T^*] \;=\; E^*[\frac{1}{b}\sum_{k=1}^{b} F_{N_k}] \;=\; E^*[F_{N_1}] \\
&= \frac{1}{T-\ell+1}\sum_{t=0}^{T-\ell} F_t \;=\; \frac{1}{T-\ell+1}\sum_{t=0}^{T-\ell}\frac{1}{\ell}\sum_{i=1}^{\ell} w_{t+i} \\
&= \frac{1}{T}\sum_{t=1}^{T} w_t + O_p(\ell T^{-1}) \;=\; G_T + O_p(\ell T^{-1}).
\end{aligned}$$

Therefore, (A.96) follows from $G_T - G_0 = O_p(T^{-1/2})$.

Next, we will prove (A.97). By definition, it follows that

$$\begin{aligned}
S_T^* &\equiv \text{Var}^*\,(m_T^*) \;=\; \text{Var}^*(\frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k}) && \text{(A.98)} \\
&= E^*\left(\frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k} - \sqrt{b}E^*(B_{N_1})\right)\left(\frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k} - \sqrt{b}E^*(B_{N_1})\right)' && \text{(A.99)} \\
&= E^*\left(\frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k}\right)\left(\frac{1}{\sqrt{b}}\sum_{k=1}^{b} B_{N_k}\right)' && \text{(A.100)} \\
&= \frac{1}{b}\sum_{k=1}^{b} E^*(B_{N_k}B_{N_k}^{'}) = E^*(B_{N_1}B_{N_1}^{'}) = \frac{1}{T-\ell+1}\sum_{t=0}^{T-\ell} B_t B_t^{'}. && \text{(A.101)}
\end{aligned}$$

It follows from Lemma A.8 that

$$S_T^* - E[S_T^*] = O_p(b^{-1/2}). \tag{A.102}$$

Since $\mu_T^* = O_p(T^{-1/2})$, we have

$$E[S_T^*] \;=\; E[B_t B_t^{'}] \;=\; \sum_{j=-\ell}^{\ell}(1-|j|/\ell)E[v_0 v_{-j}'] \;=\; S + O(\ell^{-1}). \tag{A.103}$$

Thus (A.97) follows from (A.101),(A.102) and (A.103). *Q.E.D.*

Lemma A.10: Let

$$\begin{aligned}
\alpha_T^* &= T^{1/2}\kappa_1^*(g_T^*), \\
\gamma_T^* &= (T/\ell)(\kappa_2^*(g_T^*)-1) = (T/\ell)(E^*(g_T^{*2}) - [E^*(g_T^*)]^2 - 1), \\
\kappa_T^* &= T^{1/2}E^*(g_T^{*3}) = T^{1/2}\{\kappa_3^*(g_T^*) + 3E^*(g_T^{*2})E^*(g_T^*) - 2[E^*(g_T^*)]^3\}, \\
\zeta_T^* &= (T/\ell)(\kappa_4^*(g_T^*) - 3).
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_T^* &= \alpha_\infty + T^{1/2}\mathbf{b}^{*\prime}E^*[\text{vec}(G_T^* - G_0^*)\otimes m_T^*] + T^{1/2}\mathbf{c}^{*\prime}E^*[\text{vech}(\tilde{S}_T^* - S_T^*)\otimes m_T^*] \\
&\quad + T^{1/2}\mathbf{c}^{*'}E^*[\text{vech}(\hat{S}_T^* - \tilde{S}_T^*)\otimes m_T^*] + o_p^*(\ell T^{-1/2}) \\
&= \alpha_\infty + O_p(\ell^{-1}) + O_p(b^{-1/2}) + o_p^*(\ell T^{-1/2}), && \text{(A.104)} \\
\gamma_T^* &= \gamma_\infty + 2(T/\ell)E^*\{\mathbf{a}^{*\prime}m_T^*\mathbf{b}^{*\prime}[\text{vec}(G_T^* - G_0^*)\otimes m_T^*]\} \\
&\quad + 2(T/\ell)E^*\{\mathbf{a}^{*\prime}m_T^*\mathbf{c}^{*\prime}[\text{vech}(\tilde{S}_T^* - S_T^*)\otimes m_T^*]\} \\
&\quad + 2(T/\ell)E^*\{\mathbf{a}^{*\prime}m_T^*\mathbf{e}^{*\prime}[\text{vech}(\tilde{S}_T^* - S_T^*)\otimes \text{vech}(\tilde{S}_T^* - S_T^*)\otimes m_T^*]\}
\end{aligned}$$

$$\begin{aligned}
& +(T/\ell)E^*\{\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\}^2 + o_p^*(1), \\
= \;& \gamma_\infty + o_p(1) + o_p^*(1), && \text{(A.105)} \\
\kappa_T^* = \;& \kappa_\infty + T^{1/2}E^*[(\mathbf{a}^{*\prime}m_T^*)^3] + 3T^{1/2}E^*\{(\mathbf{a}^{*\prime}m_T^*)^2\mathbf{b}^{*\prime}[\mathrm{vec}(G_T - G_0)' \otimes m_T]\} \\
& +3T^{1/2}E^*\{(\mathbf{a}^{*\prime}m_T^*)^2\mathbf{c}^{*\prime}[\mathrm{vech}(\tilde{S}_T^* - S_T^*) \otimes m_T^*]\} \\
& +3T^{1/2}E^*\{(\mathbf{a}^{*\prime}m_T^*)^2\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\} + o_p^*(\ell T^{-1/2}) \\
= \;& \kappa_\infty + O_p(\ell^{-1/2}) + O_p(b^{-1/2}) + o_p^*(\ell T^{-1/2}), && \text{(A.106)} \\
\zeta_T^* = \;& \zeta_\infty + 4(T/\ell)E^*\{(\mathbf{a}^{*\prime}m_T^*)^3\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\} \\
& +4(T/\ell)E^*\{(\mathbf{a}^{*\prime}m_T^*)^3e'[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes \mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\} \\
& +6(T/\ell)E^*\left((\mathbf{a}^{*\prime}m_T^*)^2\{\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\}^2\right) \\
& -12(T/\ell)E^*\{\mathbf{a}^{*\prime}m_T^*\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\} \\
& -12(T/\ell)E^*\{\mathbf{a}^{*\prime}m_T^*\mathbf{e}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes \mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\} \\
& -6(T/\ell)E^*\{\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\}^2 + o_p^*(1) \\
= \;& \zeta_\infty + o_p(1) + o_p^*(1), && \text{(A.107)}
\end{aligned}$$

where α_∞, γ_∞, κ_∞, and ζ_∞ are defined in Lemma A.4.

Proof of Lemma A.10: The first equalities in (A.104)–(A.107) follow from Lemmas B.1 and B.2. Thus we will show that the second equalities hold in the rest of the proof.

Part (a): Proof of (A.104). First, we introduce some notation for the proof. Let

$$\begin{aligned}
\alpha_{1T}^* &= T^{1/2}\mathbf{b}^{*\prime}E^*[\mathrm{vec}(G_T^* - G_0^*) \otimes m_T^*], \\
\alpha_{2T}^* &= T^{1/2}\mathbf{c}^{*\prime}E^*[\mathrm{vech}(\tilde{S}_T^* - S_T^*) \otimes m_T^*], \\
\alpha_{3T}^* &= T^{1/2}\mathbf{c}^{*\prime}E^*[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*], \\
\alpha_{1T} &= T^{1/2}\mathbf{b}'E[\mathrm{vec}(G_T - G_0) \otimes m_T], \\
\alpha_{2T} &= T^{1/2}\mathbf{c}'E[\mathrm{vech}(\tilde{S}_T - \bar{S}_T) \otimes m_T], \\
\alpha_{3T} &= T^{1/2}\mathbf{c}'E[\mathrm{vech}(\hat{S}_T - \tilde{S}_T) \otimes m_T], \\
\alpha_{1\infty} &= \mathbf{b}'\sum_{i=-\infty}^{\infty} E[w_0 \otimes v_i], \\
\alpha_{2\infty} &= \mathbf{c}'\sum_{i,j=-\infty}^{\infty} E[\mathrm{vech}(v_0v_i') \otimes v_j], \\
\alpha_{3\infty} &= \mathbf{c}'\sum_{i=-\infty}^{\infty} E\{\mathrm{vech}[\nabla\bar{S}(E(w_0)'VE(w_0))^{-1}E(w_0)'Vv_0] \otimes v_i\}.
\end{aligned}$$

Next, we will prove that

$$\begin{aligned}
\alpha_{1T}^* - \alpha_{1\infty} &= O_p(\ell^{-1}) + O_p(\ell^{1/2}b^{-1/2}), && \text{(A.108)} \\
\alpha_{2T}^* - \alpha_{2\infty} &= O_p(\ell^{-1}) + O_p(\ell^{1/2}b^{-1/2}), && \text{(A.109)} \\
\alpha_{3T}^* - \alpha_{3\infty} &= O_p(\ell^{-1}) + O_p(\ell^{1/2}b^{-1/2}). && \text{(A.110)}
\end{aligned}$$

Since $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty} + \alpha_{3\infty}$ and $\alpha_T^* = \alpha_{1T}^* + \alpha_{2T}^* + \alpha_{3T}^*$, (A.104) follows from (A.108), (A.109) and (A.110).

First, we will prove (A.108). From Lemma A.9, we have $\mathbf{b}^* = \mathbf{b} + O(\ell^{-1}) + O_p(b^{-1/2})$ and thus

$$\begin{aligned}
\alpha_{1T}^* &= \sqrt{\ell}\mathbf{b}^{*\prime}E^*\{[F_{N_1} - E^*(F_{N_1})] \otimes B_{N_1}\} \\
&= \mathbf{b}^{*\prime}E^*[\widetilde{F}_{N_1} \otimes B_{N_1}] \\
&= \mathbf{b}'E^*[\widetilde{F}_{N_1} \otimes B_{N_1}] + O_p(\ell^{-1}) + O_p(b^{-1/2}) \\
&= \alpha_{1\infty}^* + O_p(\ell^{-1}) + O_p(b^{-1/2}), \qquad \text{say.} && \text{(A.111)}
\end{aligned}$$

By combining (A.111) with

$$E\left[\alpha_{1\infty}^{*}\right]=\mathbf{b}'E\left[\widetilde{F}_t\otimes B_t\right]=\sum_{j=-\ell}^{\ell}(1-|j|/\ell)\mathbf{b}'E[w_0\otimes v_{-j}]=\alpha_{1\infty}+O(\ell^{-1}). \tag{A.112}$$

and $\alpha_{1\infty}^{*}-E\left[\alpha_{1\infty}^{*}\right]=O_p(b^{-1/2})$ from Lemma A.8, we obtain (A.108).

Second, we will prove (A.109). Similarly, we have $\mathbf{c}^{*}=\mathbf{c}+O(\ell^{-1})+O_p(b^{-1/2})$ from Lemma A.9 and thus

$$\begin{aligned}
\alpha_{2T}^{*} &= \sqrt{\ell}\mathbf{c}^{*\prime}E^{*}[\mathrm{vech}(B_{N_1}B_{N_1}'-E^{*}(B_{N_1}B_{N_1}'))\otimes B_{N_1}] \\
&= \sqrt{\ell}\mathbf{c}'E^{*}[\mathrm{vech}(B_{N_1}B_{N_1}'-E^{*}(B_{N_1}B_{N_1}'))\otimes B_{N_1}]+O_p(\ell^{-1})+O_p(b^{-1/2}) \\
&= \sqrt{\ell}\mathbf{c}'E^{*}[\mathrm{vech}(B_{N_1}B_{N_1}')\otimes B_{N_1}]+O_p(\ell^{-1})+O_p(b^{-1/2}) \\
&= \alpha_{2\infty}^{*}+O_p(\ell^{-1})+O_p(b^{-1/2}), \qquad \text{say.}
\end{aligned} \tag{A.113}$$

By combining (A.113) with

$$\begin{aligned}
E\left[\alpha_{2\infty}^{*}\right] &= \sqrt{\ell}\mathbf{c}'E[\mathrm{vech}\left(B_tB_t'\right)\otimes B_t] \\
&= \mathbf{c}'\sum_{i,j=-\ell}^{\ell}\left(1-\frac{\min\left((\max|i|,|j|)(i\cdot j>0)+(|i|+|j|)(i\cdot j\leq 0),\ell\right)}{\ell}\right) \\
&\quad \times E[\mathrm{vech}\left(v_0v_{-i}'\right)\otimes v_{-j}] \\
&= \alpha_{2\infty}+O(\ell^{-1}).
\end{aligned} \tag{A.114}$$

and $\alpha_{2\infty}^{*}-E\left[\alpha_{2\infty}^{*}\right]=O_p(b^{-1/2})$ from Lemma A.8, we obtain (A.109).

Lastly, we will prove (A.110). Note that

$$\begin{aligned}
\hat{S}_T^{*}-\widetilde{S}_T^{*} &= \frac{1}{b}\sum_{k=1}^{b}\left(\widehat{B}_{N_k}\widehat{B}_{N_k}'-B_{N_k}B_{N_k}'\right) \\
&= \triangledown\widetilde{S}_T^{*}(\widetilde{\beta}^{*}-\hat{\beta})+\triangledown^2\widetilde{S}_T^{*}(\widetilde{\beta}^{*}-\hat{\beta})^2
\end{aligned} \tag{A.115}$$

where

$$\begin{aligned}
\triangledown\widetilde{S}_T^{*} &= \frac{\sqrt{\ell}}{b}\sum_{k=1}^{b}\left(F_{N_k}B_{N_k}'+B_{N_k}F_{N_k}'\right), \\
\triangledown^2\widetilde{S}_T^{*} &= \frac{\ell}{b}\sum_{k=1}^{b}\left(F_{N_k}F_{N_k}'\right), \\
\widetilde{\beta}^{*}-\hat{\beta} &= \left[G_T^{*\prime}V_TG_T^{*}\right]^{-1}G_T^{*\prime}V_T\frac{1}{\sqrt{T}}m_T^{*}.
\end{aligned}$$

First, note that

$$\begin{aligned}
\triangledown\widetilde{S}_T^{*} &= E^{*}\left[\triangledown\widetilde{S}_T^{*}\right]+O_p^{*}(b^{-1/2}), \\
\ell^{-1}\triangledown^2\widetilde{S}_T^{*} &= \ell^{-1}E^{*}\left[\triangledown^2\widetilde{S}_T^{*}\right]+O_p^{*}(b^{-1/2}),
\end{aligned}$$

where

$$\begin{aligned}
E^{*}\left[\triangledown\widetilde{S}_T^{*}\right] &= \sqrt{\ell}E^{*}[F_{N_1}B_{N_1}'+B_{N_1}F_{N_1}'] \;=\; E^{*}[\widetilde{F}_{N_1}B_{N_1}'+B_{N_1}\widetilde{F}_{N_1}'], \\
E^{*}\left[\triangledown^2\widetilde{S}_T^{*}\right] &= \ell E^{*}[F_{N_1}F_{N_1}'].
\end{aligned}$$

Second, note that

$$T^{1/2}(\widetilde{\beta}^* - \hat{\beta}) = \left[E^*[G_T^{*\prime}]V_T E^*[G_T^*]\right]^{-1} E^*[G_T^{*\prime}]V_T m_T^* + O_p^*(b^{-1/2}) \tag{A.116}$$

since $G_T^* - E^*[G_T^*] = O_p^*(b^{-1/2})$ and

$$G_T^{*\prime} V_T G_T^* - E^*[G_T^{*\prime}]V_T E^*[G_T^*] = O_p^*(b^{-1/2}).$$

Thus it follows from (A.115)–(A.116) that

$$\begin{aligned}
\alpha_{3T}^* &= T^{1/2}\mathbf{c}^{*\prime} E^*[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*], \\
&= \mathbf{c}^{*\prime} E^*[\mathrm{vech}\left(\bigtriangledown \widetilde{S}_T^* T^{1/2}(\widetilde{\beta}^* - \hat{\beta}) + (\widetilde{\beta}^* - \hat{\beta})' \bigtriangledown^2 \widetilde{S}_T^* T^{1/2}(\widetilde{\beta}^* - \hat{\beta})\right) \otimes m_T^*] \\
&= \mathbf{c}^{*\prime} E^*[\mathrm{vech}\left(\bigtriangledown \widetilde{S}_T^* T^{1/2}(\widetilde{\beta}^* - \hat{\beta})\right) \otimes m_T^*] + O_p^*(\ell^{1/2}T^{-1/2}) \\
&= \mathbf{c}^{*\prime} E^*\{\mathrm{vech}\left(E^*[\bigtriangledown \widetilde{S}_T^*]\left[E^*[G_T^{*\prime}]V_T E^*[G_T^*]\right]^{-1} E^*[G_T^{*\prime}]V_T m_T^*\right) \otimes m_T^*\} \\
&\quad + O_p^*(\ell^{1/2}T^{-1/2}) \\
&= \mathbf{c}' E^*\{\mathrm{vech}\left(E^*[\widetilde{F}_{N_1} B_{N_1}' + B_{N_1}\widetilde{F}_{N_1}']\left[E^*[F_{N_1}']V E^*[F_{N_1}]\right]^{-1}\right. \\
&\quad \left.\times E^*[F_{N_1}']V B_{N_1}\right) \otimes B_{N_1}\} + O_p^*(\ell^{1/2}T^{-1/2}) + O_p(\ell^{-1}) + O_p(b^{-1/2}), \\
&= \alpha_{3\infty}^* + O_p^*(\ell^{1/2}T^{-1/2}) + O_p(\ell^{-1}) + O_p(b^{-1/2}), \qquad \text{say.}
\end{aligned} \tag{A.117}$$

Since

$$\begin{aligned}
E\left[E^*[\widetilde{F}_{N_1} B_{N_1}' + B_{N_1}\widetilde{F}_{N_1}']\right] &= \sqrt{\ell} E[F_t B_t' + B_t F_t'] = \sum_{j=-\ell}^{\ell} (1 - |j|/\ell) E[w_0 v_{-j}' + v_0 w_{-j}'] \\
&= \sum_{j=-\infty}^{\infty} E[w_0 v_{-j}' + v_0 w_{-j}'] + O(\ell^{-1}) = \bigtriangledown S + O(\ell^{-1}), \qquad \text{(A.118)} \\
E\left[E^*[F_{N_1}]\right] &= E[F_t] \;=\; E[w_0], \qquad \text{(A.119)}
\end{aligned}$$

it follows that

$$\begin{aligned}
E^*[\widetilde{F}_{N_1} B_{N_1}' + B_{N_1}\widetilde{F}_{N_1}'] - E\left[E^*[\widetilde{F}_{N_1} B_{N_1}' + B_{N_1}\widetilde{F}_{N_1}']\right] &= O_p(b^{-1/2}), \qquad \text{(A.120)} \\
E^*[F_{N_1}] - E\left[E^*[F_{N_1}]\right] &= O_p(b^{-1/2}). \qquad \text{(A.121)}
\end{aligned}$$

Hence, it follows from the moment inequality, Lemma A.8, (A.120) and (A.121) that

$$\begin{aligned}
\alpha_{3\infty}^* &= E\left[\alpha_{3\infty}^*\right] + O_p(b^{-1/2}) \\
&= \mathbf{c}' \sum_{i=-\infty}^{\infty} E\{\mathrm{vech}[\nabla \bar{S}(E(w_0)' V E(w_0))^{-1} E(w_0)' V v_0] \otimes v_i\} + O_p(\ell^{-1}) + O_p(b^{-1/2}) \\
&= \alpha_{3\infty} + O_p(\ell^{-1}) + O_p(b^{-1/2}).
\end{aligned} \tag{A.122}$$

Therefore, (A.110) follows from (A.117), and (A.122).

Part (b): Proof of (A.105). Let

$$\begin{aligned}
\gamma_{1T}^* &= (T/\ell)E^*\{\mathbf{a}^{*\prime} m_T^* \mathbf{b}^{*\prime}[\mathrm{vec}(G_T^* - G_0^*) \otimes m_T^*]\}, \\
\gamma_{2T}^* &= (T/\ell)E\{\mathbf{a}^{*\prime} m_T^* \mathbf{c}^{*\prime}[\mathrm{vech}(\tilde{S}_T^* - S_T^*) \otimes m_T^*]\}, \\
\gamma_{3T}^* &= (T/\ell)E^*\{\mathbf{a}^{*\prime} m_T^* \mathbf{e}^{*\prime}[\mathrm{vech}(\tilde{S}_T^* - S_T^*) \otimes \mathrm{vech}(\tilde{S}_T^* - S_T^*) \otimes m_T^*]\}, \\
\gamma_{4T}^* &= (T/\ell)E^*\{\mathbf{c}^{*\prime}[\mathrm{vech}(\hat{S}_T^* - \tilde{S}_T^*) \otimes m_T^*]\}^2.
\end{aligned}$$

From Lemma A.9, we have $\mathbf{a}^* = \mathbf{a} + O(\ell^{-1}) + O_p(b^{-1/2})$ and thus

$$\gamma_{1T}^* = \ell^{-1/2} E^*\{\mathbf{a}^{*\prime} B_{N_1} \mathbf{b}^{*\prime}[F_{N_1} - E^*(F_{N_1})] \otimes B_{N_1}\} = o_p^*(1).$$

Similarly,

$$\begin{aligned}
\gamma_{2T}^{*} &= (T/\ell)E^{*}\{\mathbf{a}^{*\prime}m_T^{*}\mathbf{c}^{*\prime}[\text{vech}(\tilde{S}_T^{*}-S_T^{*})\otimes m_T^{*}]\} \\
&= E^{*}\{\mathbf{a}^{*\prime}B_{N_1}\mathbf{c}^{*\prime}[\text{vech}(B_{N_1}B_{N_1}^{\prime}-E^{*}(B_{N_1}B_{N_1}^{\prime}))\otimes B_{N_1}]\} \\
&= E^{*}\{\mathbf{a}^{\prime}B_{N_1}\mathbf{c}^{\prime}[\text{vech}(B_{N_1}B_{N_1}^{\prime}-E^{*}(B_{N_1}B_{N_1}^{\prime}))\otimes B_{N_1}]\}+O_p(\ell^{-1})+O_p(b^{-1/2}) \\
&= \gamma_{2\infty}^{*}+O_p(\ell^{-1})+O_p(b^{-1/2}), \qquad \text{say.}
\end{aligned}$$

It follows from the moment inequality and Lemma A.8 that

$$\begin{aligned}
\gamma_{2\infty}^{*} &= E\left[\gamma_{2\infty}^{*}\right]+\ O_p(b^{-1/2}) \\
&= \lim_{T\to\infty}\frac{1}{\ell}\sum_{j=-\ell}^{\ell}\sum_{i,k=-T}^{T}E\{\mathbf{a}^{\prime}v_0\mathbf{c}^{\prime}[\text{vech}(v_iv_{i-j}^{\prime}-\Gamma_j)\otimes v_k]\}+O_p(\ell^{-1})+O_p(b^{-1/2}) \\
&= \gamma_{2\infty}+O_p(\ell^{-1})+O_p(b^{-1/2}). \qquad \text{(A.123)}
\end{aligned}$$

The result for γ_{3T}^{*} and γ_{4T}^{*} can be proved using similar arguments, and thus the proof is omitted.

Part (c): Proof of (A.106). Let $\kappa_{1T}^{*}=T^{1/2}E^{*}[(\mathbf{a}^{*\prime}m_T^{*})^3]$ denote the second term on the RHS of (A.106). Because the proof of Part (c) is analogous to the proofs of Parts (a) and (b), we will only show that

$$\kappa_{1T}^{*}=\sum_{i,j=-\infty}^{\infty}E(\mathbf{a}^{\prime}v_0\mathbf{a}^{\prime}v_i\mathbf{a}_j^{\prime}v_j)+O_p(\ell^{-1})+O_p(b^{-1/2}). \qquad \text{(A.124)}$$

By definition, we have

$$\kappa_{1T}^{*}\ =\ \ell^{1/2}E^{*}[(\mathbf{a}^{*\prime}B_{N_1})^3]\ =\ (\ell^{1/2}/(T-\ell+1))\sum_{t=0}^{T-\ell}[\mathbf{a}^{*\prime}(\tilde{B}_t+\hat{B}_t-\tilde{B}_t-\mu_T^{*})]^3, \qquad \text{(A.125)}$$

where $\hat{B}_t$ and $\tilde{B}_t$ are defined in the proof of Lemma A.8. Thus it suffices to show that

$$\begin{aligned}
(\ell^{1/2}/(T-\ell+1))\sum_{t=0}^{T-\ell}(\mathbf{a}^{*\prime}\tilde{B}_t)^3 &= \sum_{i,j=-\infty}^{\infty}E(\mathbf{a}^{\prime}v_0\mathbf{a}^{\prime}v_i\mathbf{a}^{\prime}v_j)+o_p(\ell T^{-1/2}), & \text{(A.126)} \\
(\ell^{1/2}/(T-\ell+1))\sum_{t=0}^{T-\ell}[\mathbf{a}^{*\prime}(\hat{B}_t-\tilde{B}_t)]^3 &= O_p(\ell^{-1})+O_p(b^{-1/2}), & \text{(A.127)} \\
(\ell^{1/2}/(T-\ell+1))\sum_{t=0}^{T-\ell}(\mathbf{a}^{*\prime}\mu_T^{*})^3 &= O_p(\ell^{-1})+O_p(b^{-1/2}), & \text{(A.128)}
\end{aligned}$$

First, we will show (A.126). Since a HAC covariance matrix estimator converges at rate $O_p(\ell^{1/2}T^{-1/2})$, it follows that

$$\begin{aligned}
&(1/(T-\ell+1))\sum_{i=0}^{T-\ell}\ell^{1/2}(\mathbf{a}^{*\prime}\tilde{B}_i)^3-\ell^{1/2}E(\mathbf{a}^{*\prime}\tilde{B}_i)^3 \\
=\ &O_p(\sum_{i,j=0}^{\ell}(1-\min(\max(i,j,|i-j|),\ell)/\ell)(1/(T-\ell+1)) \\
&\times\sum_{t=0}^{T-\ell}[a^{*\prime}v_ta^{*\prime}v_{t+i}a^{*\prime}v_{t+j}-E(a^{*\prime}v_ta^{*\prime}v_{t+i}a^{*\prime}v_{t+j})]) \\
=\ &O_p(\ell^{1/2}T^{-1/2}). \qquad \text{(A.129)}
\end{aligned}$$

By the moment inequality, it follows that

$$(1/b)\sum_{i=0}^{b-1}\ell^{1/2}E(\mathbf{a}^{*\prime}\tilde{B}_i)^3=\sum_{i,j=-\infty}^{\infty}E(\mathbf{a}^{\prime}v_0\mathbf{a}^{\prime}v_i\mathbf{a}^{\prime}v_j)+o(\ell T^{-1/2}). \qquad \text{(A.130)}$$

Thus (A.126) follows from (A.129) and (A.130). Next we will show (A.127) and (A.128). Using arguments similar to the one used in the proof of Lemma A.8, we obtain

$$(\ell^{\frac{1}{2}}/(T-\ell+1))\sum_{t=0}^{T-\ell}[\mathbf{a}^{*\prime}(\hat{B}_t-\tilde{B}_t)]^3 = (\ell^2(T-\ell+1))\sum_{t=0}^{T-\ell}\{\mathbf{a}^{*\prime}[F_t(\tilde{\beta}_T-\beta_0)]\}^3 = O_p(\ell^2 T^{-\frac{3}{2}}) \qquad \text{(A.131)}$$

and

$$(\ell^{1/2}/(T-\ell+1))\sum_{t=0}^{T-\ell}(\mathbf{a}^{*\prime}\mu_T^*)^3 = \ell^{1/2}(\mathbf{a}^{*\prime}\mu_T^*)^3 = O_p(\ell^{1/2}b^{-3/2}). \qquad \text{(A.132)}$$

Thus (A.127) and (A.128) are satisfied. Therefore, (A.106) follows.

Part (d): Proof of (A.107). Part (d) can be proved using similar arguments and thus the proof is omitted. *Q.E.D.*

Proofs of Main Theorems

Lastly, we will prove the main theorems.

Proof of Theorem 1: The result for the studentized statistics, (3.2), follows from Lemmas A.5-A.7. Let $J_T^{1/2} = \hat{S}_T^{-1/2}\sum_{t=1}^{T} z_t(y_t-\hat{\beta}_T'x_t)$. Then it follows from the first-order condition that

$$J_T^{1/2} \;=\; (I_k - G_T(G_T'G_T)^{-1}G_T')S_T^{-1/2}\sum_{t=1}^{T} z_t(y_t-\hat{\beta}_T'x_t). \qquad \text{(A.133)}$$

By the singular value decomposition, there are $k\times k-q$ matrices A, B and $(k-p)\times(k-p)$ diagonal matrix Λ with positive diagonal elements such that $A'A = I_{k-p}$, $B'B = I_{k-p}$ and

$$I_k - G_T(G_T'G_T)^{-1}G_T \;=\; A\Lambda^{1/2}B'.$$

Thus we can write

$$J_T^{1/2} \;=\; A\Lambda^{1/2}B'S_T^{-1/2}\sum_{t=1}^{T} z_t(y_t-\hat{\beta}_T'x_t) \;=\; A\tilde{J}_T^{1/2} \qquad \text{(A.134)}$$

where $\tilde{J}_T^{1/2} = \Lambda^{1/2}B'S_T^{-1/2}\sum_{t=1}^{T} z_t(y_t-\hat{\beta}_T'x_t)$ is a $(k-p)$-dimensional vector. Note that

$$J_T \;=\; \tilde{J}_T'A'A\tilde{J}_T \;=\; \tilde{J}_T'\tilde{J}_T. \qquad \text{(A.135)}$$

The rest of the proof takes the following steps. First, one can show that Lemmas A.1–A.7 hold for $c'\tilde{J}_T^{1/2}$ except that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ now take different values. Second, because the characteristic function of $\tilde{J}_T^{1/2}$ can be derived from an linear combination of $\tilde{J}_T^{1/2}$, the distribution of $\tilde{J}_T^{1/2}$ can be approximated by its Edgeworth expansion in a suitable sense. Lastly, a modification of Theorem 1 of Chandra and Ghosh (1979) with $s=5$ completes the proof of (3.3). We make the following modifications: (i) the order of the third term of the Edgeworth expansion $\xi_{s-1,n}(z)$, i.e., $1/n$, is replaced by ℓ/n; (ii) the order of such approximation errors, $o(n^{-(s-3)/2})$, by $o(\ell/n)+O(1/\ell^q)$ where $n=T$; (iii) the order of the second term of the Edgeworth expansion $\psi_{m,n}$, $1/n$, is replaced by ℓ/n. These modifications do not change the parity of $R(\alpha)$ which is the crucial element of their proof (see Remark 2.5 of Chandra and Ghosh, 1979, pp.27–28). Thus their proof will carry through for our version of their theorem. *Q.E.D.*

Proof of Theorem 2: For iid observations, a modification of Theorem 1 with $\ell=1$ yields

$$\sup_{x\in\Re^p}|P(T^{1/2}(c'\hat{\Sigma}_T c)^{-1/2}c'(\hat{\beta}_T-\beta_0)\le x)-\Psi_T(x)| \;=\; o(T^{-1}), \qquad \text{(A.136)}$$

$$\sup_{x\ge 0}|P(J_T\le x)-\Psi_{J,T}(x)| \;=\; o(T^{-1}). \qquad \text{(A.137)}$$

under Assumptions 1(b)(c)(d)(i), $\ell = 1$ and Assumption 1(e) replaced by the standard Cramer condition. It suffices to show that the conditions on $R_t = (v_t', \text{vec}(w_t)')'$ required for the Edgeworth expansion of Theorem 1 are also satisfied for $Q_{N_j} = (B_{N_j}', \text{vec}(F_{N_j})')'$ for $j = 1, \ldots, b$ conditionally on the sample $\chi_T = \{(x_t', y_t, z_t')\}_{t=1}^{T}$, uniformly for all χ_T in a set whose probability tends to 1 as $T \to \infty$. Without loss of generality, we check the conditions using B_{N_j}. For Assumption A1(b), we have

$$E^*[B_{N_j}] = E^*[B_{N_1}] = \frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} E^*\left(z_i u_i - \mu_T^*\right) = 0. \tag{A.138}$$

For Assumption A1(c), it follows from Lemma A.2 that

$$E\left[E^*\left|B_{N_j}\right|^{r+\eta}\right] = \frac{1}{T-\ell+1} \sum_{t=0}^{T-\ell} E\left|\frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} v_{t+i}\right|^{r+\eta} = E\left|\frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} v_{t+i}\right|^{r+\eta} < \infty. \tag{A.139}$$

From the proof of Theorem 4.2 of Götze and Künsch (1996),

$$E^*\left|B_{N_j}\right|^{r+\eta} - E\left[E^*\left|B_{N_j}\right|^{r+\eta}\right] = O_p(b^{-1/2}). \tag{A.140}$$

Combining the two results implies that the probability of $E^*\left|B_{N_j}\right|^{r+\eta} < \infty$ tends to unity. By construction, the moving block bootstrap sample are based on the independent sampling of B_{N_j}. Therefore, Assumption A1(d) is trivially satisfied (with a probability one) using a sigma-field defined by $\sigma(N_j)$ for $j = 1, \ldots, b$, conditionally on the sample χ_T. By the same reason, we can replace Assumption A1(e) by the standard Cramér condition and we only need to show that the condition holds with probability tends to one. Using an argument that appeared in the proof of Theorem 4.2 of Götze and Künsch (1996), we have that

$$P\left\{\sup_{d<|t|<b^{1/2}} \left|E^* \exp[itB_{N_1}]\right| \leq 1 - \zeta\right\} = 1 - o(T^{-1}) \tag{A.141}$$

for some $0 < \zeta < 1/2$. *Q.E.D.*

Proof of Corollary 1: It follows from Lemmas A.3–A.5, Lemmas A.9-A.10 and Theorems 1 and 2. *Q.E.D.*

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