SUBSTITUTION AND RISK AVERSION: IS RISK AVERSION IMPORTANT FOR UNDERSTANDING ASSET PRICES?[1]

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The standard power utility function is widely used to explain asset prices. It assumes that the coefficient of relative risk aversion is the inverse of the elasticity of substitution. Here I use the Kihlstrom and Mirman (1974) expected utility approach for separating between risk aversion and the elasticity of substitution to show that monotonic transformations of the standard power utility function do not change the predictions about asset prices by much. This says that for the purpose of understanding asset prices we should adopt the elasticity of substitution interpretation of the power coefficient.

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1. INTRODUCTION

Since the discovery of the risk premium puzzle by Mehra and Prescott (1985) there has been a debate about the magnitude of risk aversion. In his presidential address Lucas (2003) followed Mehra and Prescott in using the standard power utility function: $\sum_t \beta^t U(C_t)$ where U(C) = $C^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and U(C) = ln(C) for $\gamma = 1$. He argues for a coefficient of $\gamma = 1$ (the logarithmic function) by using the formula (Equation [6] in Lucas [2003] derived for the power utility function):

(1) $$r = r^s + \gamma g,$$

where r is the interest rate, g is the growth rate of consumption, $r^s = 1/\beta - 1$ is the subjective interest rate and γ is the parameter of the power utility function. Lucas argues that "...this formula makes it clear why fairly low γ values must be used. Per capita consumption growth in the United States is about 0.02 and the after-tax return on capital is around 0.05, so the fact that the subjective interest rate must be positive requires that γ be at most 2.5. Moreover, a value as high as 2.5 would imply much larger interest rate differential than those we see between fast-growing economies like Taiwan and mature economies like the United States. This is the kind of evidence that leads to the use of γ values at or near 1 in applications."[3]

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[3] Pages 6 and 7 in Lucas (2003) with some modifications due to difference in notation.

The inter-temporal log utility function implies a relative risk aversion of 1 and an elasticity of substitution of 1. In general, the power utility function imposes a relationship between the inter-temporal elasticity of substitution and relative risk aversion: The coefficient of relative risk aversion is γ and the inter-temporal elasticity of substitution is $1/\gamma$. This problem of distinguishing between the ordinal and the cardinal properties of the utility function has been well recognized and led to the generalization of the expected utility function by Selden (1978), Kreps and Porteus (1978) and Epstein and Zin (1989, 1991). Here I follow the expected utility approach suggested by Kihlstrom and Mirman (1974).

According to the Kihlstrom-Mirman approach we can change risk aversion without changing the ordinal properties of the utility function by applying a monotonic transformation to the utility function. I start with a monotonic transformation of the inter-temporal log (IL) utility function: The inter-temporal Cobb-Douglas (ICD). The ICD function allows for changes in risk aversion while holding constant the elasticity of substitution at the level of unity. It is shown that changes in risk aversion do not affect the expected rate of return on the market portfolio and have only a small effect on the risk free return.

I then consider monotonic transformations of the standard power utility function (with $\gamma \neq 1$): The constant elasticity (ICE) function. It is shown that also in this case asset prices do not change much under monotonic transformations but changes in the elasticity of substitution have a large effect on asset prices.

Monotonic transformations of the standard power utility function are time-non-separable. Other time non-separable utility functions were used. A well-known example is the habit persistent function used by Constantinides (1990) and others to account for the equity premium puzzle. Here I use monotonic transformations of the standard power utility function that allow for a clean separation between the elasticity of substitution and risk aversion.

To appreciate the difference between the ordinal and the cardinal properties of the utility function, I now turn to distinguish between aversion to fluctuations and aversion to risk.

2. FLUCTUATIONS AVERSION AND RISK AVERSION

Would you prefer a smooth consumption path to a path that fluctuates around the same mean? In terms of Figure 1 the smooth consumption path *a* promises 3 units of consumption in every period. The fluctuating consumption path *d* starts from 3.5 units and then fluctuates between 3.5 and 2.5. If you prefer the path *a* then a time separable utility function predicts that you will also prefer a smooth consumption path of 2 (*e* in Figure 1) to a bet between a smooth consumption path of 3 and a smooth consumption path of 1 (*a* and *b* in Figure 1). In the time separable utility function aversion to fluctuations implies aversion to risk.

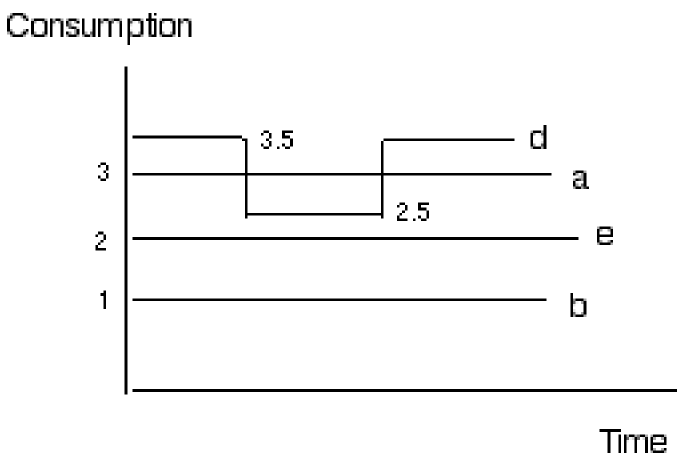

Figure 1

But aversion to fluctuations need not imply aversion to risk. It is possible that a consumer does not like fluctuations because they require changes in durables. To implement the path *d* one needs to change his house every period or to suffer from a mismatch between his house size and other components of consumption. In addition there are some irreversible choices (like the number of children) that have to be made early on (in most cases). For example, when facing a smooth consumption path one may choose to have 1 child if his permanent consumption is 1, 2 children if his permanent consumption is 2 and 3 children if his permanent consumption is 3. When facing the fluctuating consumption path *d* he may choose to have 3 children but may not enjoy them as much because they will complain whenever his consumption level drops to 2.5 and he has to cut on say the number of movies that they go to.

On the other hand if after a lottery between *a* and *b* he gets to know his permanent consumption early on he will make the optimal choice of the number of children: He will choose one child if his permanent consumption turns out to be 1 and 3 children if it turns out to be 3.

Since a bet with early resolution of uncertainty allow the consumer to make the optimal choice the consumer may accept such a bet even when he has aversion to fluctuations.

This leads to the result that risk premium does not require risk aversion. To illustrate, I assume a representative consumer who is risk neutral but does not like fluctuations. He will prefer the risk free asset to the market portfolio if both promise the same expected rate of return, because investing in the risk free asset allows for perfect smoothing. Therefore a risk premium is required to make the fluctuation averter hold the risky asset.[4]

It is also true that risk premium does not require fluctuations aversion. A consumer with a utility function $U = \left(\sum_{t=0}^{T} C_t\right)^{1/2}$ will be indifferent between the paths *d* and *a* in Figure 1 and will therefore exhibit no fluctuation aversion. But this consumer will prefer the non-random path *e* to a bet between *a* and *b*. He will thus exhibit risk aversion. This consumer will therefore require a premium for holding the risky asset.

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[4] A related argument is in Postlewaite, Samuelson and Silverman (2004). They show that consumption commitments can cause risk neutral agents to care about risk, creating incentives to both insure risks and bunch uninsured risks together. The argument is also similar to Eden (1977, 1979) where I argued that insurance type phenomena does not require risk aversion.

We therefore expect the risk premium to depend on both the ordinal and the cardinal properties of the utility function. This is because holding a risky asset leads to a random, non-smooth consumption path. Roughly speaking, the cost of non-smoothness depends on the ordinal properties of the utility function while the cost of randomness depends on the cardinal properties.

The numerical example in this paper suggests that the main difference between the expected utility approach used here and the generalized expected utility approach used by Selden, Epstein and Zin (SEZ) is about the behavior of the risk premium. Here it depends on both the elasticity of substitution and risk aversion. In the SEZ approach it depends almost entirely on risk aversion.

To better understand why the expected utility approach implies a risk premium that depends on both the ordinal and the cardinal properties of the utility function I decompose the demand for perfect insurance into a demand for eliminating risk that is resolved early and a demand for information that allows for smoothing. This is done in Appendix F.

I now turn to a related distinction.

3. BETS IN TERMS OF MONEY AND BETS IN TERMS OF CONSUMPTION

Bets in terms of money (wealth) are resolved immediately before any irreversible consumption choice is made. Introspections about money bets require an assumption about borrowing and lending opportunities.

Bets in terms of dated consumption require a different thought experiments. We start from a non-random consumption path and then consider a bet that makes date t consumption a random variable holding consumption at all dates other than t constant. The attitude towards this type of bets does not require any assumption about borrowing and lending. But introspection seems more difficult.

The distinction between the two types of bets can be illustrated with the help of Figure 2 that assumes a two-period horizon (t = 0,1) and a zero interest rate. The maximum utility that the consumer can get when having the wealth 9, 10 or 11 is a, e and b respectively. From observing the indifference map we know that: $a < e < b$. But we do not know by how much. The consumer will prefer a wealth of 10 with certainty to a random wealth {9 or 11 with equal probabilities} if $e > (1/2)a + (1/2)b$. This will occur for example, if $a = 2$, $e = 9$ and $b = 10$. Otherwise, he will prefer the bet (if for example, $a = 8.5$, $e = 9$ and $b = 10$).

This is different from a bet in terms of second period consumption that holds first period consumption constant. For example, a bet in terms of future consumption that holds current consumption at the level $C_0 = 5$ and is of the same relative size as the money bet just described has the outcomes: $C_1 = \{4.5 \text{ or } 5.5\}$. In terms of Figure 2, the money bet has the possible outcomes point A or point B. The consumption bet has the possible outcomes: point G or point F.

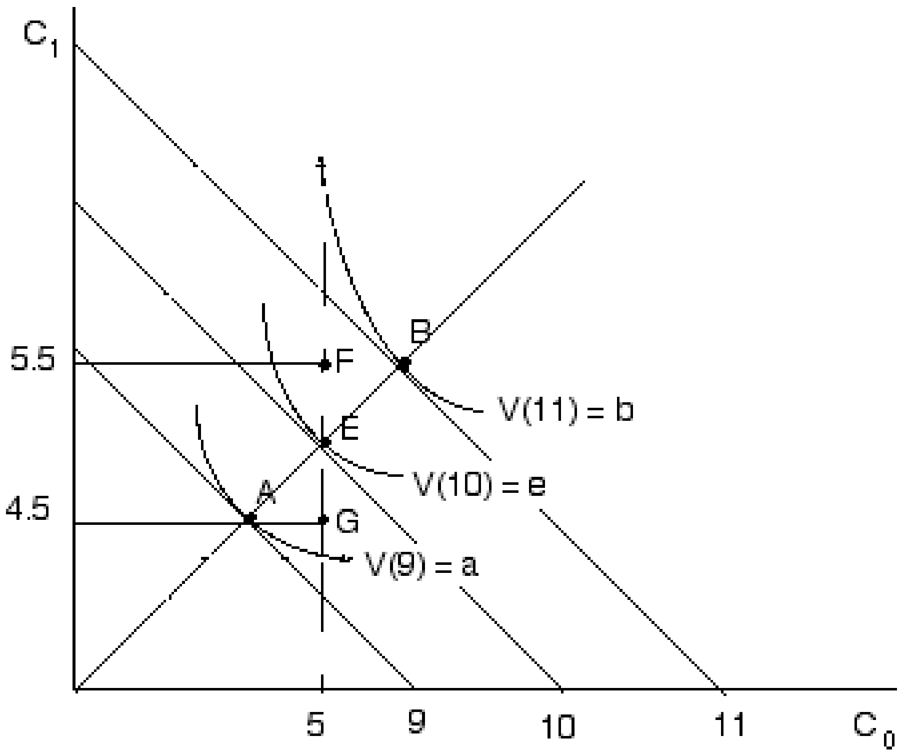

Figure 2

I now compare the relative risk aversion measures to the two kinds of bets under the assumption that the market interest rate is equal to the subjective interest rate and therefore under certainty, the consumer wants to smooth consumption. I start by showing that in the time separable case the coefficient of relative risk aversion is the same for the two kinds of bets.

I assume a T+1 periods horizon. The consumer single period strictly concave utility function is U(C) and the discount factor is $0 < \beta \le 1$. The consumer can lend and borrow at the gross interest rate $R = 1/\beta$. The consumer's problem when starting with the wealth w is:

$$(2) \qquad V(w) = \max_{C_t} \sum_{t=0}^{T} \beta^t U(C_t) \text{ s.t. } \sum_{t=0}^{T} R^{-t} C_t = w.$$

The attitude towards bets in terms of money is determined by the property of the value function V(w). The solution to (2) is the smooth path: C_t = kw, for all t where k = $1/\sum_{t=0} R^{-t}$. Therefore:

$$V(w) = \sum_{t=0}^{T} \beta^t U(kw) = U(kw) \sum_{t=0}^{T} \beta^t = U(kw)/k \tag{3}$$

Taking derivatives leads to:

$$-V''(w)w/V'(w) = -U''(kw)(kw)/U'(kw) = -U''(c)c/U'(c) \tag{4}$$

Thus under the time separable utility function, the relative risk aversion for bets in terms of money is the same as the relative risk aversion to bets in terms of consumption (at any date). An immediate implication is that relative risk aversion to money bets does not depend on age: When the individual advances with age, the horizon, T+1, gets shorter but consumption per period does not change and therefore relative risk aversion does not change.

I now turn to show that (4) is special to the time-separable case.

4. THE ATTITUDE TOWARDS RISK UNDER THE COBB-DOUGLAS FUNCTION

I consider the following utility function:

(5) $$U(C_1,...,C_T;\ \alpha) = (1/\alpha)\prod_{t=0}^{T}(C_t)^{\alpha\beta^t},\ \alpha \neq 0,\ \alpha < 1\ \text{(ICD)}$$

$$U(C_1,...,C_T;\ \alpha) = \sum_{t=0}^{T}\beta^t \ln(C_t),\ \alpha = 0\ \text{(IL)}$$

where T+1 is the horizon, $0 < \beta \leq 1$ is the discount factor, and $\alpha < 1$ is a parameter. Since a change in α is a monotonic transformation of the utility function (5), Kihlstrom and Mirman (1974, page 366) definition of risk aversion can be applied here to show that risk aversion is decreasing in α.

To get a quantitative measure of the attitude towards risk implied by (5) I define the value function:

(6) $$V(w) = \max U(C_1,...,C_T;\ \alpha)\ \text{s.t.}\ \sum_{t=0}^{T}R^{-t}C_t = w.$$

As before I assume $R = 1/\beta$ and therefore the solution to the maximization problem in (6) is $C_t = kw$ and the value function is:

(7) $$V(w) = (1/\alpha)(kw)^{\alpha\sum_{t=0}^{T}\beta^t},\ \text{(ICD)}$$

$$V(w) = \ln(kw)\sum_{t=0}^{T}\beta^t,\ \text{(IL)}$$

The coefficient of relative risk aversion to bets in terms of money (RAM) is:

(8) $$- V''(w)w/V'(w) = 1 - \alpha\sum_{t=0}^{T}\beta^{t}$$

The coefficient of relative risk aversion to bets in terms of consumption (RAC) is:

(9) $$-U_{tt}C_{t}/U_{t} = 1 - \alpha\beta^{t}$$

Comparing (8) to (9) we see that when the utility is not time separable the measure of risk aversion to proportional bets in terms of money is different from the measure of risk aversion to proportional bets in terms of consumption. Note that the assumption $\alpha < 1$ insures RAC > 0.

To appreciate the difference between RAM and RAC consider the two periods case with $\beta = 1$. In this case, RAC = $1 - \alpha$ and RAM = $1 - 2\alpha$. Assume for example that $\alpha = 2/3$. In this case RAM < 0 implying that the consumer is willing to accept any actuarially fair money bet. But since RAC > 0 he will buy any actuarially fair insurance to eliminate risk about future consumption. This is the argument used in Eden (1979) to account for the behavior of the insurance-buying gambler.

Note also that in the ICD case RAM changes with age. At age τ, RAM = $1 - \alpha\sum_{t=\tau}^{T}\beta^{t}$. When $\alpha > 0$, RAM increases with age reaching a maximum of $1 - \alpha\beta^{T}$ in the last period of one's life. When $\alpha < 0$, RAM decreases with age reaching a minimum of $1 - \alpha\beta^{T}$ in the last period of one's life. When α approaches zero RAM approaches 1 (the log utility case).

5. A TWO PERIODS SINGLE TREE ECONOMY

I now turn to assess the importance of the RAM coefficient for understanding asset prices - the question in the title. I start with a single-asset version of Lucas (1978) tree economy.

There is a representative consumer who lives for two periods. He is born with an endowment of a tree that yields y units of consumption in the first period of his life and d_s units in the second period state s. There is a market for trees after the distribution of first period dividends. The price of a tree is p and the representative consumer chooses (in the first period of his life) present consumption (C_0) and the amount of trees (A) subject to the budget constraint:

(10) $$C_0 + pA = y + p$$

Consumption in the second period in state s is given by:

(11) $$C_{1s} = Ad_s$$

The consumer chooses A and C_0 to solve:

(12) $$\max_{A,C_0} \sum_{s=1}^{S} \Pi_s U(C_0, C_{1s}) \text{ s.t. (10) and (11),}$$

where Π_s is the probability of state s. The first order condition to (12) is:

(13) $$\sum_{s=1}^{S} \Pi_s (U_{0s} - U_{1s} d_s / p) = 0$$

where $U_{0s} = \partial U(C_0, C_{1s})/\partial C_0$ and $U_{1s} = \partial U(C_0, C_{1s})/\partial C_{1s}$.

The ICD-IL case: I now assume the Cobb-Douglas case:
$U(C_0, C_1) = (1/\alpha)(C_0)^{\alpha}(C_1)^{\delta}$, where $\delta = \alpha\beta$. In this case:

$$(14) \qquad U_{0s} - d_s U_{1s}/p = (1/\alpha)\left(\frac{\alpha}{C_0} - \frac{\delta}{y+p-C_0}\right)(C_0)^{\alpha}\left(\frac{(y+p-C_0)d_s}{p}\right)^{\delta}$$

Therefore the first order condition (13) requires
$\left(\frac{\alpha}{C_0} - \frac{\delta}{y+p-C_0}\right) = 0$ and $C_0 = \alpha(y+p)/(\alpha+\delta)$.

To solve for p we substitute the market clearing condition $C_0 = y$ in $C_0 = \alpha(y+p)/(\alpha+\delta)$. This leads to:

$$(15) \qquad p = (\delta/\alpha)y = \beta y.$$

The asset pricing formula (15) can also be obtained for the IL case. The rate of return on the asset is:

$$(16) \qquad D/p = D/\beta y = G/\beta,$$

where $D = \sum_{s=1}^{S} \Pi_s d_s$ is expected dividends and $G = 1 + g = D/y$ is the expected gross rate of consumption growth. Since (8) implies $RAM = 1 - \alpha(1 + \beta)$, varying α will change it without affecting the expected returns on the asset. We have thus shown, Claim 1: When the representative agent's utility function is ICD-IL, the expected rate of return on the asset does not depend on the RAM measure of relative risk aversion and does not depend on the variance of the return. It depends only on the expected rate of consumption growth (G) and the time preference parameter β.

Claim 1 is generalized in the Appendix to the finite horizon case. It follows directly from Kihlstrom and Mirman (1974) who show that in the ICD case uncertainty does not affect savings.

6. A TWO PERIODS MANY ASSETS ECONOMY

I now turn to the many assets case. I endow the representative agent with n trees. These n trees yield a total of y units of consumption (fruits) in the first period. Tree i yields d_{is} units in the second period in state s. The budget constraint of the representative agent is now:

(18) $$C_0 + \sum_{i=1}^{n} p_i A_i = y + \sum_{i=1}^{n} p_i$$

(19) $$C_{1s} = \sum_{i=1}^{n} d_{is} A_i$$

The agent problem is:

(20) $$\max_{A_i} \sum_{s=1}^{S} \Pi_s U(C_0, C_{1s}) \text{ s.t. (18) and (19).}$$

The first order conditions for this problem are:

(21) $$\sum_{s=1}^{S} \Pi_s(-U_{0s}p_i + U_{1s}d_{is}) = 0$$

I use $D_s = \sum_{i=1}^{n} d_{is}$ for aggregate dividends. I also assume that we can write the dividends of asset i in state s as a linear function of D_s:

(22) $$d_{is} = a_i + b_iD_s + e_{is},$$

where $\sum_{i=1}^{n} e_{is} = 0$ for all s; $\sum_{i=1}^{n} b_i = 1$ and $\sum_{i=1}^{n} a_i = 0$. The error terms e_{is} is determined by a zero sum purely distributive lottery, has zero mean and is independent of D_s. A risk-less asset is an asset with non-random dividends $d_{is} = 1$. The market portfolio is an asset for which $d_{is} = D_s$.

<u>The ICD case:</u> I now turn to the ICD case: $U(C_0, C_1) = (1/\alpha)(C_0)^{\alpha}(C_1)^{\delta}$. Using the first order conditions (21), the market clearing conditions $C_0 = y$, $C_{1s} = D_s$ and (22) we arrive at the equilibrium condition:

(23) $$p_i = \beta y \frac{\sum_{s=1}^{S} \Pi_s d_{is}(D_s)^{\alpha\beta-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\alpha\beta}} = \beta y \frac{\sum_{s=1}^{S} \Pi_s (a_i + b_iD_s)(D_s)^{\alpha\beta-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\alpha\beta}}$$

When $a_i = 0$, $p_i = \beta b_i y$, and $d_{is}/p_i = (b_iD_s + e_{is})/\beta b_i y$. Taking expectations leads to the following Claim.

<u>Claim 2:</u> The expected gross rate of return on an asset with $a_i = 0$ and $b_i > 0$ is G/β.

Thus the expected rate of return on all assets with $a_i = 0$ and

$b_i > 0$ is the same and is equal to the expected rate of return on the market portfolio.

I now turn to show that risk premium does not require risk aversion.

Claim 3: When $\delta < 1$, the rate of return on the risk free asset is less than G/β.

The proof of Claim 3 is in Appendix C. The intuition is in section 2. I now repeat this intuition for the special case of the ICD utility function. When $\delta < 1$, RAC = $1 - \delta > 0$ and the representative consumer is averse to uncertainty about future consumption. He will therefore hold the market portfolio rather than the risk free asset only if there is a risk premium. Note that when $\delta = \alpha\beta < 1$ the coefficient of risk aversion RAM = $1 - \alpha(1 + \beta)$ may be positive or negative. For example if $\beta = 1$ and $\alpha = 0.5$ then RAM = 0. Therefore risk premium does not require risk aversion to money bets.

I now turn to a numerical example. It is assumed that the rate of growth in aggregate dividends (consumption) is 1 or 1.04 with equal probabilities and $\beta = 1$. I use the following notation:

R^b = the return on the risk-free asset (with $a_i = 1$ and $b_i = 0$);

R^1 = the return on the market portfolio (with $a_i = 0$ and $b_i = 1$).

As we can see from Table 1 the rate of return on the market portfolio R^1 does not depend on the RAM coefficient and is equal to $G/\beta = 1.02$ in our example. The rate of return on the risk free asset is lower and decreases with our measure of risk aversion. The net rate of return on the risk free asset is is 1.98% when RAM = 0 and 1.92% when

RAM = 3. The risk premium is accordingly, 0.02% when RAM = 0 and 0.08% when RAM = 3.

We may conclude that the RAM coefficient has very little effect on asset prices but has a considerable effect on risk premium: a change in RAM from 0 to 3 increases risk premium by 300%. But since the risk premium is so small we may not care about its exact magnitude.

I now turn to examine the robustness of this conclusion to changes in the elasticity of substitution.

7. THE CONSTANT ELASTICITY FUNCTION

I consider the inter-temporal constant elasticity (ICE) utility function:

$$U(C_0, C_1) = (1/\psi)[(C_0)^\rho + \beta(C_1)^\rho]^{\psi/\rho}. \tag{24}$$

The standard power utility function is a special case of (24) that assumes: $\psi = \rho$. Thus, (24) is a monotonic transformation of the standard power utility function.

In Appendix A I discuss extensions to the multi-period horizon case. I now turn to the interpretation of the two parameters in (24). The parameter ρ governs IES = $1/(1 - \rho)$. To interpret the coefficient ψ in (24) I consider the problem under certainty:

$$V(w) = \max\ (1/\psi)[(C_0)^\rho + \beta(C_1)^\rho]^{\psi/\rho} \text{ s.t. } C_0 + C_1/R = w, \tag{25}$$

where R = 1/β is the gross interest rate. The solution to (25) is the smooth consumption: $C_0 = C_1 = kw$, where k = R/(R + 1). Substituting the solution in (25) leads to:

(26) $$V(w) = (1/\psi)(1+\beta)^{\psi/\rho}(kw)^{\psi}$$

It follows that:

(27) $$- V''(w)w/V'(w) = 1 - \psi.$$

Thus, ψ is the parameter that governs the RAM coefficient.[5]

In the standard power (SP) utility function case, when ρ = ψ, RAM = 1 - ψ = 1/IES = 1 - ρ. This is the well-known restriction imposed by the SP function. We can therefore use the ICE function to see whether the effect of a given change in the power parameter γ is due mainly to the implied change in RAM or the implied change in IES. To answer this question I turn now to decompose the effect of a given change in the power parameter γ to a RAM effect and an IES effect.

The asset pricing formula (21) under the ICE function (24) is:

(28) $$p_i = \beta y^{1-\rho} \frac{\sum_{s=1}^{S} \Pi_s [(y)^{\rho} + \beta (D_s)^{\rho}]^{\frac{\psi}{\rho}-1} (D_s)^{\rho-1} d_{is}}{\sum_{s=1}^{S} \Pi_s [(y)^{\rho} + \beta (D_s)^{\rho}]^{\frac{\psi}{\rho}-1}}$$

[5] The RAC coefficient depends on both parameters. Therefore, risk premium depends on both IES and RAM as discussed in section 2. A sufficient (but not necessary) condition for RAC > 0 is: ψ < ρ < 0. This implies IES < 1 and RAM > 1.

I now turn to apply this formula for our example. As before, I assume that $\beta = 1$ and consumption growth is: 1 and 1.04 with equal probabilities. Table 2 calculates the gross expected rate of return on the market portfolio for alternative values of the elasticity of substitution parameter and the risk aversion parameter. In this example, changes in the elasticity of substitution have a large effect on the gross expected interest rate while changes in risk aversion have a relatively small effect and in the opposite direction.

Under the SP utility function a change in the power parameter from $\gamma = 1$ to $\gamma = 3$, implies a change in IES from 1 to 0.333 and a change in RAM from 1 to 3. This change leads to an increase the net rate of return on the market portfolio by 200%, from 2% to 6%. A decrease in IES from 1 to 0.333 holding RAM = 1 constant increases the net rate by slightly more than 200%, from 2% to 6.08%. An increase in RAM from 1 to 3 holding IES constant at the level of unity does not have any effect on the interest rate. An increase in RAM from 1 to 3 holding IES constant at the level of 0.333 reduces the net rate by about 1% (from 6.08% to 6%). We may therefore say that the effect of an increase in γ from 1 to 3 is due to the change in IES. Decomposing the effect of changes in γ from (close to) zero to 1 and from 3 to 10 yields the same conclusion.

Table 3 computes the price of the risk-less asset (by substituting $d_{is} = 1$ in [28]). The results support the claim that changes in IES are relatively more important also for the risk free return. When we change IES from 1 to 3 we get an increase in the net rate of return of about 200%. But when we change RAM from 1 to 3 the rate of return decreases only by about 2%.

Table 4 subtracts Table 3 from Table 2 to get the risk premium. The risk premiums are small and depend on both the IES and the RAM coefficients. As suggested by the discussion in section 2, risk premium is positively related to RAM and negatively related to IES. Risk premium is positively related to RAM because an increase in RAM implies more demand for the elimination of risk (that is resolved prior to the choice of first period consumption). Risk premium is negatively related to IES because an increase in the elasticity implies less demand for smoothing.

Quantitatively the two effects are of equal importance in our example. A change in the RAM coefficient from 1 to 3 leads to a change in the risk premium from 0.04% to 0.08%. A "comparable" change in IES from 1 to 0.333 leads to roughly the same change in risk premium.

In Appendix E I compare the implications of the ICD and ICE expected utility functions (Tables 1 - 4) with the implications of the generalized expected utility approach taken by Selden (1978) and Epstein and Zin (1991). The results with respect to the return on the market portfolio are almost identical in the two approaches. The results about risk premium are different. In the Selden-Epstein-Zin approach, risk premium depends almost entirely on the risk aversion parameter. But in both cases risk premium is small. Therefore we may say that both approaches support the elasticity of substitution interpretation of the power parameter γ.

8. CLAIMS ON MACROECONOMICS AGGREGATES UNDER THE ICD-IL FUNCTION

To get a sense of the importance of the RAM coefficient for the returns on empirically relevant claims, I turn now to the estimation of the relationship (22) using NIPA data on flows.

Equation (22) is conditional on all the information available at time t. Omitting the index s and adding a time index I write:

$$d_{it+1} = a_{it+1} + b_{it+1}D_{t+1} + e_{it+1}, \tag{29}$$

where the coefficients are time dependent. I normalize y = 1 and assume: $a_{it+1} = a_i d_{it}$ and $b_{it+1} = b_i d_{it}$. This means that the coefficients of asset i are proportional to its last period share in aggregate dividends. Dividing (29) by d_{it} transforms (22) to an equation in terms of rates of growth:

$$G_{it+1} = a_i + b_i G_{t+1} + \varepsilon_{it+1}, \tag{30}$$

where $G_{it+1} = d_{it+1}/d_{it}$ is the gross rate of growth in asset i dividends, $G_{t+1} = D_{t+1}$ is the gross rate of growth in consumption and $\varepsilon_{it+1} = e_{it+1}/d_{it}$ is an error term. I assume that ε_{it+1} has a zero mean and is not correlated with G_{t+1}. The time invariant coefficients a_i and b_i can therefore be estimated from running the regression (30).

Using (23) and assuming that the rate of growth of aggregate consumption is iid, we write the price of asset i at time t as:

(31) $$p_{it} = \beta \frac{d_{it} \sum_{s=1}^{S} \Pi_s (a_i + b_i G_s)(G_s)^{\alpha\beta - 1}}{\sum_{s=1}^{S} \Pi_s (G_s)^{\alpha\beta}}$$

The expected gross rate of return is:

(32) $$R^i = \frac{d_{it} \sum_{s=1}^{S} \Pi_s (a_i + b_i G_s)}{p_{it}} = \frac{\sum_{s=1}^{S} \Pi_s (a_i + b_i G_s) \sum_{s=1}^{S} \Pi_s (G_s)^{\alpha\beta}}{\beta \sum_{s=1}^{S} \Pi_s (a_i + b_i G_s)(G_s)^{\alpha\beta - 1}}$$

Note that R^i does not depend on d_{it}. This is because the price and the expected dividends are both proportional to d_{it}. More generally, our model allow prices to change as a function of current dividends but the rate of return remains constant and depends only on the asset specific coefficients (a_i and b_i) and on the stationary distribution of the rate of growth of aggregate consumption.

I use (30) to estimate the coefficients a_i and b_i and then plug these coefficients in (32) to get the predicted rate of return. This approach is similar to the consumption based CAPM model. For example, Breeden, Gibbons and Litzenberger (1989) derive a linear relationship between the rate of return on an asset and the rate of aggregate consumption growth. To compute the rate of return we must observe asset prices. But the price of many claims is not readily observable. For example, we cannot observe the price of human capital. The price of unincorporated equity is not readily observable and so on. Our approach allow us to predict the rate of return on such claims because equation (30) requires data on the rates of change of flows that are easier to get than data on prices.

I now consider hypothetical claims on: (a) GDP, (b) the wage bill, (c) non-wage income (profits) and (d) corporate profits, all in real-per capita terms. These claims are of interest because a variety of real world claims are likely to have coefficients that are similar to one of the above claims on macro aggregates. For example, a claim on the wage bill is likely to be correlated with the average claim on ones own salary.

I use NIPA US post war data (from January 1948 to January 2004) taken from the Saint Louis Fed web page to compute the rate of growth in real per capita terms of the following variables: consumption (c), wage earnings (w), corporate profits (pr), GDP (y) and non-wage income (y-w). The details of the calculations of these variables and the description of the data are in Appendix D.

Table 5 provides summary statistics for the annual data. All rates of change are close to 2%. The smallest rate is for the wage bill (1.6%) and the highest is for corporate profits (2.2%). The standard deviation is in the range 0.02 - 0.04 except for corporate profits where it is much higher (0.16).

Table 6 provides the regression results from running (30). Most intercepts are small and barely significant. The intercept on corporate profits is an exception. The expected return on the market portfolio (R^1) is therefore a good approximation for the returns on claims on the wage bill, non-wage income and GDP. The expected return on a claim on corporate profits is estimated by using a = -2 and b = 3 in a portfolio labeled R^2.

Table 7 provides the predictions of the model using $\beta = 1/1.025$. The expected rates of returns are consistent with the estimates in

McGrattan and Prescott (2003) who took an explicit account of taxes and frictions and found average returns in the 4-5 percent range. The expected rate of return on the market portfolio is 1.0455. The expected rate of return on a claim on corporate profits is 1.0459 when RAM = 0 and 1.05 when RAM = 10. The corresponding risk premia on the more risky portfolio (corporate profits) are 0.05% and 0.6% respectively.

9. CONCLUDING REMARKS

This paper studies the asset pricing implications of monotonic transformations of the standard power utility function. These monotonic transformations yield time-non-separable expected utility functions that allow for clear separation between IES and RAM.

It is shown that when markets are complete the expected rate of return on the market portfolio depends mostly on the IES and not on the RAM coefficient. Using the log utility as our bench-mark (with IES = RAM = 1), we reduce IES from 1 to 0.333. This leads to an increase in the net rate of return on the market portfolio by about 200%. When we increase RAM from 1 to 3 the rate of return on the market portfolio does not change at all.

Similar results are obtained with respect to the risk free return. When IES changes from 1 to 0.333, the risk free rate goes up by about 200%. When RAM changes from 1 to 3 the net risk free rate goes down by about 2%. The risk free rate is close to the rate on the market portfolio and therefore the risk premium is small. The risk premium depends on both the IES and the RAM coefficients in a more or less symmetric way. A reduction in the IES from 1 to 0.333 leads to the doubling of the risk premium from 0.04% to 0.08%. An increase in the RAM coefficient from 1 to 3 yields similar change in the risk premium.

The standard power (SP) utility function yields the same predictions as the ICE utility function if we impose RAM = 1/IES. Since asset prices are not sensitive to changes in RAM this supports the interpretation that changes in the parameter of the SP utility function have large effect on asset prices because of the implied changes in IES and not because of the implied changes in RAM.

This suggests that we should devote efforts to estimate IES. Hall (1988), Campbell and Mankiw (1989) and Beaudry and Wincoop (1996) provide estimates of the IES between zero and one. Introspection may also help. We may imagine that we start from a non-random consumption path that specifies a 10% increase in consumption between time t and t+1. Under the SP utility function the slope of the indifference curve at the point $C_{t+1} = 1.1C_t$ is: $(1.1)^{\gamma}$. When $\gamma = 1$ this implies that we are willing to borrow at an interest rate of 10%. When $\gamma = 2$ and IES = $\frac{1}{2}$ we are willing to borrow at 21%. Situations in which one expects an increase in consumption by 10% are quite common. We also experience more extreme episodes in which one expects say a 100% increase in his consumption as at the end of graduate school. When $C_{t+1} = 2C_t$ we should be willing to borrow at 100% when IES = 1 and at 300% when IES = $\frac{1}{2}$. This type of introspection suggests to me IES that is above unity as estimated by Vissing-Jorgensen and Attanasio (2003).

I use post war US NIPA data on flows to estimate the rates of return on claims on various macroeconomic aggregates. Claims on the wage bill and on total profits are close to a claim on the market portfolio (aggregate consumption). But a claim on corporate profits is more risky.

The predictions of the ICD utility function are consistent with the findings in McGrattan and Prescott (2003).

I use the Kihlstrom-Mirman approach for separating between risk aversion and the elasticity of substitution because it is done within the expected utility framework. The importance of the expected utility assumption is still debated (see Gollier [2001, pages 11-14] for a good summary). Fortunately, the main results here hold also for the Generalized Expected Utility preferences approach taken by Selden (1978), Kreps and Porteus (1978) and Epstein and Zin (1989, 1991). Using the Selden-Epstein-Zin (SEZ) preferences with a log utility function aggregator yields the same results as the ICD function (when the RAC measure of risk aversion is the same in both specifications). Using the SEZ preferences with ICE aggregator yields very similar results about asset prices and return but qualitatively different results about the risk premium. The risk premium in the SEZ utility function depends only on risk aversion while under the expected utility approach taken here it depends on both IES and risk aversion. For RAC ≤ 3 the risk premium is small (less that 0.1%) under both specifications.

Our main results assume a RAM coefficient in the range 0 - 10. This is the range assumed originally by Mehara and Prescott (1985). It is possible that by allowing much higher RAM coefficients one can "solve" the original equity premium puzzle. Under the ICD function higher RAM coefficients will not change the rate of return on the market portfolio but will lower the risk-free rate. Therefore, very high RAM coefficients can lead to a large risk premium. This "solution" is possible because unlike the standard power utility function, the ICD function allows risk aversion and IES to be high simultaneously. See Kocherlakota (1996).

APPENDIX A: THE TIME CONSISTENCY ISSUE

Some have argued that extending the analysis to the many periods case, must run into time inconsistency problems. Here I argue against this apparently widely held perception. Expected utility guarantees time consistency if the only thing that changes over time is the list of choice variables and the probabilities of the states of nature that are updated as new information becomes available.

I start by visiting Strotz (1956) original article that focus on the certainty case. This article is sometimes interpreted as saying that exponential discounting is a necessary condition for time consistency. Here I follow the interpretation in Peleg and Yaari (1973) and argue that there are no time consistency problems if preferences do not change over time. To illustrate, I assume a 3 periods horizon and a utility function: $U(C_0, C_1, C_2)$. Note that the discounting is implicit in this utility function and need not be exponential. It is assumed that only the list of choice variables changes over time. At t = 0, the consumer chooses (C_0, C_1, C_2) to maximize $U(C_0, C_1, C_2)$. At t = 1 he will choose (C_1, C_2) to maximize $U(\overline{C_0}, C_1, C_2)$, treating his first period choice, $\overline{C_0}$, as given. This standard formulation in which preferences do not change over time insures that the optimal plan chosen at t = 0 is time consistent in the sense that the agent will not wish to change it at t = 1.

Age and distance from present discounting: Strotz (1956) assumes a time separable utility function. At time τ the consumer discounts the instantaneous utility at time t by the discount function $\lambda(t-\tau)$. The discount function thus depends on the time-distance from the present. Strotz also allows for the effect of calendar time (or age) on the "instantaneous utility function". In our three periods formulation Strotz's specification may be written as follows. At $\tau = 0$ the utility is given by:

$U^0(C_0, C_1, C_2) = \lambda(0)u(C_0, 0) + \lambda(1)u(C_1, 1) + \lambda(2)u(C_2, 2)$.

At $\tau = 1$ it is given by:

$U^1(C_0, C_1, C_2) = \lambda(-1)u(C_0, 0) + \lambda(0)u(C_1, 1) + \lambda(1)u(C_2, 2)$.

Strotz argues that in general, U^0 is different from U^1 and therefore it will lead to time consistency problems unless $\lambda(t-\tau) = \beta^{t-\tau}$. In this exponential discounting case $U^1 = \beta U^0$ and therefore the change in the utility function does not change behavior.

Modern analysis typically uses $u(C_t)$ instead of $u(C_t, t)$. In this case an age discounting specification is:

(A1) $$U^0(C_0, C_1, C_2) = \lambda(0)u(C_0) + \lambda(1)u(C_1) + \lambda(2)u(C_2)$$

and

(A2) $$U^1(\overline{C_0}, C_1, C_2) = \lambda(0)u(\overline{C_0}) + \lambda(1)u(C_1) + \lambda(2)u(C_2)$$

A distance from the present discounting is (A1) and

(A3) $$U^1(\overline{C_0}, C_1, C_2) = \lambda(-1)u(\overline{C_0}) + \lambda(0)u(C_1) + \lambda(1)u(C_2).$$

Note that in the special case $\lambda(t-\tau) = \beta^{t-\tau}$, there is no real difference between age discounting and time from present discounting.

To choose between (A2) and (A3) we may entertain the following thought experiment. We imagine that we start from a smooth consumption path and then consider the following deviations from it:

(a) An increase in consumption at the age of 20 by 1 unit and a reduction in consumption at the age of 30 by 2 units.

(b) An increase in consumption at the age of 60 by 1 unit and a reduction in consumption at the age of 70 by 2 units.

Consumers with an age discounting may go for (a) and refuse (b). They may argue for example that at 70 it seems difficult to adjust to a downward change in consumption but at 30 this is not so much of a problem. Time from present discounting implies that if you accept (a) at 20 you must also accept (b) at 60.

I interpret Strotz (1956) as saying that if you want time from present discounting and time consistency then the discounting must be exponential. But this does not imply that exponential discounting is necessary for time consistency. We can have age discounting that leads to time consistency even if it is not exponential.

Hyperbolic discounting: Another example in which changes in preferences leads to time inconsistency problems is the hyperbolic discounting utility function suggested by Laibson (1997) to model intrapersonal dynamic conflict. He uses: $U^t = u(c_t) + \beta \sum_{\tau=1} \delta^{\tau} u(c_{t+\tau})$.

In the three period case, Laibson's preferences may be written as:

$U^0(C_0, C_1, C_2) = u(C_0) + \beta\delta u(C_1) + \beta\delta^2 u(C_2)$

$U^1(C_0, C_1, C_2) = u(C_0) + u(C_1) + \beta\delta u(C_2)$.

In this example, the utility function, $U^t(C_0, C_1, C_2)$, changes over time. The marginal rate of substitution between C_1 and C_2 also changes over time. At t = 0 it is: $MRS^0(C_1,C_2) = u'(C_1)/\delta u'(C_2)$. At t = 1 it is: $MRS^1(C_1,C_2) = u'(c_1)/\beta\delta u'(c_2)$. Since $MRS^0(C_1,C_2) \neq MRS^1(C_1,C_2)$ the consumer at t = 1 will not want to stick to his original plan made at t = 0.

We may sum the discussion for the certainty case by saying that if the only thing that changes over time is the list of choice variables, then a consumer who makes a plan at t = 0 will not want to change it at t = 1.

Uncertainty: In the case of uncertainty, the general formulation by Arrow (1964) may lead to a problem of time consistency because new information will typically lead to the updating of probabilities. But under the expected utility hypothesis the utility is linear in the probabilities and therefore updating the probabilities does not lead to time inconsistency problems.

To show this well-known claim, I assume a three periods horizon: t = 0, 1, 2. Events at each date may take S possible realizations. The probability that "state of nature" k will occur at t = 0 is denoted by π_k. The probability that "state of nature" i will occur at date 1 given that "state of nature" k has occurred at t = 0 is denoted by π_{ki} and the probability that "state of nature" j will occur at date 2 given that "state of nature" k has occurred at t = 0 and "state of nature" i has occurred at t = 1 is denoted by π_{kij}. Similarly, C_{0k} denotes consumption at t = 0 state k, C_{1ki} denotes consumption at t = 1 state (k,i) and C_{2kij} denotes consumption at t = 2 state (k,i,j). The most general formulation used in Arrow (1964) assumes that the consumer evaluates consumption plans by the utility function:

(A4) $$Z(C_{01},...,C_{0S};C_{111},...,C_{1SS};C_{2111},...,C_{2SSS}).$$

At t = -1 he faces the budget constraint:

(A5) $$\sum_{k=1}^{S} P_{0k}C_{0k} + \sum_{k=1}^{S}\sum_{i=1}^{S} P_{1ki}C_{1ki} + \sum_{k=1}^{S}\sum_{i=1}^{S}\sum_{j=1}^{S} P_{2kij}C_{2kij} = w,$$

where P are the prices of the contingent commodities. He maximizes (A4) subject to (A5). The first order condition requires:

(A6) $$Z_{2msr} / Z_{1ms} = P_{2msr} / P_{1ms},$$

where $Z_i = \partial Z/\partial C_i$. In this general formulation an agent that learns about the state at t = 0 will update the probabilities and as a result the utility function Z will change. He will therefore want to change his plans.

I now turn to the expected utility case assuming that there exists a function U such that:

(A7) $$Z = \sum_k \pi_k \sum_i \pi_{ki} \sum_j \pi_{kij} U(C_{0k}, C_{1ki}, C_{2kij})$$

In this case, the marginal utilities are:

(A8) $$Z_{1ms} = \pi_m \pi_{ms} \sum_j \pi_{msj} U_1(C_{0m}, C_{1ms}, C_{2msj}), \quad Z_{2msr} = \pi_m \pi_{ms} \pi_{msr} U_2(C_{0m}, C_{1ms}, C_{2msr})$$

The marginal rate of substitution (MRS) is:

(A9) $$Z_{2msr}/Z_{1ms} = \pi_{msr}U_2(C_{0m},C_{1ms},C_{2msr}) \;/\; \sum_j \pi_{msj}U_1(C_{0m},C_{1ms},C_{2msj})$$

Suppose now that at t = 0 the consumer learns that k = m. Then his utility function will become:

(A10) $$Z^1 = \sum_i \pi_{mi} \sum_j \pi_{mij} U(\overline{C_{0m}},C_{1mi},C_{2mij})$$

Along the optimal plan when $C_{0m} = \overline{C_{0m}}$, the MRS, Z^1_{2msr}/Z^1_{1ms}, is the same as (A9). Thus the MRS does not change when at t = 0 the consumer learns that state m has occurred. In this sense the expected utility assumption is sufficient for guaranteeing time consistency.

APPENDIX B: A FINITE HORIZON SINGLE ASSET ICD ECONOMY

I now consider an economy in which the representative agent lives for T periods. At t = 0 he gets endowment of one tree that provides fruits for T periods and then dies (together with the agent).

I allow a general dividend (income) process. It is assumed that the representative agent at t = 0 assigns positive probabilities, π_s, to all states s = 1,...,S. Over time he updates these probabilities when he learns that some states did not occur. The set of possible states at time t (the information available at time t) is denoted by I_t. The updated probability of state s is denoted by $(\pi_s|I_t)$. Note that $(\pi_s|I_t) = 0$ if $s \notin I_t$. The agent also knows the information that he will have at time j > t if state s occurred. This information is denoted by I_{js}. At time t the choices of $(A_0,\ldots,A_{t-1})$ was already made. Since there is one tree per agent we assume $A_j = 1$ for j < t. The agent chooses A_t

and makes a contingent plan that specifies the amount of trees he will own at future dates: $(A_{t+1s},\ldots,A_{T-1s})$. The agent has to choose $A_{js} = A_{js'}$ if at time j he cannot distinguish between the two states. Thus, he faces the informational constraint: $A_{js} = A_{js'}$ if $s,s' \in I_{js}$. Assuming an ICD utility function we can state the time t problem as follows.

(B1) $$V_t(k_{t-1},I_t) = \max_{A_t,A_{t+1s},\ldots,A_{T-1s}}$$

$$k_{t-1}(d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - A_{t+1s}p_{t+1s}]^{\alpha\beta^t}\, k_{t+1s}$$

s.t.

$$k_{t-1} = \prod_{j=0}^{t-1} (d_j)^{\alpha\beta^j}$$

$$k_{t+1s} = \prod_{j=t+2}^{T} [A_{j-1s}(d_{js} + p_{js}) - A_{js}p_{js}]^{\alpha\beta^j}$$

$$A_{js} = A_{js'} \text{ if } s,s' \in I_{js}$$

I now define equilibrium as follows.

Equilibrium at time t is a vector $(A_t,\ A_{t+11},\ldots,A_{T-11},\ldots,A_{t+1S},\ldots,A_{T-1S};$ $p_t,\ p_{t+11},\ldots,p_{T-11},\ldots,p_{t+1S},\ldots,p_{T-1S})$ such that

(a) given prices $(p_t,\ p_{t+1s},\ldots,p_{T-1s})$, the quantity vector $(A_t\ ,A_{t+1s},\ldots,A_{T-1s})$ solves (A1) and

(b) market clearing: $A_t = 1$ and $A_{js} = 1$ for all $j > t$ and all s.

I now generalize the asset pricing formula (15) to the finite horizon case.

Claim B1: Equilibrium prices at time t are given by:

(B2) $$p_t = (\beta + \beta^2 + ...+ \beta^{T-t})d_t \text{ and}$$

$$p_{js} = (\beta + \beta^2 + ...+ \beta^{T-j})d_{js} \text{ for all } t < j < T$$

Note that when $T = \infty$ (B2) implies $p_t = d_t/\rho$, where the subjective interest rate $1 + \rho = 1/\beta$. This formula is in the logarithmic preference example in Ljungqvist and Sargent (2000, page 239).

Proof: When $T = 1$, there is trade in the asset only in period $t = T - 1 = 0$ and (B2) coincides with (15). We now proceed by induction. We assume that equilibrium prices when the horizon is T-t-1 (at time t+1) satisfy (B2) and show that equilibrium prices when the horizon is T-t (at time t) satisfy (B2).

Given our induction hypothesis we can write the problem (B1) as:

(B3) $$V(k_{t-1}; I_t) =$$

$$\max_{A_t} k_{t-1}(d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s}$$

Now $k_{t+1s} = \prod_{j=t+2}^{T}(d_{js})^{\alpha\beta^j}$ is a constant and $p_{t+1s} = (\beta + \beta^2 + ... + \beta^{T-t-1})d_{t+1s}$. Note that the assumption $A_{t+1s} = 1$ follows from the induction hypothesis.

The first order condition for the problem (B3) is:

(B4) $$-\alpha\beta^t p_t(d_t + p_t - A_t p_t)^{\alpha\beta^t - 1} \sum_{s=1}^{S}(\pi_s \mid I_t)\,[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s}$$
$$+ \ (d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^{S}(\pi_s \mid I_t)\,\alpha\beta^{t+1}(d_{t+1s} + p_{t+1s})[A_t(d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1} - 1} k_{t+1s} = 0$$

Substituting $A_t = 1$ and $p_{t+1s} = (\beta + \beta^2 + ... + \beta^{T-t-1})d_{t+1s}$ in (B4) leads to:

(B5) $$p_t = (\beta + \beta^2 + ... + \beta^{T-t})d_t$$

This completes the proof. □

We can now use Claim B1 to compute the rate of return on the asset as follows.

(B6) $$(d_{t+1s} + p_{t+1s})/p_t =$$

$$= (1 + \beta + \beta^2 + ... + \beta^{T-t-1})d_{t+1s} / (\beta + \beta^2 + ... + \beta^{T-t})d_t = d_{t+1s}/\beta d_t$$

Using $G_t = \sum_{s=1}^{S}(\pi_s \mid I_t)(d_{t+1s}/d_t)$ to denote the expected consumption growth we can write the expected rate of return at time t as:

(B7) $$G_t/\beta = G_t(1 + \rho),$$

where ρ is the subjective rate of interest. This is exactly the formula (16) that we got in the two periods horizon.

APPENDIX C: PROOF OF CLAIM 3

The rate of return on asset i is:

(C1) $$(a_i + b_i D_s + e_{is})/p_i = (1/\beta y)(a_i + b_i D_s + e_{is})\, G(a_i,b_i),$$

where $G(a_i,b_i)=\dfrac{1}{b_i+a_i\sum_{s=1}^{S}\Pi_s(D_s)^{\delta-1}/\sum_{s=1}^{S}\Pi_s(D_s)^{\delta}}$ is a non linear term. Since we assume δ < 1, the covariance between D and $D^{\delta-1}$ is negative and

$$\text{(C2)}\quad G(1,0)=\frac{\sum_{s=1}^{S}\Pi_s(D_s)^{\delta}}{\sum_{s=1}^{S}\Pi_s(D_s)^{\delta-1}}=\frac{\sum_{s=1}^{S}\Pi_s(D_s)^{\delta-1}D_s}{\sum_{s=1}^{S}\Pi_s(D_s)^{\delta-1}}=\frac{Cov(D^{\delta-1},D)}{\sum_{s=1}^{S}\Pi_s(D_s)^{\delta-1}}+\sum_{s=1}^{S}\Pi_s D_s$$

$$<\sum_{s=1}^{S}\Pi_s D_s .$$

Substituting this in (C1) and taking expectations leads to the conclusion that the expected rate of return on any asset with $b_i = 0$ is less than G/β. □

APPENDIX D: DATA

I took the following series from the St. Louis Fed web site.

Population (POP): Civilian Labor Force (M, SA),

Wage bill (NW): Compensation of Employees: Wages and Salary Accruals (Q, SAAR),

Consumption (NC): Personal Consumption Expenditures (Q, SAAR)

Price level (P): Gross Domestic Product Chain-type Price Index

Corporate Profits (NPR): Corporate Profits After Tax with Inventory Valuation Adjustment (IVA) and Capital Consumption Adjustment (CCADi)

Nominal GDP (NGDP): Gross Domestic Product, 1 Decimal

These data are available from January 1948 until January 2004. The data are available on a quarterly basis (except for population which is given on a monthly basis and was converted to a quarterly series). The data are in billions of current dollars and were divided by the price level and by population to obtain real per capita magnitudes:

W = NW/P(POP) real per capita wage earnings

C = NC/P(POP) real per capita consumption

PR = NPR/P(POP) real per capita Corporate Profits

Y = NGDP/P(POP) real per capita GDP

Y-W = (NGDP-NW)/P(POP) real per capita non wage income

I computed the following gross rates of change: $c_t = C_t/C_{t-1}$, $w_t = W_t/W_{t-1}$, $pr_t = PR_t/PR_{t-1}$, $y_t = GDP_t/GDP_{t-1}$, $(y-w)_t = (Y-W)_t/(Y-W)_{t-1}$.

APPENDIX E: COMPARISON WITH THE SELDEN-EPSTEIN-ZIN UTILITY FUNCTION

Selden (1978), Kreps and Porteus (1978) and Epstein and Zin (1989, 1991) use a non-expected utility function to separate between risk aversion and IES. I show here that their approach yields similar predictions about the expected rate of return on the market portfolio. The difference arises only when IES < 1 and only about the predicted risk premium. The Selden-Epstein-Zin (SEZ) utility function yields risk premium that depends almost entirely on the risk aversion coefficient and not on the elasticity of substitution.

I start with the special case of IES = 1. In this case there is no difference between the prediction of the SEZ utility function and the prediction of the ICD utility function. I use Selden's formulation which is identical to the Epstein-Zin formulation for the two periods case.

Selden evaluates consumption paths in two stages. He first uses a "certainty equivalence function" to substitute a certainty equivalent for the random future consumption and then an "aggregator function" to evaluate current consumption and the certainty equivalence of future consumption.

To illustrate, let C denotes current consumption and x denotes a random future consumption. The consumer first uses the certainty equivalence function μ to convert x to a scalar: $Z = \mu(x)$. He then uses the aggregator function $G(C, Z)$ to evaluate the consumption path. In this formulation IES is determined by the properties of the aggregator function G while RAC is determined by the properties of the certainty equivalence function μ.

I now turn to the special case:

(E1) $$G(C, Z) = \log(C) + \log(Z); \quad Z = (Ex^{\sigma})^{1/\sigma} \text{ where } 0 \neq \sigma < 1.$$

In (E1) the aggregator function is logarithmic and as in Epstein and Zin (1991), the certainty equivalence function is of the CES type. The RAC coefficient can be derived from the parameter of the certainty equivalent function. It is: RAC = $1 - \sigma$.

For the single asset case, the consumer's problem is:

(E2) $$\max_{A} \log(y + p - pA) + \beta \log\{[\sum_{s=1}^{S} \Pi_s (D_s A)^{\sigma}]^{1/\sigma}\}$$

The first order condition for this problem is (15). Thus as in the log expected utility case the price of the asset depends only on current dividends ($p = \beta y$) and not on the certainty equivalent of future consumption. Therefore risk aversion and aggregate risk do not affect the price of the asset and the expected return.

For the many asset case, the consumer problem under Selden's utility function is:

(E3) $$\max_{A_i} \log[y + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i A_i] + \beta \log\{[\sum_{s=1}^{S} \Pi_s (\sum_{i=1}^{n} d_{is} A_i)^{\sigma}]^{1/\sigma}\}$$

The equilibrium prices (which we obtain after substituting A_i = 1 and in the first order conditions) are:

(E4) $$p_i = \beta y \frac{\sum_{s=1}^{S} \Pi_s d_{is} (D_s)^{\sigma-1}}{\sum_{s=1}^{S} \Pi_s (D_s)^{\sigma}}$$

This is exactly the formula (24) when σ is used instead of $\alpha\beta$. We have thus shown the following Claim.

Claim E1: The Selden-Epstein-Zin utility function (E1) and the ICD utility function are observationally equivalent (in the sense that they both yield the same asset prices) when $\sigma = \alpha\beta$.

I now turn to the often used case in which the aggregator function is given by the ICE utility function: G(C, Z) = $[C^{\rho} + \beta Z^{\rho}]^{1/\rho}$ and the representative agent's problem is given by:

(E5) $$\max_{A_i} \left([y + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i A_i]^{\rho} + \beta\{[\sum_{s=1}^{S} \Pi_s (\sum_{i=1}^{n} d_{is} A_i)^{\sigma}]^{1/\sigma}\}^{\rho} \right)^{\frac{1}{\rho}}$$

This leads to the following asset pricing formula.

(E6) $$p_i = \beta\, y^{1-\rho} \{[\sum_{s=1}^{S} \Pi_s (D_s)^{\sigma}]^{1/\sigma}\}^{\rho-1} [\sum_{s=1}^{S} \Pi_s (D_s)^{\sigma}]^{\frac{1}{\sigma}-1} \sum_{s=1}^{S} \Pi_s (D_s)^{\sigma-1} d_{is},$$

where $D_s = \sum_{i=1}^{n} d_{is}$ is consumption in state s.

Table E1 uses (E6) to derive the predictions about the expected rate of return on the market portfolio (assuming $d_{is} = D_s$). The rates of returns are almost identical to the rates of return under the ICE function in Table 2. Thus the result that the ordinal properties of the utility function (IES) dominate the predicted rate of return on the market portfolio is robust.

Table E2 uses (E6) to calculate the expected rate of return on the risk free asset (assuming $d_{is} = 1$ for all s). Here we see a difference from the ICE utility function in Table 3. But it is still true that the predicted risk free return is dominated by IES.

Table E3 uses (E6) to calculate the risk premium: the difference between Tables E1 and E2. Risk premium are still small but here there is a qualitative difference from the results of the ICE utility function in Table 4: The risk premium depends almost entirely on the risk aversion measure and not on IES.

APPENDIX F: INSURANCE PREMIUM

To better understand why risk premium under the expected utility approach depends on both the cardinal and the ordinal properties of the utility function I turn now to the analysis of the related concept of insurance premium.

I assume a zero interest rate and a random wealth constraint W. The consumer's budget constraint is: $C_{1s} = w_s - C_0$, where w_s (s = 1,...,S) is the realization of the wealth constraint in state s which occurs with probability Π_s. It is assumed that the realization of W is known after the choice of C_0 has already been made.

Under the expected utility hypothesis the maximum amount that the consumer will pay for the elimination of risk that is resolved late (after the choice of C_0) is INS defined by:

$$\text{(F1)} \qquad \max_{c_0} \sum_{s=1}^{S} \Pi_s U(C_0, w_s - C_0) = \max_{C_0} U(C_0, EW - INS - C_0)$$

Let us define the certainty equivalents INF and M by:

$$\text{(F2)} \qquad \max_{c_0} \sum_{s=1}^{S} \Pi_s U(C_0, w_s - C_0) = \sum_{s=1}^{S} \Pi_s \max_{c_{0s}} U(C_{0s}, w_s - INF - C_{0s})$$

$$= \max_{c_0} U(C_0, EW - INF - M - C_0)$$

Note that INF is the maximum amount that the consumer will pay to buy information about his wealth constraint that allows for "smoothing". M is the maximum amount that the consumer will pay to avoid the resulting money bet. Note that (F1) and (F2) imply:

$$\text{(F3)} \qquad INS = INF + M$$

We may thus say that the demand for insurance (INS) can be decomposed into a demand for information and the demand for the elimination of the money bet. The demand for information depends on the ordinal properties of the utility function. The demand for eliminating the money bet depends on the cardinal properties. Therefore the demand for insurance depends on both.

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Table 1: Expected gross rates of Returns under the ICD-IL utility function (β = 1)

RAM = $1-2\alpha$	R^1 ; d_i={1,1.04}	R^b ; d_i={1,1}	100(R^1 - R^b)
0	1.02	1.0198	0.02
1	1.02	1.0196	0.04
3	1.02	1.0192	0.08
10	1.02	1.0179	0.21

Table 2: Expected gross rates of returns on the market portfolio (D/p) under the ICE utility function (β = 1)

IES\RAM	RAM = 0	RAM = 1 (SP)	RAM = 3	RAM = 10
IES=0.99	1.0202	1.0202	1.0202	1.0202
IES=0.5	1.0404	1.0402	1.0398	1.0384
IES=0.333	1.0612	1.0608	1.0600	1.0573

Table 3: The risk free return (R^b) under the ICE utility function (β = 1)

IES\RAM	RAM = 0	RAM = 1	RAM = 3	RAM = 10
IES=0.99	1.0200	1.0198	1.0194	1.0180
IES=0.5	1.0400	1.0396	1.0388	1.0361
IES=0.333	1.0606	1.0600	1.0588	1.0547

Table 4: Risk premium in percentage terms ($100[R^1 - R^b]$) under the ICE utility function (β = 1)

IES\RAM	RAM = 0	RAM = 1	RAM = 3	RAM = 10
IES=0.99	0.020	0.039	0.079	0.215
IES=0.5	0.040	0.060	0.100	0.237
IES=0.333	0.062	0.082	0.122	0.260

Table 5: Summary Statistics about Annual Per Capita Gross Rates of Change

	Average	Standard deviation
Consumption (c)	1.019	0.02
GDP (y)	1.018	0.03
Wage earnings (w)	1.016	0.03
Profits (y-w)	1.020	0.04
Corporate Profits (pr)	1.022	0.16

Table 6*: Regressions of the rate of change of asset i on the rate of change in consumption

Dependent var.	Intercept	Slope	Rsquare
y	-0.19 (0.10)	1.18 (0.10)	0.73
w	0.07 (0.14)	0.93 (0.14)	0.46
y-w	-0.45 (0.14)	1.44 (0.14)	0.67
pr	-2.06 (0.93)	3.03 (0.91)	0.17

* Standard errors in parentheses.

Table 7: Predicted Rates of Returns (IES = 1; $1/\beta$ = 1.025)

RAM = $1-2\alpha$	R^1 $a_i = 0;\ b_i = 1$	R^2 $a_i=-2; b_i=3$	R^b $a_i=1;\ b_i=0$	$100(R^1-R^b)$	$100(R^2-R^b)$
0	1.0455	1.0459	1.0453	0.02%	0.06%
1	1.0455	1.0463	1.0451	0.04%	0.12%
3	1.0455	1.0470	1.0447	0.08%	0.23%
10	1.0455	1.0497	1.0433	0.22%	0.64%

Table E1: Rates of return for the market portfolio (R^1: $d_{is} = D_s$) using SEZ utility function with ICE aggregator

IES/RAC=1-σ	RAC= 0.5	RAC=0.99	RAC=3	RAC=10
IES=0.99	1.0202	1.0202	1.0202	1.0202
0.5	1.0403	1.0402	1.0398	1.0384
0.333	1.0611	1.0609	1.0600	1.0572

Table E2: Rates of return for the risk free asset (R^b: $d_{is} = 1$) using SEZ utility function with ICE aggregator

IES/RAC	RAC=0.5	RAC=0.99	RAC=3	RAC=10
IES=0.99	1.0200	1.0198	1.0190	1.0163
0.5	1.0401	1.0398	1.0386	1.0345
0.333	1.0609	1.0605	1.0588	1.0531

Table E3: Risk Premium in percentage terms ($100[R^1-R^b]$) using SEZ utility function with ICE aggregator

IES/RAC	RAC=0.5	RAC=0.99	RAC=3	RAC=10
IES=0.99	0.02	0.04	0.12	0.39
0.5	0.02	0.04	0.12	0.39
0.333	0.02	0.04	0.12	0.40