This chapter is taken from the book:

"A Course in Monetary Economics: Sequential Trade, Money and Uncertainty"

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CHAPTER 2

Money in the Utility Function

Figure 1.1 establishes a connection between the average rate of change in the money supply and the average rate of inflation. There is little dispute about this long run relationship. The question is whether we want to adopt a policy of low money supply growth and low inflation or a policy of high money supply growth and high inflation. Most economists will favor the low money growth low inflation rate long run equilibrium. How low should we go?

We will examine this question using a variety of models starting from the money in the utility function approach used among others by Patinkin (1965), Sidrauski (1967) and Friedman (1969). This approach assumes that money is held because it yields some services and the way to model it is to assume a utility function in which real balances enter as an argument. It has been criticized because it does not provide an explicit description of the role of money. We will nevertheless exposit this model and derive a policy implication. In chapter 5 we will examine the robustness of the policy implication using models that are more explicit about the role of money.

The exposition here borrows from Friedman's (1969) original optimum quantity of money article and can be regarded as a diagrammatic exposition of the main ideas. We will conduct the discussion around the question of the optimum rate of change in the price level and in the money supply.

2.1 MOTIVATING THE MONEY IN THE UTILITY FUNCTION APPROACH: THE SINGLE-PERIOD, SINGLE-AGENT PROBLEM

To motivate the money in the utility function assumption, we start from the problem of an agent who comes to period t with M_{t-1} dollars and an endowment of z goods: $\bar{x}_t = (\bar{x}_{t1}, \ldots, \bar{x}_{tz})$. In addition, he gets a transfer-payment from the government of G_t dollars. The amount of money before the beginning of trade at time t is: $M_{tb} = M_{t-1} + G_t$ dollars.

We start from the case in which money is the only asset (there are no bonds and no physical capital). The agent faces the dollar prices: $p_t = (p_{t1}, \ldots, p_{tz})$, where p_{ti} denotes the dollar price of good i. Nominal spending for the period is given by: $I_t = M_{tb} - M_t + \sum_i \bar{x}_{ti} p_{ti}$, where M_t are end of period balances and $\sum_i \bar{x}_{ti} p_{ti}$ is the dollar value of the endowment. It is assumed that I_t and M_t are exogenous at this stage. (Later in the multi-period problem the agent will be able to choose these variables.) The agent's budget constraint for period t is thus:

$$\sum_i x_{ti}p_{ti} = I_t = M_{tb} - M_t + \sum_i \bar{x}_{ti}p_{ti}, \tag{2.1}$$

where $x_t = (x_{t1}, \ldots, x_{tz})$ denote quantities consumed in period t.

It takes time to exchange one vector of goods for another. The amount of time (labor) required for executing a shopping list, $x_t - \bar{x}_t = (x_{t1} - \bar{x}_{t1}, x_{t2} - \bar{x}_{t2}, \ldots, x_{tz} - \bar{x}_{tz})$, that satisfies the budget constraint (2.1) is:[1]

$$L_t = F(x_t - \bar{x}_t, p_t, M_{tb}). \tag{2.2}$$

Starting with more money reduces the amount of time required for executing a given shopping list and therefore the function F( ) is decreasing in M_{tb}. This assumption may be justified in terms of a model in which agents meet each other sequentially and bilateral trade takes place until all agents complete their desired exchange. An agent who does not have enough money will have to sell first, accumulate nominal balances and buy later. This is a constraint on the exchange process and therefore on average more time is required to complete a given exchange, when M_{tb} is low.[2]

Consider now a change in all nominal magnitudes by a factor λ: Instead of (I, p, M_{tb}) we now have (λI, λp, λM_{tb}). This does not change the set of vectors x which satisfy (2.1). It is also true that the bilateral trades that a consumer can do with M_{tb} dollars at the prices p are exactly the same as the trades that he can do with λM_{tb} dollars at the prices λp. For example, suppose that a consumer wants to buy 5 units of a single good. He can do it if he has 10 dollars and the price of the good is 2. He can also do it if he has 20 dollars and the price of the good is 4. For this reason, we assume:

$$F(x - \bar{x}, p, M_b) = F(x - \bar{x}, \lambda p, \lambda M_b) \quad \text{for all } \lambda > 0, \tag{2.3}$$

where the time index is omitted.

The consumer's single period utility function is given by $u(x, L)$, where u( ) is strictly increasing in x and strictly decreasing in L. The consumer chooses x to maximize $u(x, L)$ subject to (2.1) and (2.2). Let $V(\bar{x}, I, p, M_b)$ denote the maximum single period utility the consumer can get when facing the exogenously given magnitudes $(\bar{x}, I, p, M_b)$. Thus,

$$V(\bar{x}, I, p, M_b) = \max_{x,L} u(x, L), \quad \text{s.t. (2.1) and (2.2)}. \tag{2.4}$$

The function V( ) is sometimes called an indirect utility function. The consumer does not receive utility from income or the beginning of period balances directly, but these magnitudes affect the set of feasible choices of x and L and therefore affect the maximum utility that he can achieve.

Since changing (I, p, M_{tb}) by the same proportions does not affect the set of vectors (x, L) that satisfy the constraints (2.1) and (2.2) we have:

$$V(\bar{x}, I, p, M_b) = V(\bar{x}, \lambda I, \lambda p, \lambda M_b) \quad \text{for all } \lambda > 0. \tag{2.5}$$

We now choose $\lambda = 1/p_1$ and write:

$$V(\bar{x}, Y, 1, p_2/p_1, \ldots, p_z/p_1; m_b), \tag{2.6}$$

where $Y = I/p_1$, is total expenditures in terms of good 1 and $m_b = M_b/p_1$ is the purchasing power of the nominal balances held at the beginning of trade, in terms of good 1.

For the purpose of analyzing fully anticipated changes in monetary policy, it is useful to assume that *relative* prices $(p_2/p_1, \ldots, p_z/p_1)$ and *real* endowments, $\bar{x}$, are constant and write (2.6) as:

$$V(Y, m_b), \tag{2.7}$$

where the same symbol is used to denote different functions.

2.2 THE MULTI-PERIOD, SINGLE-AGENT PROBLEM

We are now ready to discuss the choice of real balances. We assume that there exist functions f( ) and U( ) such that:

$$V(Y_t, m_{tb}) = U[Y_t + f(m_{tb})], \tag{2.8}$$

where U( ) has the standard properties of a single period utility function and f( ) has the standard properties of a production function, with $f(0) = 0$. Indeed we may think of real balances as an input in the production of consumption (liquidity services). Although money is useful only if there are many goods, we simplify the discussion by assuming that there is only one non-storable good: Corn. Under (2.8) we can define consumption as the sum of corn consumption and "liquidity services":

$$C_t = Y_t + f(m_{tb}). \tag{2.9}$$

This consumption measure is in units of corn. For example, if $m_{tb} = 4$, $Y_t = 5$ and $f(m_{tb}) = (m_{tb})^{1/2}$ then the liquidity services from 4 units of real balances are equivalent to 2 units of corn and the total consumption level is 7. This level of consumption can also be achieved with 7 units of corn and no real balances.

We start from the case in which prices are stable and there are no transfer-payments from the government. In this case, the level of real balances at the beginning of time t trade is: $m_{tb} = m_{t-1}$.

We consider the problem of an infinitely lived representative agent whose utility function is:[3]

$$\sum_{t=1}^{\infty} \beta^t U(C_t), \tag{2.10}$$

where U( ) is a single period utility function and $0 < \beta < 1$ is a discount factor. It is useful to define the subjective interest rate ρ by: $\beta = 1/(1 + \rho)$. The function U( ) is differentiable, strictly monotone and strictly concave.

The endowment of corn is constant over time and is given by $\bar{Y}$ units, per period. Since money is the only asset, the *individual* agent's real balances evolve according to:

$$m_t - m_{t-1} = \bar{Y} - Y_t. \tag{2.11}$$

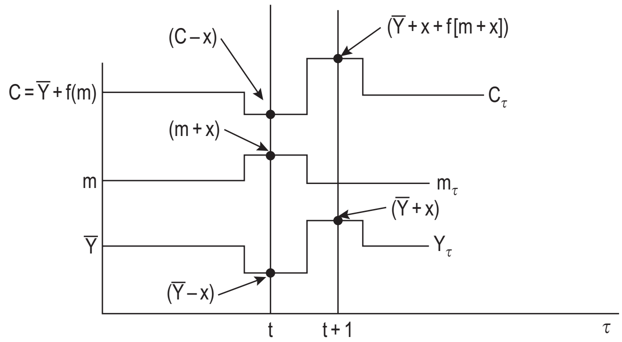

Figure 2.1 A feasible deviation from a smooth path

The representative agent's problem is to choose the levels of real balances m_t which maximize (2.10) subject to (2.9), (2.11) and

$$m_{tb} = m_{t-1} \geq 0, \quad \text{and } m_0 \text{ is given.} \tag{2.12}$$

Under what conditions will the agent want to hold his initial endowment of real balances (m_0) forever? To answer this question we define a smooth consumption path that is characterized by the level of real balances m by:

$$Y_t = \bar{Y} \quad \text{and} \quad m_t = m_0 = m \qquad \text{for all } t. \tag{2.13}$$

When this smooth consumption path maximizes (2.10) subject to (2.9), (2.11) and (2.12) an agent who starts with m units of real balances will not change the amount of real balances over time.

First order condition: It must be the case that any small deviation from an optimal path does not change the value of the objective function. We now use this basic principle from calculus to derive first order conditions.

The representative agent can deviate from a smooth consumption path in the following way. He can reduce corn consumption at t by x units and accumulate x units of real balances. He can then use the additional real balances to increase corn consumption at $t + 1$ by x units. Thus, if a smooth consumption path which is characterized by m is feasible then the path:

$$\begin{aligned} Y_t = \bar{Y} - x; \quad Y_{t+1} = \bar{Y} + x; \\ Y_\tau = \bar{Y} \quad \text{for } \tau < t \text{ and } \tau > t + 1 \\ m_t = m + x \quad \text{and} \quad m_\tau = m \qquad \text{for all } \tau \neq t; \end{aligned} \tag{2.14}$$

is also feasible. Figure 2.1 illustrates the proposed deviation (2.14).

If the agent follows the deviation (2.14), then in addition to more corn he will have $f(m + x) - f(m)$ additional units of liquidity services at time $t + 1$ because of the increase in the beginning of period real balances. It follows that giving up x units of corn at t yields:

$$\Delta C_{t+1} = C_{t+1} - C = x + f(m + x) - f(m), \tag{2.15}$$

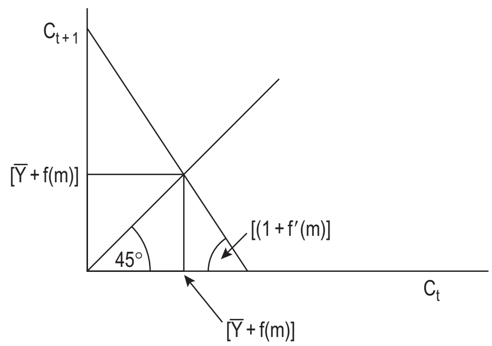

Figure 2.2 The "budget line"

units of consumption at $t + 1$, where $C = \bar{Y} + f(m)$ denotes consumption along the smooth path and ΔC is the change from the smooth path.

Dividing (2.15) by $\Delta C_t = -x$, yields the price of C_t in terms of C_{t+1}:

$$-\Delta C_{t+1}/\Delta C_t = 1 + [f(m + x) - f(m)]/x. \tag{2.16}$$

When x is small, we may approximate:

$$[f(m + x) - f(m)]/x \approx f'(m). \tag{2.17}$$

This approximation is used in figure 2.2, where the price of current consumption in terms of future consumption is: $1 + f'(m)$.[4]

The marginal product of real balances, $f'(m)$, plays the role of the interest rate in the standard Fisherian model. In Fisher's model if you deposit an amount of money which can buy a unit of corn in the bank you will get next period an amount that can buy $1 + r$ units of corn, where r is the real interest rate. In our model, if you add to your money holdings an amount that can buy a unit of corn, you will get the equivalent of $1 + f'(m)$ units of corn next period. Thus, here the relevant interest rate is the marginal product of money and this rate of return changes with m.

It is assumed that there exists a satiation level $\bar{m}$ such that $f'(m) > 0$ when $m < \bar{m}$; $f'(\bar{m}) = 0; f'(m) < 0$ when $m > \bar{m}$. The marginal product is very large for small levels of the input ($f'(0) = \infty$) and declines: $f''(m) < 0$ everywhere.[5] Figure 2.3 illustrates these properties.

Figure 2.4 illustrates the "budget lines" in the (C_{t+1}, C_t) plane. There are three "budget lines" defined for three levels of real balances: $\bar{m} > m' > m''$. Note that the slope of the budget lines goes down with m and is equal to unity when $m = \bar{m}$.

The slope of the indifference curves in the (C_{t+1}, C_t) plane can be computed by taking a full differential of (2.10) and setting $dC_\tau = 0$ for $\tau < t$ and for $\tau > t + 1$. This yields:

$$(1 + \rho)U'(C_t)/U'(C_{t+1}), \tag{2.18}$$

and is equal to $1 + \rho$ along the 45° line, when $C_t = C_{t+1}$.

We can now use figure 2.4 to determine whether the consumer will want to stay on the 45° line. If he starts with m'' units of real balances, he will move to a point like A and accumulate

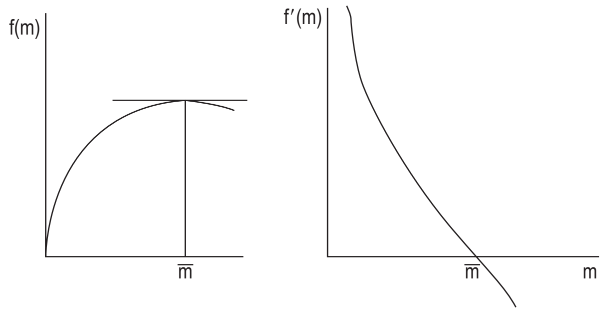

Figure 2.3 The liquidity service function

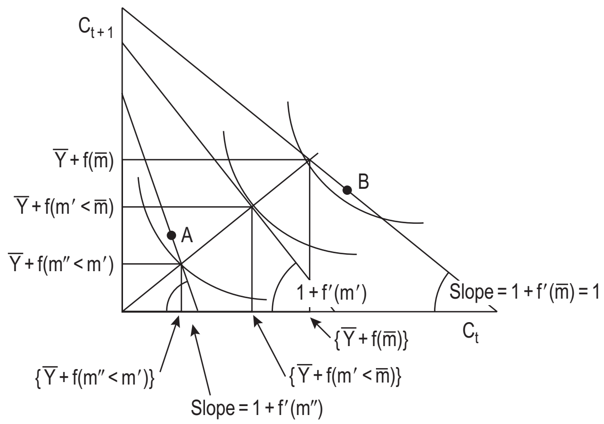

Figure 2.4 Varying m : $m'' < m' < \bar{m}$

real balances. If he starts with $\bar{m}$ units he will move to a point like B and decumulate real balances. If he starts with m' units he will not change the amount of real balances. Thus only m' characterizes an optimal smooth consumption path.

Formally if m characterizes an optimal smooth consumption path then it must satisfy the first order condition, $1 + \rho = 1 + f'(m)$, or:

$$\rho = f'(m). \tag{2.19}$$

Since $f'' < 0$ and $f'(0) = \infty$ there is a unique solution m' to (2.19), as in figure 2.5.

An alternative way of deriving (2.19): A small deviation from an optimal path should not change the level of the objective function. We now consider the following alternative deviation from a smooth path:

$$\begin{aligned} Y_t &= \bar{Y} - x; \\ Y_\tau &= \bar{Y} \quad \text{for all } \tau \neq t; \\ m_\tau &= m + x \quad \text{for all } \tau \geq t \quad \text{and} \quad m_\tau = m \quad \text{for all } \tau < t. \end{aligned} \tag{2.20}$$

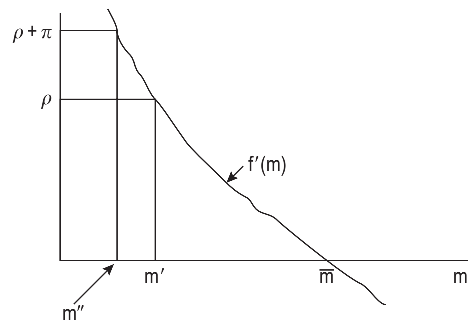

Figure 2.5 The first order condition

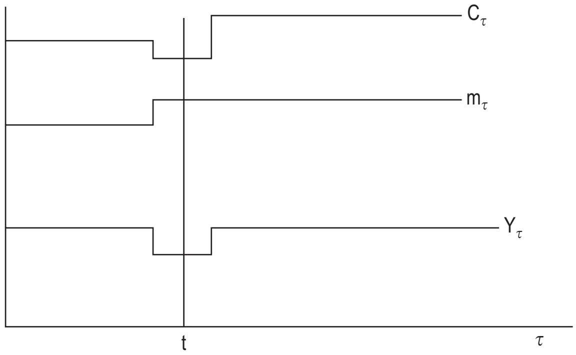

Figure 2.6 An alternative feasible deviation

In (2.20) we cut corn consumption at time t by x units ($\Delta C_t = -x$) accumulate x units of real balances and hold $m + x$ units forever. This amounts to a tradeoff between current consumption, C_t, and permanent consumption, C_p. The additional x units of real balances yield an infinite stream of $\Delta C_p = f(m+x) - f(m)$ units of consumption per period. Figure 2.6 illustrates this deviation from a smooth consumption path.

The price of current consumption in terms of permanent consumption is:

$$-\Delta C_p/\Delta C_t = \{f(m + x) - f(m)\}/x \approx f'(m). \tag{2.21}$$

The slopes of the "budget lines" in the (C_p, C_t) plane are therefore $f'(m)$ as in figure 2.7.

To find the slope of the indifference curves in the (C_p, C_t) plane, we take a full differential of (2.10) and equate it to zero: $\sum_{\tau=t}^{\infty} \beta^{\tau-t} U'(C_\tau) dC_\tau = 0$. We now substitute $dC_\tau = dC_p$ for $\tau > t$ to get:

$$-U'(C_t)\, dC_t = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} U'(C_\tau)\, dC_p. \tag{2.22}$$

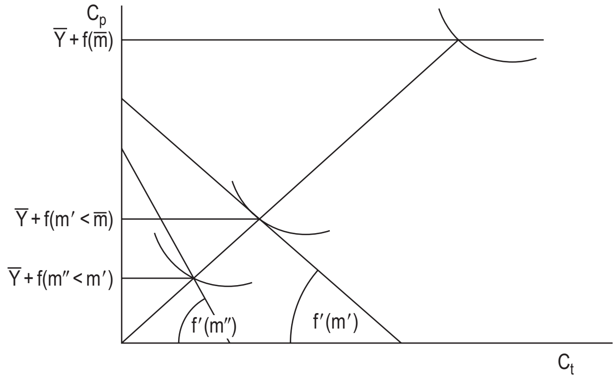

Figure 2.7 Tradeoffs between current and permanent consumption

This leads to:

$$-dC_p/dC_t = U'(C_t)\Big/\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}U'(C_\tau). \tag{2.23}$$

Along the 45 degrees line $C_\tau = C_t$ and (2.23) implies:

$$-dC_p/dC_t = \rho. \tag{2.24}$$

Thus the slope of the indifference curves in the (C_t, C_p) plane along the 45 degrees line is ρ.

If m characterizes an optimal smooth consumption path, then at the point: $C_t = C_p = \bar{Y} + f(m)$, an indifference curve must be tangent to a budget line. Thus, (2.21) = (2.24) which is equivalent to (2.19).

2.3 EQUILIBRIUM WITH CONSTANT MONEY SUPPLY

The nominal quantity of money, M, is determined by the government. The price level P (an index of the money price of goods or the dollar price of corn in our model) is determined in equilibrium by the condition:

$$f'(M/P) = \rho. \tag{2.25}$$

Thus the price level adjusts so that the money supply M is willingly held. To see the forces that operate to achieve (2.25), suppose that P' is a solution to (2.25) and we start with $P'' > P'$. In this case, real balances $m'' = M/P''$ are lower than the desired level: $m' = M/P'$. All agents will therefore try to exchange goods for money (move to a point like A in figure 2.4) but since the supply of money (M) is fixed, it is not possible for all agents to do it.

The attempt to exchange goods for money will drive the price of money up (and the price of corn down). This will stop when the price level reaches the equilibrium level: P'.

Note that the government determines the nominal quantity of money but the private sector determines the real quantity of money. This leads to the distinction between the social and the private cost for accumulating real balances, which is the main point of Friedman's article.

2.4 THE SOCIAL AND PRIVATE COST FOR ACCUMULATING REAL BALANCES

We start with an equilibrium in which all agents hold the equilibrium level of real balances m'. An individual agent, Mr. Loser, lost some money and as a result found himself at the beginning of period t with a quantity of real balances $m < m'$. Mr. Loser will find himself at point A in the left upper diagram of figure 2.8. He will choose to accumulate real balances by giving up x units of corn and moving to point B. At time $t + 1$ he will find himself at point D (the right upper diagram in figure 2.8) and he may choose either to continue and accumulate real balances (if $m + x < m'$) or maintain the level he has (if $m + x = m'$). In any case, to accumulate real balances Mr. Loser must give up corn consumption as in the lower diagram of figure 2.8.

Consider now the case in which all the agents in the economy lost money or paid some money as lump sum taxes to the government. In this case, they will all try to move to point B by selling corn for money. Since the money supply is constant, this leads only to a decline in the price level. A new equilibrium is achieved with the same level of real balances and the same level of corn consumption. Figure 2.9 illustrates this case: If everyone tries to accumulate real balances and move to point B they will actually move along the 45 degrees line and reach point E.

Thus, to the individual, there is a cost for accumulating real balances: he must give up corn consumption. To society, there is no such cost.

For an additional illustration, consider the trade-off between current corn consumption Y_t and permanent real balances m_p. An agent who starts from a smooth consumption path which is characterized by the level of real balances m, can cut his corn consumption and increase his

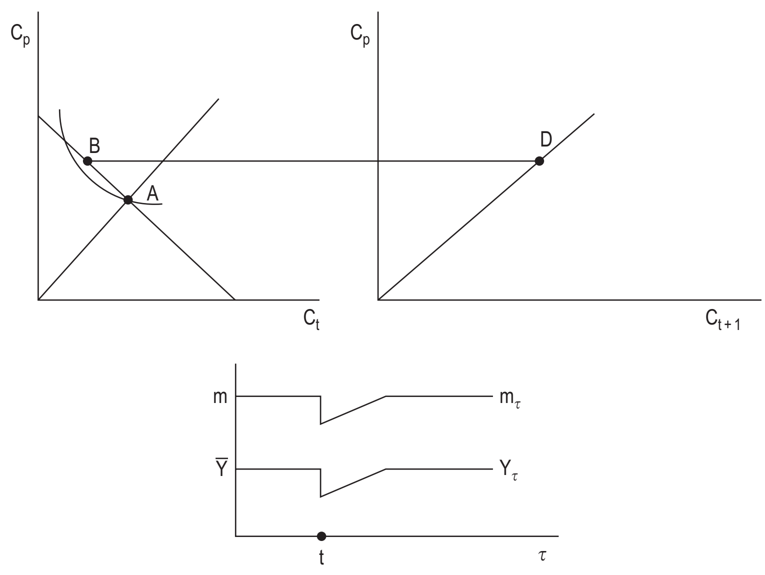

Figure 2.8 The cost of accumulating real balances from the individual's point of view

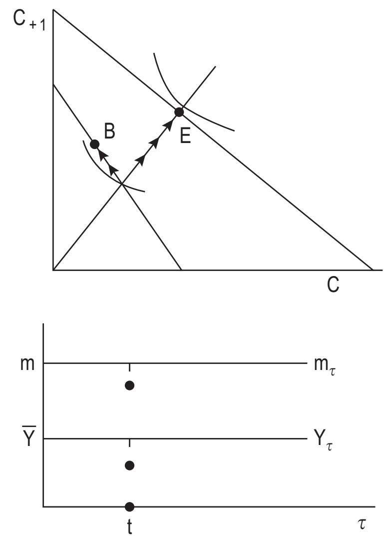

Figure 2.9 The social point of view

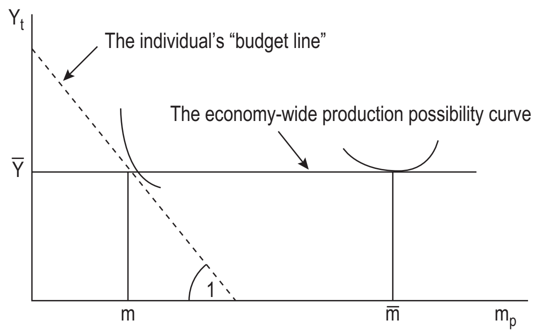

Figure 2.10 The difference between the individual and social point of view

real balances permanently as described by the deviation (2.20). From the individual point of view the trade is one for one: a unit of corn buys a unit of permanent real balances, as in the broken line of figure 2.10. From the point of view of the economy as a whole, increasing real balances is costless. The economy "production possibility curve" is the horizontal solid line in figure 2.10.

The slope of the indifference curves in the (Y_t, m_p) plane at the point of a smooth consumption path characterized by m is $f'(m)/\rho$.[6] If $m_{t-1} = m$ characterizes an equilibrium, then the indifference curve must be tangent to the budget line:

$$f'(m)/\rho = 1. \tag{2.26}$$

This is another way of deriving (2.19).

Since $f'(m)$ declines with m, the slope of the indifference curve declines with m and reaches zero at $\bar{m}$. We can therefore improve the welfare of all agents by increasing the level of real balances along the economy "production possibility curve" until we attain the maximum at the point $(\bar{Y}, \bar{m})$.

2.5 ADMINISTRATIVE WAYS OF GETTING TO THE OPTIMUM

We have shown that a competitive equilibrium with stable money supply is not Pareto efficient. This is surprising because the standard reasons for the failure of the first welfare theorem are absent: There are no external effects, no distortive taxes and no monopoly power. Nevertheless, the reason for the inefficiency is the standard reason: A divergence between the individual and the social cost.

What can we do to improve matters? It may be possible to require by law that on average agents hold $\bar{m}$ units of real balances. If this law is enforced, the price level will decline to $\bar{P}$ where $M/\bar{P} = \bar{m}$.

Alternatively, when the amount of money held by each individual is observable and lump sum taxes are possible, the government may pay interest on the holding of money. Under this subsidy scheme, an agent who increases his holding of money permanently by giving up current corn consumption, will get interest in addition to the increase in the flow of liquidity services. The slope of the budget line in the (C_t, C_p) plane is now: $f'(m) + r$, and the new equilibrium level of real balances must satisfy:

$$f'(m) + r = \rho. \tag{2.27}$$

When the policy-maker chooses $r = \rho$, $\bar{m}$ solves (2.27) and welfare is maximized.

We now turn to a solution of the efficiency problem that does not require information about the amount of money held by each individual. We start by discussing the effects of changes in the money supply.

2.6 ONCE AND FOR ALL CHANGES IN M

There are two distinct thought experiments: a once-and-for-all change in M and a change in the rate of change of M. A once-and-for-all change in M leads to an equi-proportional change in P. To see this point, let P' solve the equilibrium condition:

$$f'(M/P) = \rho. \tag{2.28}$$

Assume now that the government prints an additional M dollar per capita and transfers it to the consumers as lump sums. At the initial equilibrium position, individuals were holding the desired level of real balances. Now after they have got the transfer payment from the government, they hold more money than they want to. They will therefore try to exchange money for corn. However, since the supply of money is constant, it is not possible for all agents to do so. The attempt to exchange money for corn will only increase the price of corn. This increase in the price level will lower real balances until the quantity of real balances is equal

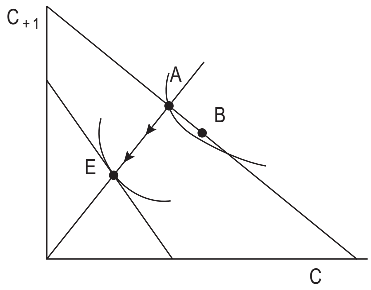

Figure 2.11 A once and for all change in the money supply

to the desired one – that is, the quantity before the transfer. The process does not necessarily require time: we may jump to the new equilibrium price level instantaneously.

In terms of our diagram, after the transfer payment all agents find themselves in a point like A in figure 2.11. They all try to move to a point like B. Since it is not possible for all agents to move to B they end up moving to the previous equilibrium: point E. The mathematics of this argument is quite simple. If P' solves the equilibrium condition $f'(M/P) = \rho$ then $2P'$ solves $f'(2M/2P) = \rho$.

What will happen if the government announces a policy of doubling the money supply each month? We will see that this thought experiment is not trivial. We start from some technical aspects of this question.

2.7 CHANGE IN THE RATE OF MONEY SUPPLY CHANGE: TECHNICAL ASPECTS

We start from the continuous time case which is a useful approximation for the discrete time case used in the theoretical analysis which follows.

The government changes the money supply at the constant rate:

$$d\ln(M)/dt = (dM/dt)/M = \mu, \tag{2.29}$$

where M denotes the (per agent) money supply and μ is the rate of change in the money supply. The first equality in (2.29) says that the time derivative of the logarithm of a variable is equal to the rate of change in the variable.[7] For example, in figure 2.12 the government starts to increase the money supply at time t, at the rate of μ.

The rate of inflation is $\pi = d\ln P/dt$. The purchasing power of one dollar is $1/P$ and the rate of change in the purchasing power of a dollar is:

$$d\ln(1/P)/dt = -d\ln P/dt = -\pi. \tag{2.30}$$

Thus, real balances depreciate at the rate of π. When the representative agent holds m units of real balances he loses πm per unit of time.

The government changes the money supply by printing money and transferring it to the agents in the economy. The real value of the transfer payment is:

$$(dM/dt)/P = [(dM/dt)/M][M/P] = \mu m. \tag{2.31}$$

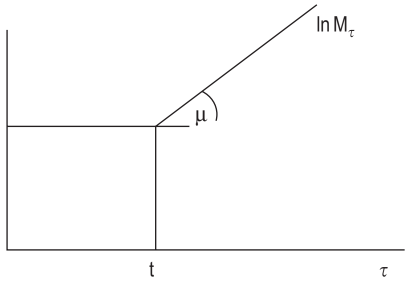

Figure 2.12 Consistent rate of growth in M

When the real value of the transfer payment is higher than the depreciation, real balances go up. The rate of change in real balances is:

$$d\ln(m)/dt = d\ln(M/P)/dt = d\ln M/dt - d\ln P/dt = \mu - \pi. \tag{2.32}$$

Thus real balances are accumulated (decumulated) if the rate of change in M is greater (lower) than the rate of change in P.

Discrete time: We use the same symbol π to denote the discrete rate of inflation: $\pi_t = (P_t - P_{t-1})/P_{t-1}$. The discrete rate of change in the purchasing power of money (the real rate of return on money) is:

$$r_{mt} = \{(1/P_t) - (1/P_{t-1})\}/(1/P_{t-1}) = [1/(1+\pi_t)] - 1. \tag{2.33}$$

Since Taylor's expansion leads to: $1/(1+\pi) = 1 - \pi + \pi^2 - \pi^3 + \cdots$, we can approximate r_m by $-\pi$ for small π.

The real value of the transfer payment is:

$$g_t = (M_t - M_{t-1})/P_t = m_t - m_{t-1}(P_{t-1}/P_t) = m_t - m_{t-1}(1 + r_{mt}). \tag{2.34}$$

It is also true that:

$$g_t = (M_t - M_{t-1})/P_t = \mu(M_{t-1}/M_t)m_t = \mu m_t/(1+\mu), \tag{2.35}$$

where here $\mu = (M_t - M_{t-1})/M_{t-1}$ is the discrete rate of change in the money supply. The discrete rate of change in real balances is:

$$(m_t - m_{t-1})/m_{t-1} = (1+\mu)/(1+\pi) - 1 = (1+\mu)(1+r_m) - 1. \tag{2.36}$$

Note that when $\mu = \pi$, the level of real balances is constant regardless of whether we use discrete or continuous time.

2.8 CHANGE IN THE RATE OF MONEY SUPPLY CHANGE: ECONOMICS

We use a discrete time analysis. At the beginning of each period, before the beginning of trade, the government transfers to the representative agent G_t dollars in a lump sum form. The agent forms expectations with respect to the sequences $\{G_t, P_t\}_{t=1}^{\infty}$ and chooses $\{M_t, Y_t\}_{t=1}^{\infty}$ to maximize:

$$\sum_{t=1}^{\infty} \beta^t U(C_t = Y_t + f[(M_{t-1} + G_t)/P_t])$$

$$\text{s.t.} \quad M_t + P_t Y_t = P_t \bar{Y} + M_{t-1} + G_t;$$

$$M_t \geq 0, \ \ Y_t \geq 0, \quad \text{and } M_0 \text{ is given.} \tag{2.37}$$

The sequences $\{G_t, M_t, Y_t, P_t\}_{t=1}^{\infty}$ is an *equilibrium* if (a) given the sequence $\{G_t, P_t\}_{t=1}^{\infty}$, the sequence $\{M_t, Y_t\}_{t=1}^{\infty}$ solves the representative consumer's problem (2.37); (b) markets are always cleared: $Y_t = \bar{Y}$ and $M_t = M_0(1 + \mu)^t$, for all t and (c) the transfer payment is equal to the change in the money supply: $G_t = M_t - M_{t-1}$.[8]

The sequence $\{G_t, M_t, Y_t, P_t\}_{t=1}^{\infty}$ is a steady-state equilibrium (SSE) if it is an equilibrium with the property that M_t/P_t is the same for all t.[9] I now turn to characterize the steady state equilibrium.

Smooth consumption path: It is useful to write the problem (2.37) in real terms. We use the discrete definition of π together with (2.33) to get:

$$\max \sum_{t=1}^{\infty} \beta^t U[Y_t + f(m_{tb})]$$

$$\text{s.t.}$$

$$m_{tb} = m_{t-1}(1 + r_{mt}) + g_t$$

$$m_t = \bar{Y} - Y_t + m_{t-1}(1 + r_{mt}) + g_t$$

$$m_t, \ Y_t \geq 0 \quad \text{and } m_0 \text{ is given.} \tag{2.38}$$

where $g_t = G_t/P_t$ is the real value of the transfer from the government.

It is assumed that the real rate of return on money is constant over time and is given by r_m. Furthermore, the consumer starts with the level of real balances $m_0 = m$ and the real value of the transfer payment is constant and is given by:[10]

$$g_t = G_t/P_t = -mr_m, \quad \text{for all t.} \tag{2.39}$$

Under (2.39), the smooth consumption path: $Y_t = \bar{Y}$ and $m_t = m$ is feasible. To see this claim we substitute (2.39) and $m_0 = m$ in the first period budget constraint of problem (2.38) to get: $m_1 = \bar{Y} - Y_1 + m(1 + r_m) - mr_m = \bar{Y} - Y_1 + m$. We now substitute $Y_1 = \bar{Y}$ to get $m_1 = m$. This is substituted in the second period budget constraint to get: $m_2 = \bar{Y} - Y_2 + m(1 + r_m) - mr_m = \bar{Y} - Y_1 + m = m$. We keep doing it to show that $m_t = m$ for all t is feasible under (2.39).

Note that if (2.39) is not satisfied then a smooth consumption path characterized by m is not feasible. Note also that for this smooth path (2.34) implies: $m_{tb} = m(1 + r_m) + g = m$ and the liquidity services per period are given by f(m).

Under what conditions will this smooth path be optimal? To answer this question, we consider the following deviation:

$$Y_t = \bar{Y} - x; \quad Y_{t+1} = \bar{Y} + x(1 + r_m);$$
$$Y_\tau = \bar{Y} \quad \text{for } \tau < t \text{ and for } \tau > t + 1;$$
$$m_t = m + x \quad \text{and} \quad m_\tau = m \qquad \text{for } \tau \neq t. \tag{2.40}$$

According to the proposed deviation (2.40), we cut corn consumption at t by x units and use the accumulated real balances to increase corn consumption at $t + 1$. The change in consumption from the smooth path $C = \bar{Y} + f(m)$ is:

$$\Delta C_t = C_t - C = -x;$$
$$\Delta C_{t+1} = C_{t+1} - C = x(1 + r_m) + f[m + x(1 + r_m)] - f(m);$$
$$\Delta C_\tau = 0 \quad \text{for } \tau < t \text{ and for } \tau > t + 1. \tag{2.41}$$

And therefore:[11]

$$-\Delta C_{t+1}/\Delta C_t = (1 + r_m)[1 + f'(m)]. \tag{2.42}$$

If the smooth path is optimal, it must be the case that the slope of the indifference curve (2.18) at the point $C_t = C_{t+1}$, must equal the slope of the budget line (2.42). This leads to:

$$1 + \rho = (1 + r_m)[1 + f'(m)]. \tag{2.43}$$

When $f'(m)$ and r_m are small, (2.43) can be approximated by:

$$f'(m) \approx \rho - r_m \approx \rho + \pi. \tag{2.44}$$

To derive the first order condition (2.43) in an alternative way, consider another possible deviation from the smooth path:

$$Y_t = \bar{Y} - x;$$
$$Y_\tau = \bar{Y} + xr_m, \quad \text{for } \tau > t;$$
$$Y_\tau = \bar{Y} \quad \text{for } \tau < t \quad \text{and}$$
$$m_\tau = m + x \quad \text{for } \tau \geq t; \qquad m_\tau = m \quad \text{for } \tau < t. \tag{2.45}$$

According to the proposed deviation in (2.45) the representative agent cuts his consumption of corn at time t by x units and uses it to increase the level of his real balances permanently. To maintain his real balances at the level $m + x$, he must increase his corn consumption permanently by xr_m. When r_m is negative this means that he must cut his corn consumption to cover the depreciation induced by inflation.

Under (2.45) $\Delta C_t = -x$ and the representative consumer's permanent consumption increases by:

$$\Delta C_p \approx f'(m)x + xr_m \approx f'(m)x - \pi x. \tag{2.46}$$

Therefore,

$$-\Delta C_p/\Delta C_t \approx f'(m) - \pi. \tag{2.47}$$

The slope of the indifference curve (2.23) at the point $C_p = C_t$ is ρ. If the smooth path is optimal, this must equal the slope of the budget line (2.47). Thus, $\rho \approx f'(m) - \pi$.

2.9 STEADY-STATE EQUILIBRIUM (SSE)

It is assumed that the supply of money changes at a constant rate μ. Thus,

$$G_t = (M_t - M_{t-1}) = \mu M_{t-1}. \tag{2.48}$$

The real value of the transfer payment is given by (2.34) and (2.35) which are repeated here for convenience:

$$g_t = G_t/P_t = m_t - m_{t-1}(1 + r_{mt}) = \mu m_t/(1 + \mu). \tag{2.49}$$

Substituting the steady state requirement $m_t = m$ in (2.49) leads to: $g = -r_m m = \mu m/(1+\mu)$ and $1+r_m = 1/(1+\mu)$. Since $g = -r_m m$ the transfer payment covers the depreciation (appreciation) of real balances due to inflation (deflation) and a smooth consumption path characterized by m is feasible.

Since in the steady state $1 + r_m = 1/(1 + \mu)$ we can write the first order condition (2.43) as:

$$(1 + \rho)(1 + \mu) = 1 + f'(m). \tag{2.50}$$

When μ and ρ are small (2.50) can be approximated by:

$$f'(m) \approx \mu + \rho. \tag{2.51}$$

Figure 2.5 can be used to show that when μ goes up the SSE level of real balances goes down and welfare goes down.

2.10 TRANSITION FROM ONE STEADY STATE TO ANOTHER

We consider an economy that before time t was in a SSE with a rate of change in the money supply $\mu = 0$. At time t the government announces a new rate of change $\mu = 0.1$. We assume that at $t' > t$ the economy will reach a new SSE with a new inflation rate $\pi = 0.1$. Can we say anything about the rate of inflation during the transition period (the period from t to t')?

For the sake of concreteness, we assume that in the initial steady state (before time t): $m = 10$, $M = 50$ and $P = 5$. In the new SSE $\mu = \pi = 0.1$ and therefore the new SSE level of real balances is lower and is given by $m = 5$.

We start with the case in which the economy reaches the new steady state equilibrium immediately at time $t\,(=t')$. At time t only the rate of change in the money supply has changed: the level of the money supply is still 50. Therefore to get to the lower level of real balances, the price level must jump to $P = 10$. The price level must then increase at the rate of $\pi = 0.1$ to maintain the desired level of 5 units of real balances. Figure 2.13 illustrates the instantaneous adjustment to the new steady state.

It is possible that because of long term contracts, price rigidities and imperfect information it will take time until we get to the new SSE. We now consider this case in which $t' > t$. The money supply at the end of the transition period satisfies: $\ln M_{t'} = \ln 50 + 0.1(t' - t)$. In order

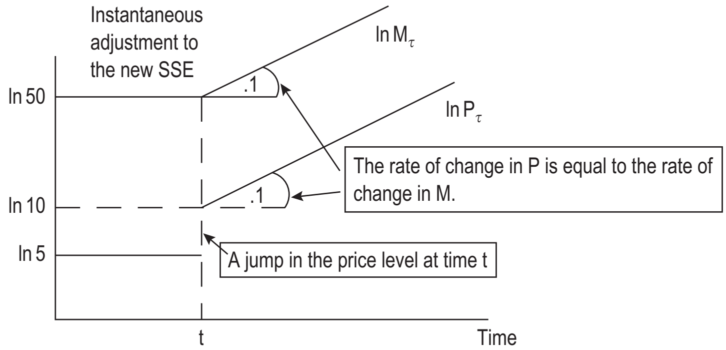

Figure 2.13 Instantaneous adjustment

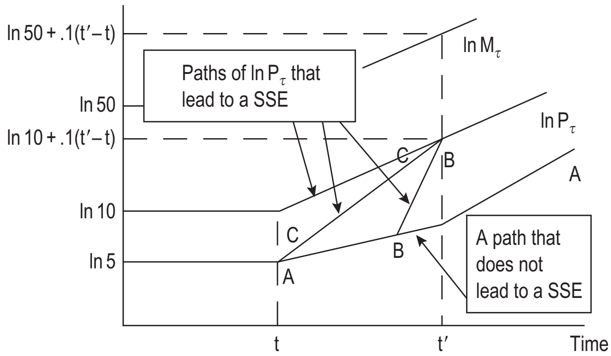

Figure 2.14 Slow adjustment

to get real balances at the desired level of 5, the price level at the end of the transition period satisfies: $\ln P_{t'} = \ln 10 + 0.1(t' - t)$. This price level at time t' can be reached if we follow a path like CC in figure 2.14. In this path, the rate of inflation during the transition period (the slope of the $\ln P_\tau$ line) is always greater than 0.1. We can also reach the SSE price level if we follow a path like ABB in which initially the rate of inflation is less than the steady state rate. We cannot reach the SSE price level if we follow a path like AA in which the rate of inflation is less than the SSE rate during the entire transition period.

It follows that we must have $\pi > 0.1$ at least during part of the transition period. This overshooting result can be explained as a consequence of the technical relationship (2.32). Since in the new steady state agents want to hold less real balances, they must decumulate real balances during the transition period. Since the rate of change in m is $\mu - \pi$, this can be done only if $\pi > \mu$ at least during part of the transition period.

Figure 2.15 illustrates the possible rates of inflation during the transition period. The path that is labeled "not possible" is the slope of the AA path in figure 2.15. (In this path the rate of inflation is always less than 0.1.) The two paths that are labeled "possible" are paths in which the rate of inflation is higher than 0.1 at least during part of the transition period.

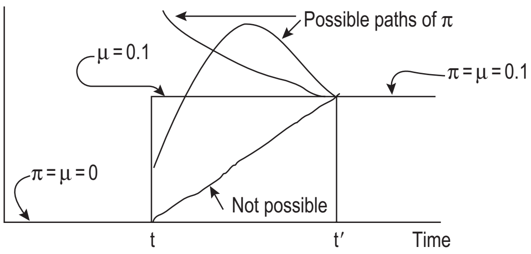

Figure 2.15 Possible inflation paths

(The instantaneous adjustment case cannot be described in this graph because it requires an infinite rate of inflation at time t.)

The "overshooting" result is relatively "model free" because it does not require a complete understanding of the transition period (the modeling of the various frictions which lead to a transition period).

2.11 REGIME CHANGES

In the previous analysis of a regime change (change in μ) the consumer was completely surprised by the announced change in policy. This is problematic because an agent with rational expectations assigns the true probability for all possible events. If the true probability of a regime change is zero (or close to zero) then regime changes are rare and do not deserve our attention.

A more satisfactory treatment would therefore assume that the agent takes into account the possibility of a regime change before it actually occurs. One way of doing it is to incorporate the probability of a regime change in the agent's maximization problem. Another approach may assume that the agent knew from the "beginning of time" that a regime change would occur at time t. Here I consider the second simpler case.

When agents expect a regime change at time t, the paths in figure 2.15, which assume zero rate of inflation until time t are no longer equilibrium paths. To see this point, note that if agents expect that the price level will go up at time t they will want to reduce their holding of real balances at time $t - 1$. Therefore the price level will go up at time $t - 1$, and by the same logic at time $t - j$, for $j = 1, 2, 3 \ldots$, as illustrated by figure 2.16.[12] Here the "transition period" starts before time t and as before, at least during part of the "transition period" π must be greater than μ.

Deflation

Deflation occurs when the government takes money from the public (by lump sum taxes) and burns it. Figures 2.17 and 2.18 describe the case in which at time t the government starts to reduce the money supply at the rate of $\mu = -0.1$. Here the level of real balances is higher in the new equilibrium and therefore $\mu - \pi$ must be positive at least during part of the transition

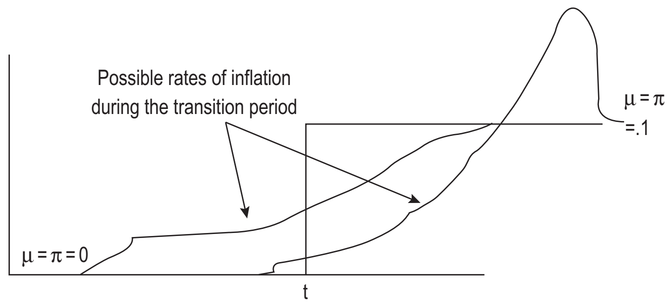

Figure 2.16 The change in policy is known in advance

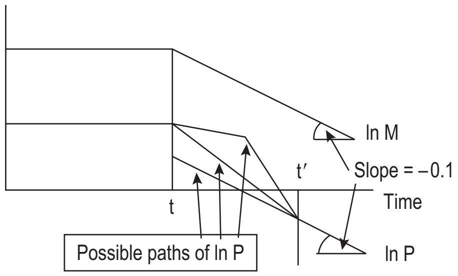

Figure 2.17 Deflation (logs of levels)

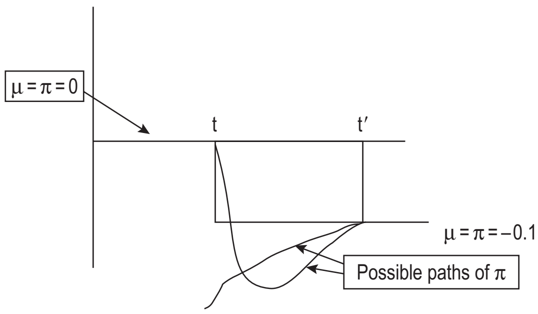

Figure 2.18 Deflation (rates of change)

period. There is also "overshooting" here in the sense that during the transition period the rate of inflation goes down below the steady-state level.

The optimum rate of inflation

The SSE condition (2.50) is: $(1 + \rho)(1 + \mu) = 1 + f'(m)$. At the social optimum f( ) is maximized and therefore: $f' = 0$. This can be achieved by setting $\mu = (1 + \rho)^{-1} - 1 \approx -\rho$. Thus, at the social optimum real balances appreciate approximately at the rate of ρ.

2.12 INTRODUCING PHYSICAL CAPITAL AND BONDS

We now add the option to sow corn and refer to corn in the soil as capital. The harvest at time $t+1$ is given by $R(k_t)$ units of corn, where k_t denotes the amount of corn sown at time t. The marginal gross rate of return on capital is: $R'(k_t) = 1 + r(k_t)$, where $r(k_t)$ is the net rate of return. It is assumed that $r'(k) < 0$ for all $k \geq 0$ and that $r'(0) = \infty$, $r'(\infty) = 0$ as in figure 2.19.

We also add riskless private nominal bonds. We use B_t to denote the nominal quantity of bonds held by the representative consumer at time t and i_t to denote the nominal rate of return on bonds. The consumer's budget set is:

$$P_tY_t + P_tk_t + B_t + M_t = P_t\bar{Y} + P_tR(k_{t-1}) + B_{t-1}(1+i_t) + M_{t-1} + G_t; \tag{2.52}$$

$$k_t, M_t \geq 0;$$

$$k_0, M_0 \text{ and } B_0 = 0 \text{ are given} \quad \text{and} \quad \lim_{t\to\infty} B_t = 0. \tag{2.53}$$

Constraint (2.52) says that the dollar value of corn consumption and the end of period portfolio must equal the available beginning of period dollar resources (the dollar value of the endowment plus the beginning of period portfolio plus the transfer payment from the government). Constraint (2.53) specifies non-negativity constraints, initial conditions and the requirement that the consumer cannot accumulate an infinite amount of debt.

Given the sequence $\{P_t, i_t, G_t\}_{t=1}^{\infty}$, the consumer chooses $\{M_t, B_t, k_t, Y_t\}_{t=1}^{\infty}$ to maximize

$$\sum_{t=1}^{\infty} \beta^t U\{C_t = Y_t + f[(M_{t-1} + G_t)/P_t]\}$$

$$\text{subject to:} \quad (2.52) \text{ and } (2.53). \tag{2.54}$$

The sequence $\{M_t, G_t, k_t, B_t, Y_t, i_t, P_t\}_{t=1}^{\infty}$ is an equilibrium if: (a) given $\{i_t, P_t\}$ and $G_t = M_t - M_{t-1}$, the sequence $\{M_t, k_t, B_t, Y_t\}$ solves the representative consumer's problem (2.54); and (b) markets are cleared:

$$Y_t = \bar{Y} + R(k_{t-1}) - k_t, M_t = M_0(1+\mu)^t \quad \text{and} \quad B_t = 0 \quad \text{for all } t > 0.$$

A steady-state equilibrium sequence is an equilibrium sequence for which M_t/P_t and k_t do not change over time.

Steady-state equilibrium: As before, we express (2.52) in real terms by dividing both sides by P_t and use (2.33). This leads to:

$$Y_t + k_t + b_t + m_t = \bar{Y} + b_{t-1}(1 + r_{bt}) + m_{t-1}(1 + r_{mt}) + R(k_{t-1}) + g_t \tag{2.55}$$

where lower case letters denote real magnitudes and $(1 + r_{bt}) = (1 + i_t)(1 + r_{mt})$ is the gross real return on bonds.[13]

The nominal interest rate and the inflation rate do not change over time and $g = -r_m m$ covers the depreciation of m units of real balances. Under these assumptions we can write (2.55) as:[14]

$$Y_t + k_t + b_t + m_t = \bar{Y} + b_{t-1}(1 + r_b) + m_{t-1}(1 + r_m) + R(k_{t-1}) - mr_m. \tag{2.56}$$

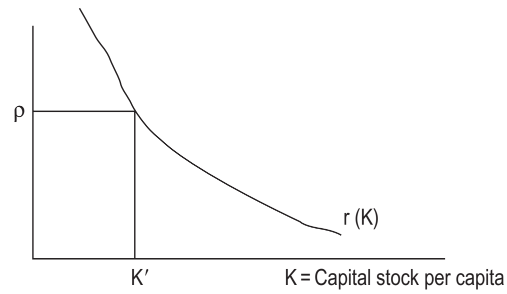

Figure 2.19 The choice of capital

The consumer starts with m units of real balances, k units of capital and no private bonds. If he does not change the amounts of assets held he will consume the smooth path:

$$Y_t = Y = \bar{Y} + R(k) - k; \quad m_t = m, \ k_t = k, \quad b_t = 0 \quad \text{for all t.}$$

To check whether or not this smooth path is optimal, we consider three possible deviations. All of the proposed deviations assume that corn consumption is cut by x units at time t and then invested either in money (2.57) or in capital (2.58) or in bonds (2.59):

$$\begin{gathered} Y_t = Y - x; \quad Y_{t+1} = Y + x(1 + r_m); \\ Y_\tau = Y \quad \text{for } \tau < t \text{ and for } \tau > t+1; \\ m_t = m + x \text{ and } m_\tau = m \quad \text{for } \tau \neq t; \qquad k_\tau = k, \ b_\tau = 0 \quad \text{for all } \tau. \end{gathered} \tag{2.57}$$

$$\begin{gathered} Y_t = Y - x; \quad Y_{t+1} = Y + x(1 + r); \\ Y_\tau = Y \quad \text{for } \tau < t \text{ and for } \tau > t+1; \\ k_t = k + x \text{ and } k_\tau = k \quad \text{for } \tau \neq t; \qquad m_\tau = m, \ b_\tau = 0 \quad \text{for all } \tau. \end{gathered} \tag{2.58}$$

$$\begin{gathered} Y_t = Y - x; \quad Y_{t+1} = Y + x(1 + r_m)(1 + i); \\ Y_\tau = Y \quad \text{for } \tau < t \text{ and for } \tau > t+1; \\ b_t = x \text{ and } b_\tau = 0 \quad \text{for } \tau \neq t; \qquad m_\tau = m, \ k_\tau = k \quad \text{for all } \tau. \end{gathered} \tag{2.59}$$

The slope of the budget line if savings are held as real balances is (2.42) which can be approximated by:

$$-\Delta C_{t+1}/\Delta C_t \approx 1 + f'(m) - \pi. \tag{2.60}$$

The slope of the budget line if savings are invested in physical capital is:

$$-\Delta C_{t+1}/\Delta C_t = 1 + r. \tag{2.61}$$

And the slope of the budget line if savings are invested in bonds is (approximately):

$$-\Delta C_{t+1}/\Delta C_t = 1 + r_b \approx 1 + i - \pi. \tag{2.62}$$

If the smooth consumption path characterized by m is optimal, it must be the case that the slope of the indifference curve (2.18) at the point $C_t = C_{t+1}$, must equal the slope of the budget line. Using the standard approximation we get: $1 + \rho = 1 + f'(m) - \pi = 1 + r(k) = 1 + i - \pi$; or

$$\rho = f'(m) - \pi = r(k) = i - \pi. \tag{2.63}$$

Without the approximation the first order conditions are:

$$1 + \rho = (1 + r_m)[1 + f'(m)] = (1 + r_m)(1 + i) = 1 + r(k). \tag{2.64}$$

Note that the first order condition that determines real balances can be written as:

$$f'(m) = i. \tag{2.65}$$

We may therefore think of $f'(m)$ as a standard demand for money function.

In a SSE $\mu = \pi$ and we can therefore express the SSE condition by substituting $\mu = \pi$ in (2.64) and (2.63). This leads to:

$$\begin{gathered} 1 + \rho = [1 + f'(m)]/(1 + \mu) = (1 + i)/(1 + \mu) = 1 + r(k) \\ \text{or, using the standard approximation,} \\ \rho = f'(m) - \mu = r(k) = i - \mu. \end{gathered} \tag{2.66}$$

When μ goes up the SSE level of real balances goes down (see figure 2.5) but the SSE level of capital does not depend on μ. It is determined by the condition: $\rho = r(k)$.

As before we get a simple relationship between the rate of change in the money supply and welfare: When $\mu \geq -\rho$ goes up, consumption goes down and welfare goes down. At the optimum, $\mu = -\rho$ and $i = 0$.

2.13 THE GOLDEN RULE AND THE MODIFIED GOLDEN RULE

Corn consumption in the steady state is given by $Y = \bar{Y} + R(k) - k$, which is a constant plus the difference between the R(k) curve in figure 2.20 below and the 45° line. This difference is maximized at $\bar{k}$ where $R'(\bar{k}) = 1$. This is the golden rule. However the optimal steady-state level of capital is at $\hat{k}$ which satisfies the first order condition: $R'(\hat{k}) = 1 + \rho$ (see [2.64]). This is the modified golden rule.

Note that unlike real balances there is no difference between the social and the private point of view when we look at the accumulation of physical capital: To accumulate capital we must give up corn consumption. Therefore, at the social optimum we have satiation in real balances but no satiation in physical capital.

To elaborate, we consider the problem of a social planner who takes the level of real balances m as given. (As we have seen the level of real balances can be chosen optimally by an appropriate choice of μ and therefore we abstract from this choice problem here.) The

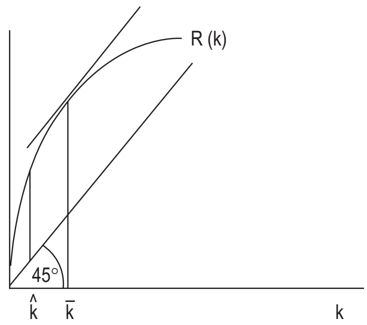

Figure 2.20 The maximum consumption level of capital and the optimum level

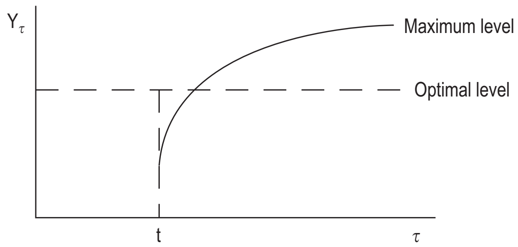

Figure 2.21 The effect of a subsidy to capital on consumption

planner's problem is:

$$\max_{k_t} \sum_{t=1}^{\infty} \beta^t U(C_t)$$

$$\text{s.t.}$$

$$C_t = Y_t + f(m)$$

$$Y_t + k_t = \bar{Y} + R(k_{t-1})$$

$$Y_{t'},\ k_t \geq 0 \quad \text{and} \quad k_0 \text{ is given.} \tag{2.67}$$

Under what conditions will the planner choose a smooth path: $k_t = k_0 = k$? To answer this question and find the first order condition we consider the following deviation: We cut consumption at t by x units, invest it in physical capital and increase corn consumption at $t+1$ by the change in the harvest $R(k+x) - R(k)$. This deviation is the same as (2.58) and the slope of the planner's budget line is given by (2.61). Equating the slope of the budget line to the slope of the indifference curve yields the first order condition $\rho = r(k)$, which is exactly the same as the relevant part of (2.64).

This means that if a planner starts with the steady state level of capital, $\hat{k}$ he should not attempt to change it. To appreciate the implication of this result consider a subsidy scheme, financed by a lump sum tax, in which the planner pays s units per unit of capital. The first order condition for the representative private agent is now: $\rho = r(k) + s$. By choosing $s = \rho$ the central planner can induce agents to accumulate capital until they reach $\bar{k}$ and at this point $r(k) = 0$. The consumption of corn path in this case may be described by the solid line in figure 2.21.

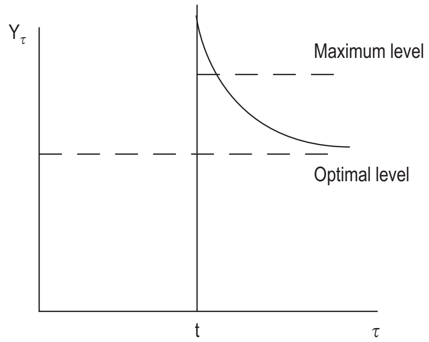

Figure 2.22 Optimal consumption when capital is above the optimal level

The above subsidy scheme reduces the welfare of the representative consumer, in spite of the fact that eventually it leads to more consumption. The reason for the reduction in welfare is that the cost of the initial reduction in consumption is too high relative to the future benefits.

Suppose now that in the past, there was a subsidy scheme of the type just described and as a result the level of capital per agent is $\bar{k}$ at time t. Should the central planner continue with the subsidy scheme? The answer is in the negative. The central planner should abolish it and as a result the private agents will choose to "eat capital" until they reach the optimal level as in the solid line of figure 2.22. This is an immediate corollary of the result that $\hat{k}$ is optimal from both the consumer's point of view and the planner's point of view.

PROBLEMS

In questions 1–5 the money supply (M) does not change over time.

1 At time t a single individual experiences a change in taste and his time preference parameter ρ goes up. What will happen to his level of corn consumption (Y_t) and total consumption (C_t)?

2 At time t a single individual experiences a change in technology: m units of real balances yield after the change $\alpha + f(m)$ units of consumption instead of the previous yield of f(m) units, where $\alpha > 0$ is a constant. Does the technological change affect the individual level of current corn consumption (Y_t)?

3 The same as 2 but now the change is to $\alpha f(m)$.

4 Answer 1–3 under the assumption that all individuals in the society experience the same changes.

5 Comparing your answer 4 to 1–3 what can you say about the cost of accumulating real balances? Distinguish between the cost to an individual and the cost to society.

6 The government increases the rate of change in the money supply and as a result the economy jumps immediately (without a transition period) from one steady state to another steady state. What will happen to the price level and the rate of inflation at the jump to the new steady state?

7 Assume that the economy was in a steady state equilibrium and at time t all individuals experience a change in taste: ρ goes up. What will be the rate of inflation in the new steady state? What will happen to the rate of inflation in the transition to the new steady state?

8 Assume that liquidity services are described by the function $f(m) = m^{\alpha}$. What do we assume about α? What is the demand for money in a steady-state equilibrium as a function of π and ρ? (Develop a logarithmic expression).

9 In the text we wrote the budget constraint in real terms as: $m_t = \bar{Y} - Y_t + m_{t-1}(1 + r_m) + g_t$, where g_t was exogenous from the consumer's point of view (it did not depend on any of the choices he made). Assume now that the transfer payment is given in proportion to the amount of money held by each individual so that $g_t = -r_m m_{t-1}$. What is the effect of changing μ on the steady-state equilibrium level of real balances in this case?

PROBLEMS WITH ANSWERS

10 The infinite horizon problem described by (2.9)–(2.12) in the text can be written as:

$$\max_{Y_t} \sum_{t=1}^{\infty} \beta^t U[Y_t + f(m_{t-1})]$$
$$\text{s.t.} \quad Y_t + m_t = \bar{Y} + m_{t-1}$$
$$m_t \geq 0, \quad \text{and} \quad m_0 \text{ is given.}$$

Find the first order conditions for this problem using the Lagrangian method.

Answer

The Lagrangian is:

$$L = \sum_{t=1}^{\infty} \beta^t \{U[Y_t + f(m_{t-1})] + \lambda_t(\bar{Y} - Y_t + m_{t-1} - m_t)\}$$

First order conditions:

(a) $\partial L/\partial m_t = \beta^{t+1} U'(C_{t+1}) f'(m_t) + \beta^{t+1}\lambda_{t+1} - \beta^t \lambda_t = 0$

(b) $\partial L/\partial Y_t = \beta^t U'(C_t) - \beta^t \lambda_t = 0$

From (b) we get: $U'(C_t) = \lambda_t$. Substituting this in (a) yields:

(c) $1 + f'(m_t) = U'(C_t)/\beta U'(C_{t+1})$.

In the steady-state: $1 + f'(m) = 1 + \rho$. This is the first order condition (2.19) derived in the text.

11 Assume two assets: Real balances and physical capital. As in chapter 2, consumption is defined by: $C_t = Y_t + f(m_{tb})$, where Y_t is corn consumption and m_{tb} is the beginning of period real balances. But unlike the specification in chapter 2, the agent's budget constraint (asset evolution equation) is:

$$k_t + m_t = \bar{Y} - Y_t + m_{t-1}(1 + r_m) + R(k_{t-1})\theta(m_{tb}) + g_t,$$

where R( ) is a monotonic function with the standard properties and the function θ( ) has the same properties as the function f( ): it is strictly concave and increasing initially. Thus we assume that in addition to yielding services to the shopper, real balances also make capital more productive.

We assume $g_t = -mr_m$ for all t.

(a) What is the consumption of corn along a smooth consumption path characterized by m?
(b) Consider the following deviation from the smooth consumption path. The consumer plans to reduce current corn consumption by x units, increase his real balances by that amount ($\Delta m_t = -\Delta Y_t = -\Delta C_t = x$) and spend it in the next period. What will be the change (ΔC_{t+1}) in $t + 1$ consumption relative to the smooth consumption path?
(c) Answer (b) under the assumption that instead of accumulating real balances, the agent increases the stock of capital at time t by x units.
(d) What is the slope of the budget constraint in the (C_t, C_{t+1}) plane. Give two expressions: One when you accumulate real balances as in (b) and one when you increase the stock of capital as in (c). Assume now that x is small.
(e) What are the two first order conditions?
(f) What happens to the steady-state level of capital when the steady state rate of inflation goes up? Assume that the steady-state real balances go down when inflation goes up.
(g) Derive the first order conditions using the Lagrangian method.

Answer

(a) A smooth consumption path that is characterized by m is:

$$m_t = m_0 = m; \quad k_t = k; \quad Y_t = \bar{Y} + R(k)\theta(m) - k \quad \text{for all t.}$$

(b) $$\Delta C_{t+1} = x(1 + r_m) + f[m + x(1 + r_m)] - f(m) + R(k)\{\theta[m + x(1 + r_m)] - \theta(m)\}$$

(c) $\Delta C_{t+1} = \theta(m)[R(k + x) - R(k)]$.

(d) $$-\Delta C_{t+1}/\Delta C_t = (1 + r_m) + (1 + r_m)f'(m) + (1 + r_m)R(k)\theta'(m) = (1 + r_m)[1 + f'(m) + R(k)\theta'(m)]$$

$$-\Delta C_{t+1}/\Delta C_t = \theta(m)R'(k).$$

(e) $(1 + r_m)[1 + f'(m) + R(k)\theta'(m)] = \theta(m)R'(k) = 1 + \rho$.

(f) Since we assume that the steady state real balances goes down, the marginal product curve $R'(k)\theta(m)$ shifts to the left and the steady state level of capital is down. This is shown in figure 2.23 when comparing two steady states: The first with $i = 0.1$ and real balances $= m(0.1)$ and the second with $i = 0.2$ and real balances $= m(0.2)$.

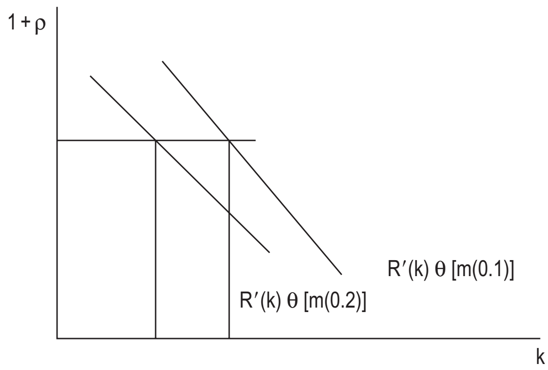

Figure 2.23 The effect of a change in the rate of inflation on the choice of capital

(g) *Solving by the lagrangian method*:

$$L = \sum_{t=1}^{\infty} \beta^t \{U(Y_t + f[m_{t-1}(1 + r_m) + g_t])$$

$$+ \lambda_t(\bar{Y} - Y_t + m_{t-1}(1 + r_m) + R(k_{t-1})\theta[m_{t-1}(1 + r_m) + g_t]$$

$$+ g_t - k_t - m_t)\}.$$

First order conditions:

(a) $\partial L/\partial m_t = \beta^{t+1}U'(C_{t+1})(1 + r_m)f'(m_{t+1b}) + \beta^{t+1}\lambda_{t+1}(1 + r_m)$
$\quad + \beta^{t+1}\lambda_{t+1}R(k_t)(1 + r_m)\theta'(m_{t+1b}) - \beta^t\lambda_t = 0$

(b) $\partial L/\partial k_t = \beta^{t+1}\lambda_{t+1}R'(k_t)\theta(m_{t+1b}) - \beta^t\lambda_t = 0$

(c) $\partial L/\partial Y_t = \beta^t U'(C_t) - \beta^t\lambda_t = 0$

From (c) we get: $U'(C_t) = \lambda_t$. Substituting this in (a) and (b) yields:

(d) $(1 + r_m)[1 + f'(m_{t+1b}) + R(k_t)\theta'(m_{t+1b})] = U'(C_t)/\beta U'(C_{t+1})$

(e) $R'(k_t)\theta(m_t + g_{t+1}) = U'(C_t)/\beta U'(C_{t+1})$.

In the steady state:

(f) $(1 + r_m)[1 + f'(m) + R(k_t)\theta'(m)] = R'(k_t)\theta(m) = 1 + \rho$.

APPENDIX 2A A DYNAMIC PROGRAMMING EXAMPLE

We now illustrate the use of dynamic programming by solving the infinite horizon problem (2.9)–(2.12) in section 2.2. We use two alternative ways of deriving the Bellman equation. One follows Stokey and Lucas with Prescott (1989) who starts with an infinite horizon problem. The other follows Sargent (1987) who starts from a finite horizon problem.

The infinite horizon problem (2.9)–(2.12) can be written as:

$$V_1(m_0) = \max_{Y_t} \sum_{t=1}^{\infty} \beta^{t-1} U[Y_t + f(m_{t-1})]$$

$$\text{s.t.}$$

$$Y_t + m_t = \bar{Y} + m_{t-1}$$

$$m_t \geq 0, \quad \text{and} \quad m_0 \text{ is given.} \tag{A2.1}$$

The value function $V_1(m_0)$ is the maximum utility the consumer can achieve at $t = 1$ if he starts with m_0 units of real balances. Suppose that after consuming in the first period the consumer is left with m_1 units of real balances. Then in the next period he will solve:

$$V_2(m_1) = \max_{Y_t} \sum_{t=2}^{\infty} \beta^{t-2} U[Y_t + f(m_{t-1})]$$

$$\text{s.t.}$$

$$Y_t + m_t = \bar{Y} + m_{t-1}$$

$$m_t \geq 0, \quad \text{and} \quad m_1 \text{ is given.} \tag{A2.2}$$

The value function $V_2(m_1)$ is the maximum utility that the consumer can achieve at $t = 2$ if he starts with m_1 units of real balances. Note that since the horizon is infinite, the problem at $t = 2$ looks the same as the problem at $t = 1$ and therefore $V_1(\ )$ and $V_2(\ )$ are the same. We use $V \equiv V_1 \equiv V_2$ to denote this value function.

We now write the problem (A2.1) as:

$$V(m_0) = \max_{Y_t} U[Y_1 + f(m_0)] + \beta \sum_{t=2}^{\infty} \beta^{t-2} U[Y_t + f(m_{t-1})]$$

$$\text{s.t.}$$

$$Y_t + m_t = \bar{Y} + m_{t-1}$$

$$m_t \geq 0, \quad \text{and} \quad m_0 \text{ is given.} \tag{A2.3}$$

This way of writing the problem separates the objective function into two components: The current utility component $(U[Y_1 + f(m_0)])$ and the future utility component $\left(\beta \sum_{t=2}^{\infty} \beta^{t-2} U[Y_t + f(m_{t-1})]\right)$. For any given choice Y_1 the level of the beginning of next period real balances is given by: $m_1 = \bar{Y} - Y_1 + m_0$. The maximum utility that the consumer can achieve at $t = 2$ with this predetermined level of real balances, is: $\beta V(\bar{Y} - Y_1 + m_0)$.

We can therefore write the problem (A2.3) as:

$$V(m_0) = \max_{Y_1} U[Y_1 + f(m_0)] + \beta V(\bar{Y} - Y_1 + m_0)$$
$$\text{s.t.} \quad \bar{Y} - Y_1 + m_0 \geq 0. \tag{A2.4}$$

Since the choice problem does not depend on the time it is made (again, because of the infinite horizon) we may omit the time index and write:

$$V(m) = \max_{Y} U[Y + f(m)] + \beta V(\bar{Y} - Y + m)$$
$$\text{s.t.} \quad \bar{Y} - Y + m \geq 0. \tag{A2.5}$$

This is a Bellman equation.

We now derive the first order condition for an interior solution to (A2.5) by taking a derivative with respect to Y and ignoring the non-negativity constraint. This leads to: $U'[Y + f(m)] - \beta V'(\bar{Y} - Y + m) = 0$. Using $C = Y + f(m)$ for current consumption and $m' = \bar{Y} - Y + m$ for the beginning of next period real balances we can write this first order condition as:

$$U'(C) = \beta V'(m'). \tag{A2.6}$$

To derive V' we use the envelope theorem which says that after solving the problem in (A2.5) and replacing the optimal values we can take a derivative from both sides of (A2.5) ignoring the max operator. See also Benveniste and Scheinkman (1979). This yields:

$$V'(m) = U'[Y + f(m)]f'(m) + \beta V'(\bar{Y} - Y + m) = U'(C)f'(m) + \beta V'(m'). \tag{A2.7}$$

Substituting (A2.6) in (A2.7) leads to:

$$V'(m) = U'(C)[1 + f'(m)]. \tag{A2.8}$$

The intuition is that at the optimum it does not matter what we do with an additional unit of the beginning of period real balances. In particular we may use it to increase current consumption. This yields $U'(C)$ utils. In addition the added unit increases liquidity services by f' yielding $U'(C)f'(m)$ utils.

Since (A2.8) holds for any m it also holds for the beginning of next period's real balances. We thus have: $V'(m') = U'(C')[1 + f'(m')]$ where prime is used to denote next period's magnitudes. Substituting this in (A2.6) leads to:

$$U'(C) = \beta[1 + f'(m')]U'(C'). \tag{A2.9}$$

Using time indices we can write (A2.9) as:

$$U'(C_t) = \beta[1 + f'(m_t)]U'(C_{t+1}). \tag{A2.10}$$

This first order condition (A2.10) must hold for any optimal consumption path. For a smooth path $C_t = C_{t+1}$ and we can write (A2.10) as: $\rho = f'(m)$. This is the first order condition (2.19) derived in the text.

Deriving the Bellman equation in an alternative way

We now consider a finite (T periods) horizon problem. We start with the last period problem, solve it as a function of the beginning of period real balances and use the solution to solve the two periods problem that the consumer faces at $T-2$. The two periods problem at $T-2$ is solved as a function of the beginning of period real balances and is used to solve the three periods problem at $T-3$. We keep doing this (going back in time) until we reach the first period where we solve the T periods horizon problem.

The consumer enters the last period with the predetermined level of m_{T-1} units of real balances. His utility in the last period of life is:

$$V_0(m_{T-1}) \equiv U[\bar{Y} + m_{T-1} + f(m_{T-1})], \tag{A2.11}$$

where here $V_0(m)$ is the maximum utility one can achieve in a one period problem when the predetermined level of real balances is m.

Note that:

$$V_0'(m_{T-1}) = U'(C_T)(1 + f'(m_{T-1})). \tag{A2.12}$$

The intuition is that an "additional unit" of real balances will increase corn consumption by one unit and liquidity services by: $f'(m_{T-1})$ units.

Next we consider the problem in period $T-1$. The consumer starts this period with the predetermined level of m_{T-2} units of real balances and solves:

$$V_1(m_{T-2}) = \max U[Y_{T-1} + f(m_{T-2})] + \beta V_0(m_{T-1})$$
$$\text{s.t.} \quad Y_{T-1} + m_{T-1} = \bar{Y} + m_{T-2} \quad \text{and non-negativity constraints.} \tag{A2.13}$$

Here $V_1(m)$ is the maximum utility achievable in a two periods problem when the predetermined level of real balances is m.

Using the constraint to substitute for m_{T-1} leads to:

$$\max U(Y_{T-1} + f(m_{T-2})) + \beta V_0(\bar{Y} + m_{T-2} - Y_{T-1}). \tag{A2.14}$$

Taking a derivative with respect to Y_{T-1} and equating it to zero, we obtain the first order condition:

$$U'(C_{T-1}) = \beta V_0'(m_{T-1}). \tag{A2.15}$$

Substituting (A2.12) into (A2.15) yields:

$$U'(C_{T-1}) = \beta U'(C_T)(1 + f'(m_{T-1})). \tag{A2.16}$$

This condition can be used to solve for the optimal Y_{T-1} and m_{T-1} as a function of m_{T-2}. To do that, we substitute $m_{T-1} = \bar{Y} + m_{T-2} - Y_{T-1}$ and $C_{T-1} = Y_{T-1} + f(m_{T-2})$ in (A2.16) to get:

$$\begin{aligned} &U'[Y_{T-1} + f(m_{T-2})] \\ &\quad = \beta U'[\bar{Y} + f(\bar{Y} + m_{T-2} - Y_{T-1})][1 + f'(\bar{Y} + m_{T-2} - Y_{T-1})]. \end{aligned} \tag{A2.17}$$

This is one equation with one unknown. We solve for $Y_{T-1}(m_{T-2})$ and then get $m_{T-1}(m_{T-2}) = \bar{Y} + m_{T-2} - Y_{T-1}(m_{T-2})$.

In general, in period t m_{t-1} is given and the consumer solves:

$$V_{T-t-1}(m_{t-1}) = \max U[Y_t + f(m_{t-1})] + \beta V_{T-t}(m_t)$$
$$\text{s.t.} \quad Y_t + m_t = \bar{Y} + m_{t-1} \quad \text{and non-negativity constraints.} \qquad \text{(A2.18)}$$

When T goes to infinity the function on both sides is the same and we get the Bellman equation (which defines the value function V):

$$V(m_{t-1}) = \max U[Y_t + f(m_{t-1})] + \beta V(m_t)$$
$$\text{s.t.} \quad Y_t + m_t = \bar{Y} + m_{t-1} \quad \text{and non-negativity constraints.} \qquad \text{(A2.19)}$$

NOTES

1 The function F( ) depends in general on what other people do and is therefore defined only in equilibrium.
2 See Kiyotaki and Wright (1993) for a formal search model.
3 Here the present value of utility is computed to date $t = 0$. For some purposes it is more convenient to discount to date $t = 1$ and write the utility function as $\sum_{t=1}^{\infty} \beta^{t-1} U(C_t)$. Since $\sum_{t=1}^{\infty} \beta^{t} U(C_t) = \beta \sum_{t=1}^{\infty} \beta^{t-1} U(C_t)$, they both describe the same preferences.
4 Strictly speaking, the slope of the "budget line" is $1 + f'(m)$ only in the neighborhood of the diagonal.
5 Friedman motivates the assumption that $f'(m) < 0$ for $m > \bar{m}$ by the need to employ guards for securing large quantity of real balances. This assumption is required only for the uniqueness of the social optimum.
6 This may be derived as follows. We take a full differential $\sum_{\tau=t}^{\infty} \beta^{\tau-t} U'(C)\, dC_\tau$, substitute $dC_t = dY_t$ and $dC_\tau = f'(m) dm_p$ for $\tau > t$ and equate to zero. This leads to: $-U'(C)\, dY_t = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} U'(C) f'(m)\, dm_p$ and to: $-dY_t/dm_p = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} f'(m) = f'(m)/\rho$.
7 The "outside" derivative $d \ln(M)/dt$ is $1/M$ and the "inside" derivative is dM/dt.
8 Thus, in equilibrium expectations about future transfer payments and prices are correct.
9 Alternatively we may define a SSE as an equilibrium with the property that the rate of inflation π does not change over time. In the present context the two definitions are equivalent.
10 For small π we can approximate (2.39) by $m\pi$.
11 This uses: $f'(m) = \lim_{x \to 0} \{f[m + x(1 + r_m)] - f(m)\}/[x(1 + r_m)]$.
12 Strictly speaking the rate of inflation should be positive from $t = -\infty$.
13 To check the interpretation of the last formula note that to invest one dollar in bonds the consumer must give up $1/P_t$ units of consumption. At the beginning of next period, the consumer will have $(1 + i)$ dollars which will buy him $(1 + i)/P_{t+1}$ units of consumption. Therefore the gross real return on bonds is: $1 + r_b = [(1 + i)/P_{t+1}]/(1/P_t) = (1 + i)(P_t/P_{t+1}) = (1 + i)(1 + r_m)$. For small π, we can use the approximation: $r_b = i - \pi$.
14 When π is small we can use the approximation:

$$Y_t + k_t + b_t + m_t = \bar{Y} + b_{t-1}(1 + i - \pi) + m_{t-1}(1 - \pi) + R(k_{t-1}) + m\pi.$$