This chapter is taken from the book:

"A Course in Monetary Economics: Sequential Trade, Money and Uncertainty"

By Benjamin Eden, Published by Blackwell, 2005.

PART III

An Introduction to Uncertain and Sequential Trade (UST)

In the Arrow–Debreu model reviewed in chapter 11 there is uncertainty about the future but no uncertainty about current demand conditions. Trade occurs before anything happens. The number of agents who participate in trade is known and we may assume that the price of all contingent commodities is known in advance to all participants.

The situation is different if demand conditions are not known before the beginning of actual trade. In this case the standard Walrasian model assumes an auctioneer who finds the market clearing prices by the following (tatonnement) process. He calls a vector of prices and asks agents to report their demand and supply for this price vector. He then checks whether markets are cleared. If not he tries another vector of prices and keeps doing it until he finds a vector of prices that clears all markets. Actual trade is prohibited until the market clearing price vector is found.

This standard formulation is problematic for three reasons. First, the description of the Walrasian auctioneer is not complete. Why does he provide the public service of finding the market clearing prices? What is his objective function? A second problem arises from the prohibition of trade: Trade is not allowed until the market clearing price vector is found.

Finally, and maybe most importantly, prices do not behave according to the standard Walrasian model. There is ample evidence against the "law of one price" and the effect of monetary shocks on prices occurs with a significant lag.

The new Keynesian (sticky price) models reviewed in chapter 8 provide an answer to the first problem. In these models agents, rather than the Walrasian auctioneer, make price choices. But new Keynesian models typically neglect the choice of quantities and typically assume that sellers satisfy demand at their preannounced prices. An attempt to relax the demand satisfying assumption was made in chapter 9 and proved to be rather difficult.

The uncertain and sequential trade (UST) model attempts to answer the second problem by allowing trade before the resolution of uncertainty about demand (and the market clearing price). Agents know in advance the prices in all potential markets, take these prices as given and make plans accordingly. In equilibrium the plans made by all agents are mutually consistent and can be executed. But unlike the Arrow–Debreu model, in the UST model there is uncertainty about the set of markets that will open (or be active).

It is also possible to think of the UST model as an answer to the first problem. As in the Arrow–Debreu model there is no need for an auctioneer who finds the market clearing price. We may simply assume that agents know the probability distribution of demand and the prices in all potential markets before the beginning of trade. We may also think of agents in the UST model as choosing price tags (not necessarily the same tags on all units).

But the major contribution of the UST model is in explaining observations which are regarded as "puzzles" from the point of view of the standard Walrasian model. We will apply the UST approach to explain the observed deviations from the law of one price, the real effects of money and the behavior of inventories. We will then turn to some policy questions. We start from a real version of the model and then turn to monetary versions.

CHAPTER 14

Real Models

UST models use ideas in Prescott (1975) and Butters (1977). Prescott considers an environment in which sellers set prices before they know how many buyers will eventually appear. He assumes that less expensive goods will be sold before more expensive ones and obtains an equilibrium trade-off between the price and the probability of making a sale. A similar trade-off arises in Butters (1977) in a model in which sellers send price offers to potential customers. In both models sellers commit to prices before the realization of demand. Prescott thinks of his example as one "which entails monopoly power on the part of sellers" (p. 1233).

In the UST approach taken by Eden (1990), trade is sequential and equilibrium distribution of prices is obtained even though sellers have no monopoly power and are allowed to change their prices during trade.

We now turn to the comparison of the UST model with the standard Walrasian model. It turns out that the main difference is in the time in which information about the realization of demand becomes public. In the UST model information about the realization of demand is being resolved sequentially during trade while in the standard model it is resolved before the beginning of trade.

14.1 AN EXAMPLE

To illustrate the difference between the two alternative spot market models we use the example in Eden and Griliches (1993) that builds on Hall (1988).

Restaurants in a certain location produce lunches. Fixed and variable labor are the only factors of production. Preparing a meal requires λ man-hours. Serving the meal requires ϕ man-hours. The wage rate is one dollar per hour.

The number of buyers that will arrive in the marketplace is uncertain: It may be N or $N + \Delta$ with equal probabilities of occurrence. Each buyer that arrives, is willing to pay up to θ dollars for a meal, where $\theta > \phi + 2\lambda$.

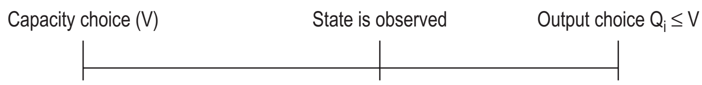

Figure 14.1 Sequence of events in the standard model

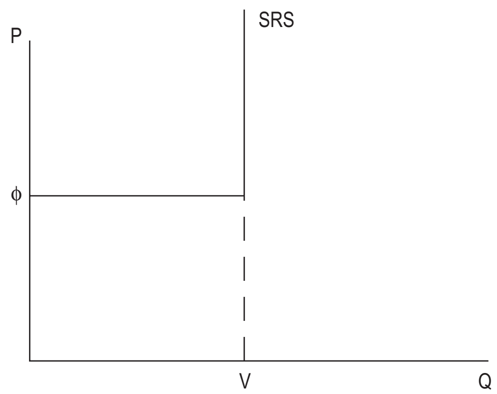

Figure 14.2 Short run supply

The standard model

There is a single price taking firm. It chooses capacity V (the number of prepared meals) on the basis of its expectations about the market-clearing price. Then buyers arrive and the market-clearing price is announced: P_1 if the demand is low (state 1) and P_2 if demand is high (state 2). The firm then chooses output (the number of served meals: $Q_i \le V$) and sells it at the market-clearing price. Figure 14.1 describes the sequence of events.

The firm's problem is to choose capacity, V, and output in state i, Q_i, to maximize expected profits:

$$\max_V \left\{ \frac{1}{2} \left[\sum_{i=1}^{2} \max_{Q_i} Q_i(P_i - \phi);\ \text{s.t. } Q_i \le V \right] - \lambda V \right\}. \tag{14.1}$$

Using the logic of dynamic programming we solve (14.1) backward starting at the stage in which capacity is given. The maximum expected variable profits that can be achieved with V units of capacity is:

$$F(V) = \frac{1}{2} \left\{ \sum_i \max_{Q_i} Q_i(P_i - \phi);\ \text{s.t. } Q_i \le V \right\}. \tag{14.2}$$

The first order conditions for these maximization problems are:

$$\begin{aligned} Q_i &= V && \text{when } P_i > \phi; \\ 0 \le Q_i &\le V && \text{when } P_i = \phi; \\ Q_i &= 0 && \text{when } P_i < \phi. \end{aligned} \tag{14.3}$$

Figure 14.2 illustrates the resulting short-run supply (SRS) curve.

Under the assumption $P_i \geq \phi$, the firm cannot do better than choosing $Q_1 = Q_2 = V$ and the expected variable profit is:

$$F(V) = \tfrac{1}{2}(P_1 - \phi)V + \tfrac{1}{2}(P_2 - \phi)V. \tag{14.4}$$

We now choose capacity by solving:

$$\max F(V) - \lambda V = \tfrac{1}{2}(P_1 - \phi)V + \tfrac{1}{2}(P_2 - \phi)V - \lambda V. \tag{14.5}$$

The first order condition for an interior solution $(0 < V < \infty)$ to (14.5) requires that the expected net revenue from an additional unit of capacity is equal to the cost of creating capacity:

$$\tfrac{1}{2}(P_1 - \phi) + \tfrac{1}{2}(P_2 - \phi) = \lambda. \tag{14.6}$$

In equilibrium, the first order conditions (14.3) and (14.6) are satisfied and the market clears. Formally, the vector (P_1, P_2, Q_1, Q_2, V) is a competitive equilibrium if: (a) given the prices (P_1, P_2), the quantities (Q_1, Q_2, V) solve (14.1) and (b) Q_i is equal to the number of buyers whose reservation price is above P_i.

Equilibrium prices are:

$$P_1 = \phi; \quad P_2 = \phi + 2\lambda, \tag{14.7}$$

and the equilibrium quantities are: $V = N + \Delta$, $Q_1 = N$ and $Q_2 = N + \Delta$.

To show this claim, we first solve (14.2) for $V = N + \Delta$. When $P_1 = \phi$, the state 1 variable profits, $Q_1(P_1 - \phi)$, are zero regardless of the choice of Q_1 and therefore the firm cannot do better than choosing $Q_1 = N < V$. Variable profits in state 2 are given by $2\lambda Q_2$ and therefore the firm will choose $Q_2 = V$. It follows that $F(V) = \lambda V$ and therefore the maximization in (14.5) yields zero profits regardless of the choice of V. Thus, the firm cannot do better than choosing: $V = N + \Delta$, $Q_1 = N$ and $Q_2 = N + \Delta$. This choice insures that the market-clearing condition (b) is satisfied.

The uncertain and sequential trade (UST) model

Buyers arrive sequentially in batches. N buyers arrive first with probability 1. After they complete trade, a second batch of Δ may arrive, with probability 1/2. The seller is a price taker. He knows that he can sell to the first batch at the price p_1. He also knows that if the second batch arrives he can sell at the price p_2. On the basis of these expectations the seller makes a contingent plan and choose to sell x_1 units to the first batch and x_2 units to the second batch if it arrives.

It helps to talk in terms of two markets. The arrival of each batch opens a market. Since the number of batches that will arrive is random, the number of markets that will open is random. The representative firm knows that if market s opens it will be able to sell at the price p_s in this market. On the basis of these prices it chooses the amount of capacity allocated to each market (x_s). Figure 14.3 describes the sequence of events.

The representative firm is a price taker. It chooses the quantities $x_s \geq 0$, to maximize:

$$q_s(p_s - \phi)x_s - \lambda x_s; \tag{14.8}$$

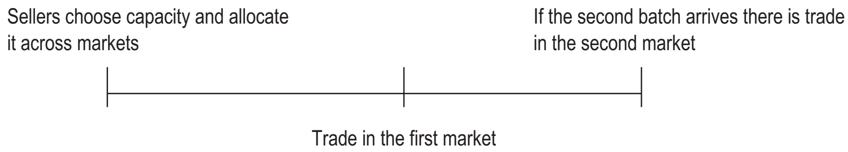

Figure 14.3 Sequence of events in the UST model

where q_s is the probability of making a sale: $q_1 = 1$ and $q_2 = 1/2$.

The vector (p_1, p_2, x_1, x_2) is an equilibrium vector if: (a) given the prices p_s the quantities x_s solve (14.8) and (b) markets that open are cleared: $x_1 = N$ if $\theta \geq p_1$ and zero otherwise; $x_2 = \Delta$ if $\theta \geq p_2$ and zero otherwise.

The UST equilibrium prices are:

$$p_1 = \phi + \lambda; \quad p_2 = \phi + 2\lambda. \tag{14.9}$$

To show this claim we substitute (14.9) in (14.8). The expected profit is zero regardless of the choice of x and therefore the firm cannot do better than satisfy demand.

It is useful to define:

$$\text{ENR} = (p_1 - \phi) = \tfrac{1}{2}(p_2 - \phi) = \lambda = \text{MCC}, \tag{14.10}$$

where ENR denotes the expected net revenue per unit of capacity and MCC is the marginal capacity cost. Condition (14.10) says that ENR is the same for both goods and is equal to MCC.

Can our equilibrium unravel? Sellers who allocated capacity to market 2 may want to offer the good at an arbitrarily low price after realizing a period of low demand. But this is too late: you cannot sell another lunch to someone who has already had lunch, even at a low price.

Comparing the predictions of the two models

We now compare the time series implications of the two models under the assumption that the number of buyers each period is an identically and independently distributed (i.i.d) random variable.

Since in the UST model there are many prices for the same commodity we should distinguish between average quoted price and average transaction price. We define average quoted price by the outcome of a price survey that asks about price offers and is given by: $\bar{p} = (p_1x_1 + p_2x_2)/(x_1 + x_2)$. Average transaction price is the outcome of a survey that asks about prices of actual transactions. This is p_1 when demand is low and $\bar{p}$ when demand is high. The solid line in figure 14.4 illustrates a possible path for the average quoted price in the UST model. The broken line is for the average transaction price.[1]

In the standard model there is a single price that fluctuates over time between ϕ and $\phi + 2\lambda$: The solid line in figure 14.5. For comparison, the broken line is the average transaction price in the UST model.

The fluctuations of the price in the standard model are larger than the fluctuations of the average price in the UST model even if we measure transaction prices. The average transaction price fluctuates between $\bar{p}$ and $\phi + \lambda$. Since $\bar{p}$ is an average between $\phi + 2\lambda$ and $\phi + \lambda$ the

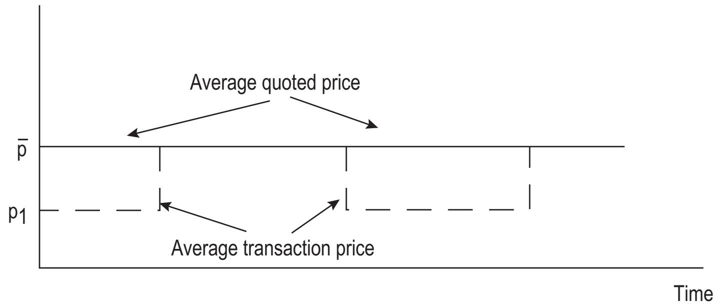

Figure 14.4 Average prices in the UST model

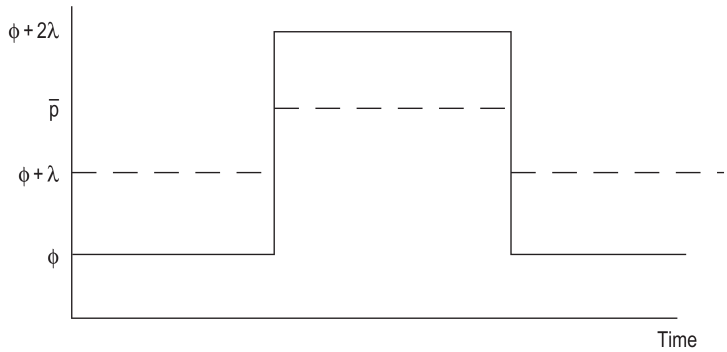

Figure 14.5 Average transaction prices in the standard model (solid line) and the UST model (broken line)

average price moves by less than λ in response to a change in demand. The price in the standard model moves by 2λ.

Therefore, if the UST is the "true" model we will reject the standard competitive model on the grounds that prices do not move much in response to changes in demand or that prices are "sticky".

Moreover, on average prices will appear to be "too high" relative to the prediction of the standard model. Average transaction price in the UST model is $(1/2)(\phi + \lambda) + (1/2)\bar{p}$ which is higher than the average price in the standard model, $\phi + \lambda$, because $\bar{p} > \phi + \lambda$. Thus if the UST is the "true" model, we may reject the standard model on the grounds that firms have market power.

A numerical example: We assume: $\lambda = 1/2$, $\phi = 1$, $N = 6$ and $\Delta = 4$. In this case the price of the 6 lunches that will be sold with probability 1 is 1.5 and the price of the 4 lunches that will be sold with probability 0.5 is 2. The data generated by the UST model is in table 14.1.

The market clearing price in the standard model is 1 if demand is low and 2 if demand is high. The data generated by the standard model is in table 14.2.

In the standard model there is one price in each period and different prices across periods. In the UST model there is a difference in prices between the two "contingent" commodities within the same period, no difference in quoted prices between periods, but no transactions in commodity 2 in the period of low demand. We get different prices for the "same" commodity

Table 14.1 Data generated by the UST model

Period	Output	Labor input	Average transaction price	Average quoted price	Wage bill	Revenue	Labor share	Profits
Low demand	6	11	1.5	1.7	11	9	1.22	−2
High demand	10	15	1.7	1.7	15	17	0.88	2

Table 14.2 Data generated by the standard model

Period	Output	Labor input	Average price	Wage bill	Revenue	Labor share	Profits
Low demand	6	11	1	11	6	1.83	−5
High demand	10	15	2	15	20	0.75	5

within the same period and smaller differences in average transaction prices across periods. Prices, labor share and profits fluctuate relatively less in the UST model.

14.1.1 Downward sloping demand

In the above example demand was inelastic and there was no difference in the predictions of both models with respect to output. We now consider the case in which all agents have the same downward sloping demand curve: $D(P)$.

Standard model: We modify the definition of equilibrium as follows.

The vector (P_1, P_2, Q_1, Q_2, V) is a competitive equilibrium if:

(a) given the prices $(P_1,\ P_2)$ the quantities (Q_1, Q_2, V) solve (14.1) and
(b) $ND(P_1) = Q_1$; $(N + \Delta)D(P_2) = Q_2$.

When $\lambda = 1/2$, $\phi = 1$, and $D(P) = 1/P$ the equilibrium vector (P_1, P_2, Q_1, Q_2, V) must satisfy the following 5 equations:

$$Q_1 = 0 \ \text{if}\ P_1 < 1; \quad Q_1 = V \ \text{if}\ P_1 > 1 \quad \text{and} \quad 0 \le Q_1 \le V \ \text{if}\ P_1 = 1;$$

$$Q_2 = 0 \ \text{if}\ P_2 < 1; \quad Q_2 = V \ \text{if}\ P_2 > 1 \quad \text{and} \quad 0 \le Q_2 \le V \ \text{if}\ P_2 = 1;$$

$$\tfrac{1}{2}(P_1 - 1) + \tfrac{1}{2}(P_2 - 1) = \tfrac{1}{2};$$

$$N/P_1 = Q_1;$$

$$(N + \Delta)/P_2 = Q_2.$$

The first equations are the first order conditions for the firm's problem (14.1) and the last two equations are the market-clearing conditions.

When $N = 6$ and $\Delta = 4$ we get an equilibrium in which capacity is always fully utilized: $P_1 = 9/8$, $P_2 = 15/8$ and $Q_1 = Q_2 = V = 16/3$. Figure 14.6 illustrates this case.

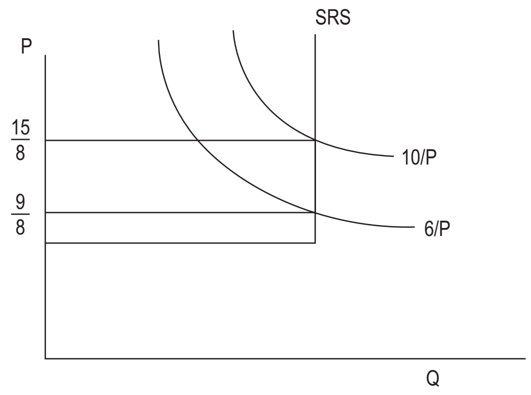

Figure 14.6 Full capacity utilization in the standard model

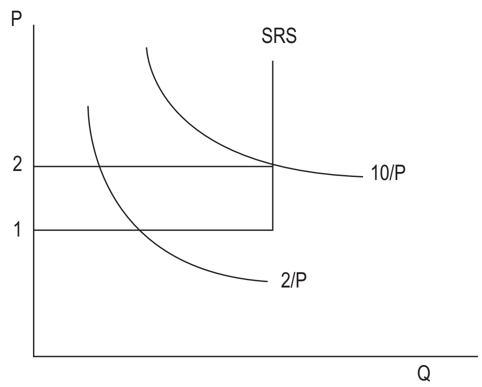

Figure 14.7 Partial utilization in the standard model

When $N = 2$ and $\Delta = 8$ we get an equilibrium in which capacity is not fully utilized in state 1: $P_1 = 1$, $P_2 = 2$, $V = 5$, $Q_1 = 2$ and $Q_2 = 5$. In this case capacity utilization in the low demand state: $Q_1/V = 2/5$. Figure 14.7 illustrates this case.

UST model: We modify the equilibrium definition as follows.

The vector (p_1, p_2, x_1, x_2) is an equilibrium vector if: (a) given the prices p_s the quantities x_s solve (14.8) and (b) $ND(p_1) = x_1$; $\Delta D(p_2) = x_2$.

When $\lambda = 1/2$, $\phi = 1$, and $D(p) = 1/p$ the equilibrium vector (p_1, p_2, x_1, x_2) must satisfy the following 4 equations:

$$p_1 - 1 = \tfrac{1}{2};$$
$$\tfrac{1}{2}(p_2 - 1) = \tfrac{1}{2};$$
$$N/p_1 = x_1;$$
$$\Delta/p_2 = x_2.$$

Table 14.3 Data generated by the UST model

Period	Output	Labor input	Av. trans. price	Capacity utilization
$N = 6$ and $\Delta = 4$				
Low demand	4	7	1.5	2/3
High demand	6	9	5/3	1
$N = 2$ and $\Delta = 8$				
Low demand	4/3	4	1.5	1/4
High demand	16/3	8	5/4	1

Table 14.4 Data generated by the standard model

Period	Output	Labor input	Price	Capacity utilization
$N = 6$ and $\Delta = 4$				
Low demand	16/3	8	9/8	1
High demand	16/3	8	15/8	1
$N = 2$ and $\Delta = 8$				
Low demand	2	9/2	1	2/5
High demand	5	15/2	2	1

The equilibrium solution for the case $N = 6$ and $\Delta = 4$ is: $p_1 = 1.5$; $p_2 = 2$, $x_1 = 4$ and $x_2 = 2$.

Capacity utilization in the low demand state: $x_1/(x_1 + x_2) = 2/3$.

When $N = 2$ and $\Delta = 8$ we get:

$$p_1 = 1.5, \quad p_2 = 2, \quad x_1 = \frac{4}{3} \quad \text{and} \quad x_2 = 4.$$

Capacity utilization in the low demand state: $x_1/(x_1 + x_2) = 1/4$.

Tables 14.3 and 14.4 summarize the data for the two different cases.

In the numerical examples, capacity utilization is lower in the UST model. In the first case ($N = 6$ and $\Delta = 4$) capacity is fully utilized in the standard model but is not fully utilized in the UST model. In the second case ($N = 2$ and $\Delta = 8$) capacity utilization is less than unity in both models (in the low demand case) but capacity utilization is lower in the UST model. When demand is high, the average transaction price in the UST model is lower than the standard model price. Since demand is always satisfied a lower average transaction price in the high demand state requires more capacity. When demand is low the UST first market price is higher than the standard model price and therefore output in the low demand state is lower in the UST model. Since capacity is higher and output in the low demand state is lower, capacity utilization in the low demand state is lower in the UST model. Since the above argument holds for any choice of parameters we may state the following claim.

Claim 1: Average capacity utilization is lower in the UST model.

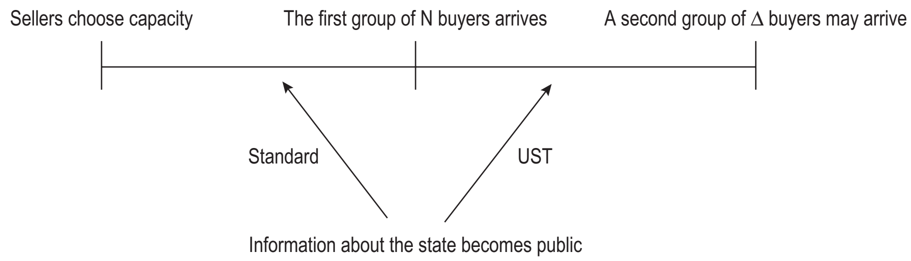

Figure 14.8 Sequence of events in the two models

Does this imply that the outcome of the standard model is better? We now turn to discuss this question.

14.1.2 Welfare analysis

To allow for welfare analysis we now provide a complete description of the economy. There are three dates ($t = 0, 1, 2$) and two goods (X and Y where lower case letters denote quantities of these goods). There are two states of nature ($s = 1, 2$) with equal probability of occurrence. There are three types of agents: one seller (S), N definite buyers (DB) and Δ possible buyers (PB).

The seller (and only the seller) can use part of his endowment of Y to produce X. It takes λ units of Y to produce a unit of capacity of X. It takes ϕ units of Y to convert a unit of capacity into output. Capacity choice must be made at $t = 0$.

The seller's utility function is: $u(x, y) = y$.

The definite buyers' utility function is: $u(x, y) = U(x) + y$, where U( ) is differentiable and strictly concave.

The possible buyers' utility function is: $u(x, y) = U(x) + y$ if the state of nature is $s = 2$ and $u(x, y) = y$ if $s = 1$.

Thus the seller does not like X, definite buyers like X and possible buyers like X only in state 2.

All agents are born at $t = 0$ with a large endowment $\bar{y}$ of the numeiraire commodity Y. Buyers (DB and PB) form a line at $t = 0$. At $t = 1$, buyers learn about their desire to consume. If $s = 1$ the Δ possible buyers drop out of the line and the N definite buyers arrive at the market-place. If $s = 2$ then all $N + \Delta$ stay in the line. The first group of N buyers go to the market at $t = 1$, complete their transactions and then go elsewhere. The second group of Δ buyers arrive at the market later, at $t = 2$.

Information about the state becomes public at time τ where the standard model assumes $0 < \tau < 1$ and the UST model assumes $1 < \tau < 2$. Figure 14.8 illustrates the sequence of events and the two alternative informational assumptions.

Let, $x(i, s)$ = quantity of X per consumer in group i if the aggregate state is s, where i = First, Second and $s = 1, 2$.

Assuming $x(2, 1) = 0$, the total utility from X derived in state 1 is $NU[x(1, 1)]$. The total utility from X derived in state 2 is: $NU[x(1, 2)] + \Delta U[x(2, 2)]$. The total utility from Y is obtained by subtracting the cost of production from the endowment. This is: $(1 + N + \Delta)\bar{y} - \lambda \max\{Nx(1,1), \; Nx(1,2) + \Delta x(2,2)\} - \phi N x(1,1)$ in state 1 and $(1 + N + \Delta)\bar{y} - \lambda \max\{Nx(1,1), Nx(1,2) + \Delta x(2,2)\} - \phi[Nx(1,2) + \Delta x(2,2)]$ in state 2. The problem of maximizing the sum of expected utilities is therefore:[2]

$$\begin{aligned} &\tfrac{1}{2}\{NU[x(1,1)] + NU[x(1,2)] + \Delta U[x(2,2)]\} + (1 + N + \Delta)\bar{y} \\ &\quad - \lambda \max\{Nx(1,1), Nx(1,2) + \Delta x(2,2)\} \\ &\quad - \tfrac{1}{2}\phi[Nx(1,1) + Nx(1,2) + \Delta x(2,2)]. \end{aligned} \tag{14.11}$$

Assuming $Nx(1,1) < Nx(1,2) + \Delta x(2,2)$, a solution to (14.11) must satisfy the first order conditions:

$$U'[x(1,1)] = \phi; \quad U'[x(1,2)] = U'[x(2,2)] = 2\lambda + \phi. \tag{14.12}$$

Claim 2: The standard model's allocation maximizes (14.11).

To show this claim (the first welfare Theorem), note that when the price of X in terms of Y is P, the consumer solves: $\max U(x) + \bar{y} - Px$. The first order condition for an interior solution to this problem is:

$$U'(x) = P. \tag{14.13}$$

Substituting the equilibrium prices (14.7), $P_1 = \phi$ and $P_2 = 2\lambda + \phi$, in (14.13) leads to (14.12).

In the UST model information about the state becomes public after $t = 1$ and therefore the social planner maximizes (14.11) subject to the informational constraint:

$$x(1,1) = x(1,2) = x_1. \tag{14.14}$$

The first order conditions to this problem are:

$$U'[x(1,1)] = U'[x(1,2)] = \lambda + \phi; \quad U'[x(2,2)] = 2\lambda + \phi. \tag{14.15}$$

These first order conditions are satisfied in the UST equilibrium because the UST equilibrium prices (14.9) imply that the first group of N buyers buys at the price $\lambda + \phi$ and the second group will buy at the price $2\lambda + \phi$ if it arrives. We have thus shown,

Claim 3: The UST equilibrium allocation maximizes (14.11) subject to the informational constraint (14.14).

It follows that each of the two models produces an efficient outcome, where efficiency is defined relative to available information.

We may say that welfare and capacity utilization (Claim 1) are higher in the standard model because the standard model assumes more information.

Figure 14.9 illustrates the difference between the two models and the value of information to a "social planner" who maximizes (14.11). When the social planner knows the state before

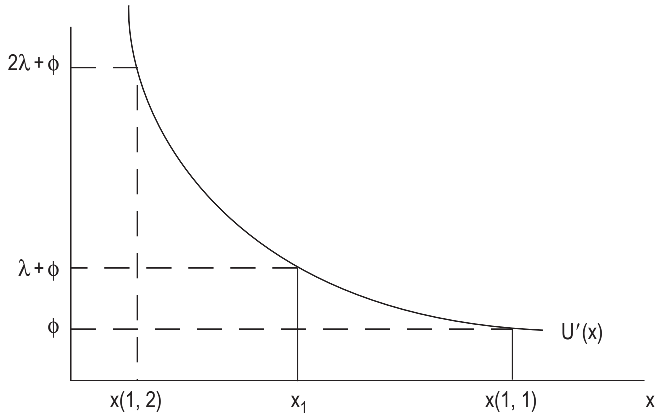

Figure 14.9 The allocation to the first group in the standard and the UST models

he chooses the allocation to the first group, he will make this choice as a function of the state: $x(1,s)$. When he does not know the state he chooses: $x(1,2) < x_1 < x(1,1)$. Thus information is useful for choosing the allocation to the first group.

Efficiency in the UST model and in the Prescott model

From a positive economics point of view it does not matter whether the prices in our model are flexible or rigid. But for the question of efficiency, it does matter. In the Prescott (1975) model prices are set before the arrival of buyers but actual sales occur after all buyers arrive and the realization of demand is known. At this point sellers may want to change their price but cannot. A central planner that has the same information as the sellers in the Prescott model can achieve the Walrasian allocation and will do better whenever the Walrasian allocation is different from the Prescott allocation.

Prescott assumes that each buyer demands one unit only and therefore he gets an equilibrium allocation that is the same as the Walrasian allocation. Allowing for a more general downward sloping demand curve (per buyer) will alter the efficiency result in the Prescott model because a planner that makes the actual allocation after he knows the realization of demand, will distribute the entire capacity to the first batch of buyers if he knows that the second batch will not arrive. A planner in a UST environment faces the same informational constraint as the sellers in the UST model and must therefore choose the amount given to the first batch before he knows whether the second batch will arrive or not. Therefore the allocation in the UST model is efficient even when buyers have a downward sloping demand curves.

For this reason it is useful to think of the UST model and the Prescott model as two different models, while keeping in mind that the resulting allocation is the same in both models.

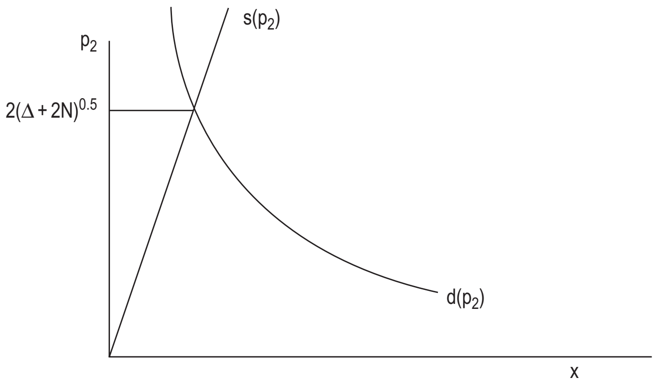

Figure 14.10 Production and prices in the UST model

14.1.3 Demand and supply analysis

Prices in the UST model are connected by an arbitrage condition. Therefore we may solve the model by standard supply and demand analysis. This will become especially useful later when we introduce storage.

To illustrate, we assume no variable costs. The cost of producing a unit of capacity is $C(x) = x^2$ and the individual demand function is $D(p) = 1/p$. The vector (x_1, x_2, p_1, p_2) is a UST equilibrium if:

$$x_1 = N/p_1; \tag{14.16}$$

$$x_2 = \Delta/p_2; \tag{14.17}$$

$$p_1 = p_2/2; \tag{14.18}$$

$$2(x_1 + x_2) = p_1. \tag{14.19}$$

The first two equations are market-clearing conditions. The third is the arbitrage condition: It says that the expected price must be the same in both markets. The fourth equation says that the marginal capacity cost is equal to the expected price (the price in the first market).

To solve for equilibrium we substitute (14.18) into (14.16) to get $x_1 = 2N/p_2$. To this quantity we add (14.17) and compute total demand as a function of p_2:

$$d(p_2) = x_1 + x_2 = (\Delta + 2N)/p_2. \tag{14.20}$$

After substituting (14.18) into (14.19) we compute supply as a function of p_2. This yields:

$$s(p_2) = x_1 + x_2 = p_2/4. \tag{14.21}$$

We can now solve for p_2 by equating supply and demand:

$$(\Delta + 2N)/p_2 = p_2/4. \tag{14.22}$$

This yields: $p_2 = 2(\Delta + 2N)^{0.5}$. Figure 14.10 illustrates this solution.

We can now do standard comparative static exercises. For example, an increase in Δ or in N will shift demand to the right and increase equilibrium prices.

PROBLEMS WITH ANSWERS[3]

1 In the UST model delivery to the first group must be made before information about the state of nature becomes public. Assume now that at $t = 1$ (before the first group arrives) sellers observe a signal about the state.

Formally the sellers observe at $t = 1$ the realization of a random variable σ. If $\sigma = 1$ the probability that the second group will arrive is $3/4$. If $\sigma = 0$ the probability that the second group will arrive is $1/4$.

A UST equilibrium for this environment is a vector of functions $[P_1(\sigma), P_2(\sigma), x_1(\sigma), x_2(\sigma)]$.

(a) Find the UST equilibrium functions for the case in which each potential buyer wants at most one unit and is willing to pay a lot for it; the fixed cost is λ per unit and the variable cost is ϕ per unit. Use the numerical example ($\lambda = 1/2$ and $\phi = 1$) to solve for the equilibrium functions numerically.

Answer

One possible solution is the following.

$$P_1(1) = P_1(0) = \phi + \lambda = \tfrac{3}{2}$$
$$[P_1(1) - \phi] = \tfrac{3}{4}[P_2(1) - \phi] = \lambda$$
$$P_2(1) = \tfrac{4}{3}\lambda + \phi = \tfrac{5}{3}$$
$$P_2(0) = 4\lambda + \phi = 3$$

(b) Compute the average quoted price as a function of σ.

Answer

$$\bar{P}(\sigma) = [NP_1(\sigma) + \Delta P_2(\sigma)]/(N + \Delta)$$
$$\bar{P}(1) = [N(\phi + \lambda) + \Delta(\tfrac{4}{3}\lambda + \phi)]/(N + \Delta) = 1.56$$
$$\bar{P}(0) = [N(\phi + \lambda) + \Delta(4\lambda + \phi)]/(N + \Delta) = 2.1$$

(c) Compare the standard deviation of the quoted price for the three informational assumptions (i) the standard model, (ii) the UST model where the signal σ is observed and (iii) the UST model where the signal σ is not observed.

Answer

(i) $SD = \lambda = 0.5$

(ii) $SD = \Delta\lambda(4 - \tfrac{4}{3})/2(N + \Delta) = \Delta\lambda\tfrac{8}{3}/2(N + \Delta) = \tfrac{8}{15}\lambda = 0.266$

(iii) $SD = 0$

(d) Does more information increase the standard deviation of quoted prices?

Answer

In this example, more information leads to a higher standard deviation of quoted prices. But this result does not seem to be general. We can choose parameters such that

$\Delta/(N + \Delta)$ is close to unity and therefore the SD in (ii) is close to $(4/3)\lambda$ which is greater than the standard deviation in (i).

2 Assume that the utility from X is: $U(x) = (v)[\min(x, 1)]$ and buyers who want to consume X maximize $U(x) + y$.

(a) What is the demand function, D(p), in this case?

Answer

The buyer's problem is: $\max v[\min(x, 1)] + y - px$. And the resulting demand function is: $D(p) = 1$ if $p \leq v$ and zero otherwise.

(b) Write the (unconstraint) planner's problem (14.11) for this special case.

Answer

$$\tfrac{1}{2}\{N(v)\min[x(1,1),1]+N(v)\min[x(1,2),1]+\Delta(v)\min[x(2,2),1]\}+(n+N+\Delta)\bar{y}- \lambda\max\{Nx(1,1),Nx(1,2)+\Delta x(2,2)\} - \tfrac{1}{2}\phi[Nx(1,1)+Nx(1,2)+\Delta x(2,2)].$$

(c) Solve for the (unconstraint) planner's problem. Distinguish between three cases: (a) $v > 2\lambda + \phi$; (b) $\lambda + \phi < v < 2\lambda + \phi$; (c) $\lambda + \phi > v$.

Answer

If the planner chooses a unit to a member of the first group he gets v and the cost for doing that is: $\lambda + \phi$. Therefore he will supply the unit to members of the first group if $v \geq \lambda + \phi$ and will not supply otherwise.

If the planner chooses to supply a unit to a member of the second group if he arrives, he gets $(1/2)v$ and the cost for doing that is: $\lambda + (1/2)\phi$. Therefore he will supply the unit to members of the first group if $(1/2)v \geq \lambda + (1/2)\phi$ and will not supply otherwise.

This considerations leads to:

(a) $x(1,1) = x(1,2) = x(2,2) = 1$
(b) $x(1,1) = x(1,2) = 1$; $x(2,2) = 0$. (The solution is not unique. Any solution in which the number of units distributed is N will do (this does not have to be the first group who gets it)
(c) $x(1,1) = x(1,2) = x(2,2) = 0$.

(d) What is the solution to the planner's problem if we add the constraint: $x(1,1) = x(1,2)$.

Answer

Since the constraint is not binding the choice of the planner will be the same as in the unconstrained problem.

(e) What is the competitive UST allocation and the monopolist UST allocation in this case?

Answer

The UST allocation is the same as in (d). The monopolist will charge v. The resulting allocation is the same.

3 Let us go back to question 1 and assume that the social planner can observe the signal (the realization of σ).

(a) Assume that as in (14.11) the social planner wants to maximize the expected value of total utilities (over all agents). Write the social planner's problem for this environment.

Answer

$x(i, \sigma, s)$ = quantity delivered to group i when the observed signal is σ and the state is s.

$$\max(\tfrac{1}{4})NU[x(1,1,1)]+(\tfrac{3}{4})\{NU[x(1,1,2)]+\Delta U[x(2,1,2)]\}+(\tfrac{3}{4})NU[x(1,0,1)]+(\tfrac{1}{4})\{NU[x(1,0,2)]+\Delta U[x(2,0,2)]\}+(n+N+\Delta)\bar{y} - \text{cost terms}$$

$$\text{s.t. } x(1,1,1) = x(1,1,2) \quad \text{and} \quad x(1,0,1) = x(1,0,2).$$

(b) Will the ability to observe the signal improve matters (increases the objective function of the social planner)? Distinguish between two cases: (a) like in question 2: $U(x) = (v)[\min(x,1)]$ and (b) $U(x)$ is a general function with $U' > 0$, $U'' < 0$ and $U'(0) = \infty$.

Answer

Without the signal the planner will face the constraint: $x(1,1,1) = x(1,1,2) = x(1,0,1) = x(1,0,2)$. This is more restrictive than the two constraints that he faces.

14.2 MONOPOLY

We now assume that all the sellers in the sequential trade economy merge into a single monopolistic firm. There are no variable costs and the fixed cost of producing x units of capacity is given by C(x), where C( ) has the standard properties of a cost function. The monopoly chooses the price to the first group (p_1) and the price to the second group (p_2).[4] Since the first group chooses the cheapest available price we require $p_1 \leq p_2$. The monopoly's problem is:

$$\max_{p_s} \; p_1 ND(p_1) + \tfrac{1}{2}p_2\Delta D(p_2) - C[ND(p_1) + \Delta D(p_2)]$$

$$\text{s.t.} \quad p_1 \leq p_2. \tag{14.23}$$

Apart from the constraint, this is the problem of a monopoly that can discriminate between two markets. An alternative formulation uses the inverse demand function $p(\bullet) = D^{-1}(\bullet)$ and write the monopoly problem as:

$$\max_{x_s} \; p(x_1/N)x_1 + (1/2)p(x_2/\Delta)x_2 - C(x_1 + x_2)$$

$$\text{s.t.} \quad x_1/N \geq x_2/\Delta \geq 0. \tag{14.24}$$

Here x_s is the amount supplied to market s. The constraint implies $p(x_1/N) \leq p(x_2/\Delta)$.

When the constraint is not binding an interior solution to (14.24) must satisfy the following first order conditions:

$$[p'(x_1/N)x_1/N + p(x_1/N)] = \tfrac{1}{2}[p'(x_2/\Delta)x_2/\Delta + p(x_2/\Delta)] = C'(x_1 + x_2). \qquad (14.25)$$

Let $MR(z) = p'(z)z + p(z)$ denote the marginal revenue from supplying z units to an individual buyer. Then we can write (14.25) as:

$$MR(x_1/N) = \tfrac{1}{2}MR(x_2/\Delta) = C'(x_1 + x_2). \qquad (14.26)$$

Since this condition implies $MR(x_1/N) < MR(x_2/\Delta)$, the constraint in (14.24) is satisfied if the marginal revenue is a decreasing function. We therefore assume that the constraint in (14.24) is not binding.

A comparison with the UST competitive outcome

A monopoly that produces according to (14.26) produces less than the UST competitive output. The proof, based on Eden (1990, Theorem 2) uses an algorithm for computing the competitive outcome and the monopoly outcome and then comparing between the two. Here is the proof for our special case.

A competitive UST equilibrium is a vector (p_1, p_2, x_1, x_2) satisfying: (a) $p_1 = (1/2)p_2 = C'(x = x_1 + x_2)$ and (b) $x_1 = ND(p_1)$ and $x_2 = \Delta D(p_2)$. To solve for a competitive equilibrium, we choose the expected revenue per unit, π, arbitrarily and set prices as: $p_1(\pi) = \pi$ and $p_2(\pi) = 2\pi$. Given these prices the competitive output is: $x(\pi) = \text{argmax}[\pi x - C(x)]$. The demand in market 1 is $ND(\pi)$ and the demand in market 2 is $\Delta D(2\pi)$. The fraction of output allocated to market 1 is

$$\mu_1(\pi) = ND(\pi)/x(\pi) \qquad (14.27)$$

and the fraction of output allocated to market 2 is:

$$\mu_2(\pi) = \Delta D(2\pi)/x(\pi). \qquad (14.28)$$

We now look at $\mu(\pi) = \mu_1(\pi) + \mu_2(\pi)$. If $\mu(\pi) = 1$ then all markets are cleared. If $\mu(\pi) > 1$ there is excess demand and if $\mu(\pi) < 1$ there is excess supply. When π goes up demand goes down and supply $x(\pi)$ goes up. Therefore, the function $\mu(\pi)$ is monotonically decreasing as in figure 14.11. The equilibrium expected profit is given by the solution $\bar{\pi}$ to $\mu(\pi) = 1$.

We can solve the monopoly problem in a similar way where the expected marginal revenue plays the role of the expected price. We choose the expected marginal revenue arbitrarily at the level of π and set the marginal revenues in the two markets: $MR_1(\pi) = \pi$ and $MR_2(\pi) = 2\pi$. Given this expected marginal revenue, the monopoly output is: $x(\pi) = \text{argmax}[\pi x - C(x)]$.[5] The demand in market 1 is $ND(\pi/[1-(1/\varepsilon_1)])/x(\pi)$ and the demand in market 2 is $\Delta D(2\pi/[1-(1/\varepsilon_2)])$, where ε_s is the absolute value of the price elasticity in market s. The fraction of output allocated to market 1 is $M_1(\pi) = ND(\pi/[1-(1/\varepsilon_1)])/x(\pi)$ and the fraction of output allocated to market 2 is $M_2(\pi) = \Delta D(2\pi/[1-(1/\varepsilon_2)])/x(\pi)$. At the optimum the monopoly satisfies demand at his announced prices and therefore the

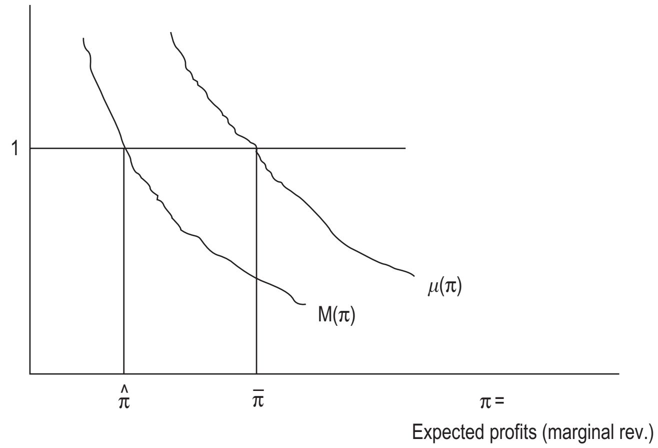

Figure 14.11 Expected marginal revenues: competition versus monopoly

expected marginal revenue for the monopoly is the solution $\hat{\pi}$ to

$$M(\pi) = M_1(\pi) + M_2(\pi) = 1.$$

We now note that the function $x(\pi) = \text{argmax}[\pi x - C(x)]$ is the same for the monopoly and the price-taker. The argument π has a different economic meaning. For the price-taker π is the expected price. For the monopoly it is the expected marginal revenue. Since for any given π, the price set by the monopoly is higher, a monopoly faces a lower demand and therefore $M(\pi) < \mu(\pi)$ for all π. It follows that the expected marginal revenue at the monopoly's optimum is less than the competitive equilibrium expected price ($\hat{\pi} < \bar{\pi}$) as illustrated by figure 14.11. Since $C'(x) = \pi$ the monopoly produces less than the price-taker. We have thus shown,

Claim 4: The UST monopoly output is less than the UST competitive output.

This claim extends the result in the standard model to a UST environment.

14.2.1 Procyclical productivity

Rotemberg and Summers (1990) start their paper with the following statement: "Productivity, no matter how it is measured, is procyclical. Leaving aside trend growth, output rises by about 1.25 percent when man-hours employed rise by 1 percent . . . The apparent contradiction with the basic principle of diminishing returns has long troubled economists." They then offer an explanation based on price rigidity and on the Prescott (1975) model for this observation. Here we use their excellent review of the procyclical productivity issue.

Procyclical productivity is not a mystery if we allow for a two-stage production process. In our restaurant example both the standard and the UST model predict a rise in output per man-hour, the common measure of productivity, in the high demand period. Using the numerical example of tables 14.1 and 14.2, output per man-hour in the high demand period is 2/3 and in the low demand period it is 6/11 which is considerably less. This is because some capacity is wasted in the low demand period. This argument is related to the modeling of labor as a quasi fixed factor of production in Oi (1962).[6]

Things get more complicated when we have many goods and use prices to achieve a measure of aggregate output. If we compute PQ/L from the data generated by the standard model in table 14.2 we will get: $PQ/L = 6/11 = 0.545$ in the low demand period and $PQ/L = 20/15 = 1.333$ in the high demand period. In this case productivity should be even more procyclical because prices are procyclical.

Alternatively, total factor productivity rises from a low demand period. Alternatively, total factor productivity rises from a low demand period if $\Delta Q/Q > (WL/PQ)(\Delta L/L)$ or if $P\Delta Q > W\Delta L$. If we write the last inequality as $P > W\Delta L/\Delta Q$ we get that productivity rises if price exceed marginal cost. If $\Delta Q/Q > (WL/PQ)(\Delta L/L)$ or if $P\Delta Q > W\Delta L$. If we write the last inequality as $P > W\Delta L/\Delta Q$ we get that productivity rises if price exceeds marginal cost.

In our example, $P\Delta Q = W\Delta L = 4$ in the standard model (table 14.2) but $P\Delta Q = 6.8 > W\Delta L = 4$ if we use the data generated by the UST model (Table 1). This is because in the UST model the price in the low demand period is higher than short run marginal cost.

By measuring the extent of procyclical productivity, Hall (1988) finds that price equals more than twice marginal cost in over half of US two-digit manufacturing industries. Eden (1990) and Rotemberg and Summers (1990) argue that his findings are consistent with a competitive version of the UST model.

14.2.2 Estimating the markup

Eden and Griliches (1993) estimated the ratio of price to marginal cost – the markup – under the UST model. They assumed that capacity is determined at the beginning of the period according to:

$$V = L^{\alpha}, \tag{14.29}$$

where $0 < \alpha < 1$ is a parameter and L is the number of workers hired. Firms and workers enter into a contingent labor contract that specifies the fixed and variable inputs that will be supplied in each state of the world. Total compensation is also state contingent. It is: W_1 if only one market opens and W_2 if both markets open. The firm chooses labor input L and the supply to market s, x_s, by solving:

$$\max_{L\, x_1\, x_2} \tfrac{1}{2}[p_1 x_1 - LW_1] + \tfrac{1}{2}[p_1 x_1 + p_2 x_2 - LW_2]$$
$$\text{s.t.} \quad x_1 + x_2 = V = L^{\alpha}. \tag{14.30}$$

An interior solution must satisfy the following first order condition:

$$R\alpha L^{\alpha-1} = W, \tag{14.31}$$

where $R = p_1 = (1/2)p_2$ is the expected revenue per unit of capacity and $W = (1/2)W_1 + (1/2)W_2$ is the expected compensation per worker. Multiplying both sides of (14.31) by L and using (14.29) leads to:

$$S \equiv WL/VR = \alpha. \tag{14.32}$$

Thus the elasticity of capacity with respect to labor, α, is approximately equal to an average of the labor share.

Output is measured according to:

$$Q = VU, \tag{14.33}$$

where Q is output and U is capacity utilization. Furthermore, we assume that capacity utilization is related to hours per worker (H) by:

$$\ln U = a + \beta \ln H, \tag{14.34}$$

where a and β are coefficients. Using $Q = VU = (L^{\alpha})U$ and $TH = (L)(H)$ for total hours leads to:

$$\ln Q = a + \alpha \ln L + \beta \ln H = a + \alpha \ln TH + (\beta - \alpha) \ln H. \tag{14.35}$$

Using Δx to denote the log difference of a variable and using $S = \alpha$ leads to:

$$\Delta Q = \gamma S \, \Delta TH + (\beta - \alpha) \, \Delta H, \tag{14.36}$$

where according to the theory $\gamma = 1$.

Following Abbot, Griliches and Hausman (1988), Eden and Griliches (1993) estimated (14.36) and could not reject the null hypothesis: $\gamma = 1$.

14.3 RELATIONSHIP TO THE ARROW–DEBREU MODEL

When buyers are ex-ante identical, we may view trade in the sequential spot markets as the execution of ex-ante contingent contracts. This interpretation of the UST model, is in the appendix of Eden (1990).

At $t = 0$, N_S ex ante identical potential buyers enter into contingent contracts with the firm. At $t = 1$, they form a line by a lottery that treats everyone symmetrically. At $t = 2$, the first $\tilde{N}$ buyers learn that they want to consume and execute their contracts according to their place in the line.

The random variable $\tilde{N}$ can take S possible realizations: $N_1 < N_2 < \cdots < N_S$ where the realization N_s occurs with probability Π_s. For notational convenience we set $N_0 = 0$. Buyers are in batch s if their place in the line is between N_s and N_{s-1}. The probability that buyers in batch s will arrive and execute their contracts is: $q_s = \sum_{i=s}^{S} \Pi_i$.

There are $N_S!$ ways of forming the line at $t = 1$ and S possible realizations of $\tilde{N}$ at $t = 2$. There are thus $S(N_S!)$ states of nature. There are two physical characteristics (X, Y) and therefore $2S(N_S!)$ Arrow-Debreu contingent commodities. But since buyers are ex-ante identical, we can use symmetry to simplify trade without loss of efficiency.

We define the following 2S goods: X_s (Y_s) is a good with physical characteristic X (Y) that will be delivered if the buyer is in batch s and the realization of $\tilde{N}$ is greater than N_s (which means that batch s arrives). We use P_{xs} (P_{ys}) to denote the price of X_s (Y_s). A buyer who buys a claim on X_s (Y_s) will get delivery with probability $\zeta_s q_s$ where $\zeta_s = (N_s - N_{s-1})/N_S$ is the probability that he is in batch s and q_s is the probability that batch s arrives.

The utility of the representative potential buyer is: $U(x) - y$ if he wants to consume and zero otherwise.[7] The function U( ) is differentiable and strictly concave.

The representative buyer maximizes expected utility by solving:

$$\max_{x_s\, y_S} \sum_{s=1}^{S} \zeta_s q_s [U(x_s) - y_s]$$
$$\text{s.t.} \quad \sum_{s=1}^{S} P_{xs} x_s = \sum_{s=1}^{S} P_{ys} y_s. \tag{14.37}$$

The firm produces k units of X at the cost of C(k) units of the numeraire commodity (say labor) where C( ) is strictly convex. A firm that promises k_s units of X_s to each of the N_S potential buyers will get $P_{xs}N_S k_s$ units of the numeraire commodity and will have to produce $(N_s - N_{s-1})k_s$ units to honor its promise. This is because at most $N_s - N_{s-1}$ buyers will exercise a contract to buy k_s units of X_s. The firm maximizes profits by solving:

$$\max_{k_s} \sum_{s=1}^{S} P_{xs} N_S k_s - C\left[\sum_{s=1}^{S} (N_s - N_{s-1}) k_s\right]. \tag{14.38}$$

Markets are cleared if:

$$k_s = x_s. \tag{14.39}$$

Sequential delivery contracts equilibrium is a vector

$(x_1, \ldots, x_S; k_1, \ldots, k_S; P_{x1}, \ldots, P_{xS}; P_{y1}, \ldots, P_{yS})$ such that

(a) given the prices $(P_{x1}, \ldots, P_{xS}; P_{y1}, \ldots, P_{yS})$ the quantities

$(x_1, \ldots, x_S)$ maximize (14.37) and the quantities $(k_1, \ldots, k_S)$ maximize (14.38);

(b) the market-clearing condition (14.39) is satisfied.

We can also define equilibrium in sequential spot markets as follows.

UST spot markets equilibrium is a vector

$(x_1, \ldots, x_S; k_1, \ldots, k_S; p_1, \ldots, p_S)$ such that

(a) $p_s q_s = C'\left[\sum_{s=1}^{S} (N_s - N_{s-1}) k_s\right]$;

(b) $U'(x_s) = p_s$;

(c) $x_s = k_s$.

The spot markets equilibrium applies to the case in which buyers and sellers meet only at $t = 2$. The condition $p_s q_s = C'$ guarantees that the UST firm produces the optimal amount and is indifferent to the way it allocates its supply across markets. The individual buyer demands in market s is determined by the first order condition $U'(x_s) = p_s$. The demand of batch s is therefore $(N_s - N_{s-1})x_s$ and in equilibrium this must be equal to the supply in market s: $(N_s - N_{s-1})k_s$. We now show the following.

Proposition 1: If $(x_1, \ldots, x_S; k_1, \ldots, k_S; p_1, \ldots, p_S)$ *is a UST spot markets equilibrium, then*

$$(x_1, \ldots, x_S; k_1, \ldots, k_S;$$
$$P_{x1} = p_1\zeta_1 q_1, \ldots, P_{xS} = p_S\zeta_S q_S;$$
$$P_{y1} = \zeta_1 q_1, \ldots, P_{yS} = \zeta_S q_S)$$

is a sequential delivery contracts equilibrium.

Note that at $t = 2$ goods with physical characteristics X are exchanged for goods with physical characteristics Y. In the spot markets case each buyer in market s receives x_s units of X in exchange of p_s units of Y. The proposition says that the execution of the sequential delivery contracts in market s (i.e. by buyers in batch s when batch s arrives) requires that each buyer will get x_s units of X and will deliver some Y. Because of risk neutrality the amount of Y delivered by buyers in market s is not determined but one possibility is that for each unit of X that the buyer gets he delivers to the firm $P_{xs}/P_{ys} = p_s$ units of Y. In this case, the Proposition says that an outside observer at $t = 2$ will not be able to tell whether these transactions are executions of contracts signed at $t = 0$ or spot markets transactions: The two models are observationally equivalent from the point of view of an outside observer at $t = 2$.

To show the Proposition, we start by solving for the sequential delivery contracts equilibrium. Under the proposed prices the representative buyer's problem (14.37) is:

$$\max_{x_s\, y_s} \sum_{s=1}^{S} \zeta_s q_s [U(x_s) - y_s]$$
$$\text{s.t.} \quad \sum_{s=1}^{S} p_s \zeta_s q_s x_s = \sum_{s=1}^{S} \zeta_s q_s y_s. \tag{14.37'}$$

The first order conditions for this problem are:[8]

$$U'(x_s) = p_s. \tag{14.40}$$

The firm's problem (14.38) under the proposed prices is:

$$\max_{k_s} \sum_{s=1}^{S} p_s \zeta_s q_s N_S k_s - C\left[\sum_{s=1}^{S} (N_s - N_{s-1}) k_s\right]. \tag{14.38'}$$

Using $\zeta_s = (N_s - N_{s-1})/N_S$, the first order conditions for this problem are:

$$p_s q_s = C'\left[\sum_{s=1}^{S}(N_s - N_{s-1})k_s\right]. \tag{14.41}$$

We now note that (14.39)–(14.41) define UST spot markets equilibrium. This completes the proof.

14.4 HETEROGENEITY AND SUPPLY UNCERTAINTY

Dana (1998) has generalized the Prescott (1975) model to the case of heterogeneous agents. In Dana's model buyers have a demand for one unit only but reservation prices and the probability of wanting to consume may be different across buyers. He concludes that the equilibrium allocation may not be efficient because of price rigidity. Dana compares the allocation in the Prescott model to the standard Walrasian allocation. This is the relevant comparison for the Prescott model but, as was argued in section 14.1.2; it is not the relevant comparison for the UST model.

Here we introduce heterogeneity and supply uncertainty to the UST model. We show that the UST outcome may not be efficient: Even a social planner who operates under the informational constraints faced by the UST firms can improve matters. Moreover, even a monopoly that faces the same informational constraints may improve matters. We start with some examples that illustrate the efficiency problems and then attempt at a more general treatment.

Example 1: As in section 14.1.2, there are two types of agents: Definite buyers and possible buyers. The number of agents from each type is 1. Definite buyers have a reservation price of $v_1 = 10$ dollars (units of Y) and possible buyers have a reservation price of $v_2 = 7$ dollars. There are two states of nature (indexed s) that occurs with equal probabilities. Definite buyers want to consume X in both states. Possible buyers want to consume X only if $s = 2$. The cost of production is $\lambda = 5$ per unit of capacity. Capacity can be costlessly converted into output, if there is demand for it.

At $P_1 > 10$, there is no demand. At $7 < P_1 \leq 10$ total demand is 1 and at $P_1 \leq 7$ total demand is 1 if $s = 1$ and 2 if $s = 2$. The number of buyers in the first batch is the minimum number that will arrive. It is:

$$\Delta_1(P_1) = 1 \quad \text{for } P_1 \leq 10 \quad \text{and} \quad \text{zero otherwise.} \tag{14.42}$$

Market 2 will open if there are buyers who wanted but could not buy at the first market price. This will happen if $P_1 \leq 7$ and $s = 2$. The probability that market 2 will open is:

$$q_2(P_1) = 1/2 \quad \text{for } P_1 \leq 7 \quad \text{and} \quad \text{zero otherwise.} \tag{14.43}$$

When $P_1 \leq 7$ the number of remaining buyers in state 2 is 1/2 of each type. When $P_2 \leq 7$, all the remaining buyers want to buy in market 2 and the size of the second batch is:

$$\Delta_2(P_1, P_2) = 1 \quad \text{for } P_1 < P_2 \leq 7. \tag{14.44}$$

When $7 < P_2 \leq 10$ only the definite buyers want to buy in market 2 and therefore:

$$\Delta_2(P_1, P_2) = \tfrac{1}{2} \quad \text{for } P_1 \leq 7 \text{ and } 7 < P_2 \leq 10. \tag{14.45}$$

When $P_2 > 10$ none of the remaining buyers want to buy in market 2 and therefore:

$$\Delta_2(P_1, P_2) = 0 \quad \text{for } P_1 \leq 7 \text{ and } P_2 > 10. \tag{14.46}$$

Since the second market opens in this example with probability 1/2, equilibrium prices are: $P_1 = \lambda = 5$ and $P_2 = 2\lambda = 10$. The number of buyers in the second batch is 1/2 according with (14.45) and production is therefore 1.5 units at the cost of 7.5. The surplus in state 1 is: $v_1 - 7.5 = 2.5$. The surplus in state 2 is: $v_1 + (1/2)v_2 - 7.5 = 6$. The average surplus over the two states is 4.25.

A monopoly will choose $P_1 = 10$ and produce one unit making a profit of 5. This profit is also the surplus in this case and is greater than the expected surplus in the competitive sequential trade.

A social planner who can set prices to maximize the expected surplus cannot do better than the monopoly: The planner will choose to produce one unit and will price it at $7 < P \leq 10$ so that only the high valuation buyers will get it. Thus the monopoly choice is efficient.

In example 1 the UST competitive firm produces more than the monopoly but this is not efficient because the additional half a unit of capacity is being used by type 2 agents whose ex-ante valuation is less than the cost of production (3.5 per unit). The reason why in example 1 the competitive firm produces too much capacity is in the failure to allocate capacity to buyers who value it the most: Low valuation buyers who arrive early are not rationed and therefore the residual demand includes high valuation buyers who arrive late. These high valuation buyers are willing to pay enough to produce goods that will be sold with probability 1/2.

Note that the probability that a market will open and the number of buyers participating in this market are endogenous. The next market will open if there are buyers who wanted but could not buy in the last market. The probability of this event depends on the prices in previous markets. The demand in the next market is the minimum size of the residual demand. It depends on the prices in previous markets and the price in the next market.

Example 2: The same as example 1 but now $v_1 = 9$ instead of 10. In this case competitive UST prices remains the same as in the previous example: $P_1 = 5$ and $P_2 = 10$. As in the previous example, market 2 will open in state 2 but since $\Delta_2(5, 10) = 0$ it will not be active. In a UST equilibrium only one unit is produced and allocated to market 1. The surplus is: 4 in state 1 and $3(= (1/2)v_1 + (1/2)v_2 - 5)$ in state 2. The average surplus is: 3.5.

A monopoly will choose $P_1 = 9$ guaranteeing a profit (surplus) of 4.

In example 2 both the monopoly and the competitive firm produce the same amount but the monopoly does a better job in allocating the existing capacity to the buyers who value it the most.

Example 3: We now add a new type to example 1: Type 3 who wants to consume only when $s = 1$ (when type 2 does not want to consume) and is willing to pay only up to $v_3 = 4$.

Adding type 3 will not change the UST equilibrium and the monopoly choice. But it will change the planner's choice. Now the planner can do better by producing two units and pricing them at $P \leq 4$.

When $s = 1$, type 1 and type 3 will buy the good and the surplus will be $10 + 4 - 10 = 4$. When $s = 2$, type 1 and type 2 will buy the good and the surplus will be $10 + 7 - 10 = 7$. The average surplus is 5.5 which is higher than the monopoly's expected profits.

In example 3 the UST firm is producing too little relative to the sequential efficient level: 1.5 instead of 2.

The above three examples show:

Proposition 1: (a) The UST allocation is not necessarily efficient; (b) A monopoly may improve matters and (c) The UST output may be either too high or too low relative to the sequential efficient level of output.

Note that a departure from zero expected profits is required for improving on the UST competitive allocation. To improve we must either raise prices and achieve a better screening of buyers (allowing only high valuation buyers to buy) or reduce prices and allow the participation of low valuation buyers who want to consume in low demand states. This requires either positive or negative expected profits and therefore cannot occur in the UST competitive environment.

We use these examples in chapter 21 to discuss the welfare consequences of international trade. We now turn to a more general formulation and to the conditions under which efficiency can be guaranteed.

14.4.1 The model

We consider an economy with two dates ($t = 0, 1$) and two goods (X and Y with lower case letters denoting quantities). There are S possible aggregate states of nature (indexed s). There are many potential sellers and out of this group actual sellers are chosen randomly. An actual seller or just a seller for short can produce as many units as he wants at the price of λ units of Y per unit of X. In state s there are M_s actual sellers. State s occurs with probability Π_s.[9] Sellers are risk neutral and derive utility from Y only.

There are J types of buyers. The number of type j buyers is n_j. All buyers are endowed with a large quantity of Y. Ex-ante the utility of a type j agent is random and is given by: $\{u_{js}(x, y)$ with probability $\Pi_s\}$. In aggregate state s the utility function that a fraction ϕ_{js} of type j buyers realize is: $u_{js}(x, y) = U_j(x) + y$, where $U_j(x)$ is strictly monotone, strictly concave and differentiable. The remaining fraction of $1 - \phi_{js}$ who "do not want to consume X" realize the utility function: $u_{js}(x, y) = y$. The random utility of a type j buyer in aggregate state s is thus:

$$u_{js}(x, y) = \{U_j(x) + y \text{ with probability } \phi_{js} \text{ and } y \text{ otherwise}\}. \tag{14.47}$$

A type j buyer demands $d_j(p)$ units of X at the price p if he wants to consume, where the individual demand function is defined by:

$$d_j(p) = \underset{x \geq 0}{\operatorname{argmax}}\, U_j(x) - px. \tag{14.48}$$

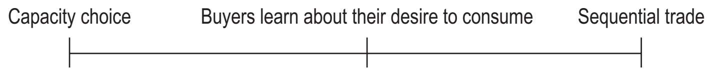

Figure 14.12

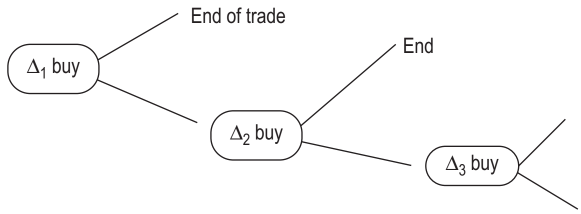

Figure 14.13

The first order condition for the problem (14.48) is:

$$U_j'(x) \leq p \quad \text{with equality if } x > 0. \tag{14.49}$$

Production occurs at $t = 0$. Then buyers realize a utility function and those who want to consume form a line. It is assumed that any batch of buyers taken from this line accurately represents the type composition of buyers who want to consume: In state s, $\sum_{j=1}^{J} \phi_{js} n_j$ buyers want to consume and the fraction of type j buyers in any batch is: $\phi_{js} n_j / \sum_{j=1}^{J} \phi_{js} n_j$. After the line is formed, buyers arrive at the market place one by one according to their place in the line and choose whether to buy at the cheapest available offer. The sequential trade does not take real time (it occurs in meta time). Figure 14.12 illustrates the sequence of events.

We start with the relatively simple case in which all types have the same demand functions (but unlike section 14.2.1 they may have different probabilities of wanting to consume).

Buyers have the same demand functions and there is no uncertainty about supply: We assume that: $U_j(x) = U(x), d_j(p) = d(p)$ for all j and $M_s = 1$ for all s. We use $N_s = \sum_{j=1}^{J} \phi_{js} n_j$ for the number of buyers and assume: $N_1 < N_2 < \cdots < N_S$.

The minimum number of buyers is: $\Delta_1 = N_1$. The first batch of Δ_1 buyers arrives with certainty. After buyers in this first batch complete trade they go away. If $s = 1$ trade ends. If $s > 1$, there are $N_s - N_1$ unsatisfied buyers. The minimum number of unsatisfied buyers is: $\Delta_2 = \min_s\{N_s - N_1\} = N_2 - N_1$. The probability that $s > 1$ is $q_2 = 1 - \Pi_1$ and this is the probability that batch 2 will arrive. After batch 2 completes trading (and disappear) there may be again two possibilities: either no additional buyers arrive or, if $s > 2$, some additional buyers do arrive. The probability that $s > 2$, is $q_3 = 1 - \Pi_1 - \Pi_2$ and this is the probability that the third batch of buyers will arrive. The minimum number of unsatisfied buyers if $s > 2$ is: $\Delta_3 = \min_s\{N_s - N_2\} = N_3 - N_2$. Proceeding in this way we define q_s and Δ_s for all $s = 1, \ldots, S$. Figure 14.13 illustrates the sequential trade process.

The seller is a price taker and behaves as if he can sell any amount at the price P_i to buyers in batch i if it arrives. He makes a contingent plan to sell x_i units to batch i if it arrives. It is convenient to think of a sequence of Walrasian markets, where batch i buys in market i and the seller supplies x_i units to market i.

Because of constant returns to scale equilibrium prices are determined by supply conditions only. The expected revenue from supplying a unit to market i is $q_i P_i$. When $q_i P_i = \lambda$ the expected profit is zero, the seller is indifferent about the quantity supplied and is willing to satisfy demand.

A UST equilibrium is a vector of prices $(P_1, \ldots, P_S)$ and a vector of supplies $(x_1, \ldots, x_S)$ such that:

(a) $P_i = \lambda/q_i = \lambda/\sum_{s=i}^{S} \Pi_s$ and
(b) $x_i = (N_i - N_{i-1})d(P_i) = \Delta_i d(P_i)$.

Thus in equilibrium markets that open are cleared. Note that prices may appear rigid because they do not respond to the realization of demand (the state). Nevertheless, sellers do not have an incentive to change prices during trade.[10]

To solve for the equilibrium quantities we substitute the equilibrium condition (a) in (b) to get:

$$x_i = \Delta_i d(\lambda/q_i). \tag{14.50}$$

We now show the following proposition which is a version of Theorem 1 in Eden (1990).

Proposition 2: The equilibrium allocation (14.50) is a solution to the following social planner's problem:

$$\max_{x_i} \sum_{i=1}^{S} q_i \Delta_i U(x_i/\Delta_i) - \lambda \sum_{i=1}^{S} x_i. \tag{14.51}$$

Note that the objective in (14.51) is the expected utility of the sum of utilities in the economy.

Proof: The first order condition for the problem (14.51) is:

$$q_i U'(x_i/\Delta_i) \leq \lambda \quad \text{with equality if } x_i > 0. \tag{14.52}$$

Substituting $P_i = \lambda/q_i$ in the first order condition (14.49) leads to:

$$q_i U'[d(\lambda/q_i)] \leq \lambda \quad \text{with equality if } d_j(\lambda/q_i) > 0. \tag{14.53}$$

In equilibrium $x_i/\Delta_i = d(\lambda/q_i)$ and therefore (14.53) implies that the UST equilibrium allocation satisfies (14.52). Since (14.52) is both sufficient and necessary condition for a solution to the problem (14.51), the UST equilibrium allocation is a solution to (14.51). □

We now turn to the general case in which buyers have different demand functions and different probabilities of wanting to consume.

The general case

As before buyers arrive in batches but here the size of each batch is endogenous. We now turn to describe an algorithm for computing the size of each batch for an arbitrarily chosen price vector ($P_1 \leq P_2 \leq \cdots \leq P_S$). This is done under the assumption that the demand of each batch that arrives is satisfied: Batch i's demand is satisfied at the price P_i.

Roughly speaking, the size of the first batch is the minimum demand at the price P_1. Market 2 opens if there are some buyers who wanted to buy in the first market but could not. In general, an additional market opens after transactions in market $i - 1$ are complete if there is residual demand and the size of batch i is the minimum residual demand per seller. We now turn to a detailed description of this algorithm.

Demand per seller in state s at the price P_1 is: $\sum_{j=1}^{J} \phi_{js} n_j d_j(P_1)/M_s$. We choose indices such that state 1 is the state of minimum demand, $1 = \text{argmin}_s\{\sum_{j=1}^{J} \phi_{js} n_j d_j(P_1)/M_s\}$. The size of the first batch (per seller) is: $D_1(P_1) = \sum_{j=1}^{J} \phi_{j1} n_j d_j(P_1)/M_1$ units and it is assumed that each seller supplies that many units at the price P_1.

If $s = 1$ then all buyers are served in the first market and trade ends. Otherwise, if $s > 1$, a demand for $\sum_{j=1}^{J} \phi_{js} n_j d_j(P_1) - M_s D_1(P_1) \geq 0$ units was not satisfied. The fraction of demand satisfied in market 1 is: $1 - \chi_s^1(P_1) = M_s D_1(P_1)/\sum_{j=1}^{J} \phi_{js} n_j d_j(P_1)$. The residual demand per seller at the price P_2 is $\chi_s^1(P_1)\sum_{j=1}^{J} \phi_{js} n_j d_j(P_2)/M_s$. We now choose the indices $s > 1$ so that $2 = \text{argmin}_s\{\chi_s^1(P_1)\sum_{j=1}^{J} \phi_{js} n_j d_j(P_2)/M_s\}$ and the minimum residual demand per seller is in state 2. The size of batch 2 is: $D_2(P_1, P_2) = \chi_2^1(P_1)\sum_{j=1}^{J} \phi_{j2} n_j d_j(P_2)/M_2$ units.

In general, we start iteration i having already computed $\chi_s^k(P_1, \ldots, P_k)$ for $k < i - 1$ and the amount per seller supplied in market $i - 1$, $D_{i-1}(P_1, \ldots, P_{i-1})$. We then compute $\chi_s^{i-1}(P_1, \ldots, P_{i-1})$ as follows. If $s > i - 1$, the demand for $\chi \sum_{j=1}^{J} \phi_{js} n_j d_j(P_i) - M_s D_{i-1}(P_1, \ldots, P_{i-1}) \geq 0$ units was not satisfied in market $i - 1$, where $\chi = \prod_{k=1}^{i-2} \chi_s^k(P_1, \ldots, P_k)$ is the fraction of buyers who did not buy in markets $1, \ldots, i - 2$. The fraction of demand satisfied in market $i - 1$ is:

$$1 - \chi_s^{i-1}(P_1, \ldots, P_{i-1}) = M_s D_{i-1}(P_1, \ldots, P_{i-1})/\chi \sum_{j=1}^{J} \phi_{js} n_j d_j(P_{i-1}). \tag{14.54}$$

The residual demand per seller at the price P_i is: $\chi' \sum_{j=1}^{J} \phi_{js} n_j d_j(P_i)/M_s$, where $\chi' = \chi\chi_s^{i-1}(P_1, \ldots, P_{i-1})$. We choose indices $s > i - 1$ such that: $i = \text{argmin}_{s>i-1}\{\chi' \times \sum_{j=1}^{J} \phi_{js} n_j d_j(P_i)/M_s\}$. The size of batch i is:

$$D_i(P_1, \ldots, P_i) = \chi' \sum_{j=1}^{J} \phi_{ji} n_j d_j(P_i)/M_i, \tag{14.55}$$

units.

Given the construction of the demand functions $D_i(P_1, \ldots, P_i)$ we can now define equilibrium as follows.

A UST equilibrium is a vector of prices $(P_1 \leq P_2 \leq \cdots \leq P_S)$ and a vector of per seller supplies $(x_1, \ldots, x_S)$ such that:

(a) $P_i = \lambda/q_i$ and
(b) $x_i = D_i(P_1, \ldots, P_i)$.

The examples at the beginning of this chapter demonstrate that efficiency cannot be guaranteed for the general case. But it is possible to show that efficiency can be guaranteed when all buyers have the same probabilities of wanting to consume.

Conclusions

The UST allocation is efficient if (1) there is no uncertainty about the number of sellers and buyers have the same downward sloping demand functions or (2) there is no uncertainty about the number of sellers and buyers have the same probabilities of "wanting to consume". Otherwise, the UST allocation may not be efficient and a monopoly may improve matters.

14.5 INVENTORIES

We have assumed that output not sold evaporates. Here we follow Bental and Eden (1993) and assume that output not sold is carried to the next period as inventories.

When demand is low the seller accumulates "undesired" inventories and next period's prices and production go down. This story has a Keynesian flavor. But here prices are completely flexible, expectations are rational and the allocation is Pareto efficient.

Our model is also different from the standard storage model in Deaton and Laroque (1992, 1996). In the standard model inventories are held only when the expected increase in price covers storage and interest costs. The UST model allows for purely speculative inventories. But in addition, inventories in the UST model are held whenever demand does not reach its highest possible realization and not all of the UST markets open.

The model

There are many identical infinitely lived risk neutral firms. Production occurs each period, before the beginning of trade. The cost of producing x units of the good is $C(x) = x^2$. The firm's discount factor is given by $0 < \beta < 1$ and for simplicity we assume that storage itself is costless.

The number of buyers (per firm) that may show up each period is an i.i.d. random variable that may take two possible realizations: N and $N + \Delta$ with equal probability of occurrence. The demand of each individual buyer that arrives is: $D(p) = 1/p$, where p is the price faced by the buyer.

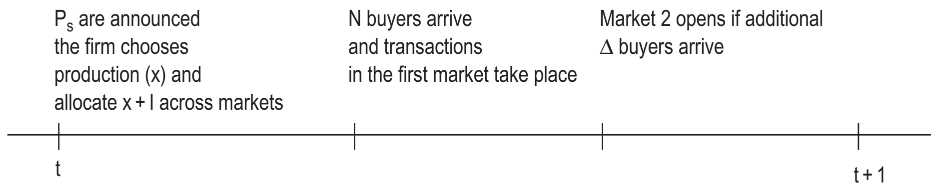

Figure 14.14 Temporary (partial) equilibrium

14.5.1 Temporary (partial) equilibrium

The representative firm starts period t with I_t units of inventories. It takes the prices in the two UST markets (p_{1t}, p_{2t}) as given and forms expectations about next period's prices.

The expected price in next period's first market is $p_{1t+1} = \alpha(I_{t+1})$, where the function $\alpha(\,)$ is decreasing and I_{t+1} is the average beginning of next period's level of inventories per firm. Since the individual firm cannot affect the average level of inventories it cannot affect next period's prices.

The quantity supplied to market s per firm is denoted by k_s. The firm may choose to store even if all $N + \Delta$ buyers arrive and market 2 opens. The quantity that the firm chooses not to sell even if market 2 opens (purely speculative inventories) is denoted by k_3. Production at the beginning of the period is denoted by x_t.

Figure 14.14 describes the sequence of events. On the basis of the announced prices (p_{st}) and expectations about future prices (α), the firm chooses production (x_t) and allocates the available supply $(k_t = x_t + I_t)$ across markets. Then the first group of N buyers arrives, trades in the first market at the price p_{1t} and goes home. If a second group of Δ buyers arrives, market 2 opens and transactions occur at the price p_{2t}.

A partial (temporary) equilibrium takes the expectation function $\alpha(\,)$ as given. It requires that markets which open are cleared and some arbitrage conditions which guarantee that in equilibrium the firm cannot increase its expected present value of profits. In the Appendix we formulate the firm's maximization problem and derive the arbitrage conditions as first order conditions.

The vector $(p_{1t}, p_{2t}, x_t, k_{1t}, k_{2t}, k_{3t})$ is a temporary equilibrium for a given expectations function, $\alpha(\,)$, and a given beginning of period inventories, I_t, if it satisfies the following conditions:

$$k_{1t} = N/p_{1t}; \tag{14.56}$$

$$k_{2t} = \Delta/p_{2t}; \tag{14.57}$$

$$p_{2t} \geq \beta\alpha(k_{3t}) \text{ with equality if } k_{3t} > 0; \tag{14.58}$$

$$p_{1t} = (1/2)p_{2t} + (1/2)\beta\alpha(k_{2t} + k_{3t}) \tag{14.59}$$

$$C'(x_t) = 2x_t = p_{1t}; \tag{14.60}$$

$$k = k_{1t} + k_{2t} + k_{3t} = x_t + I_t. \tag{14.61}$$

Conditions (14.56) and (14.57) are market-clearing conditions. The third condition governs the demand for purely speculative inventories: k_3. To develop the intuition for this condition, let us think of the seller's choice when market 2 opens. If he sells a unit he will get p_2 dollars. The alternative is to store the unit and sell it in the next period's first market. Since if both markets open this period, inventories next period are given by k_{3t}, the price in the next period's first market is: $p_{1t+1} = \alpha(k_{3t})$. The value of a unit stored in terms of current dollars is: $\beta\alpha(k_{3t})$. The market clearing condition (14.57) requires $k_2 > 0$. This can be optimal only if (14.58) holds. Otherwise, if $p_2 < \beta\alpha(k_{3t})$, it is better to carry the k_2 units to the next period as inventories. If (14.58) holds with strict inequality then we must have $k_3 = 0$, since otherwise the firm can increase its profits by selling k_3 in market 2. If (14.58) holds with equality then we may have an interior solution ($k_3 > 0$). In this case the value of inventories is the same as the revenues from selling the unit. Note that we use the assumption that market 2 opens and that the seller cannot affect average per seller magnitudes.

The fourth equation captures the main idea of the UST model. Since (14.56) and (14.57) require strictly positive k_1 and k_2, the seller must be indifferent between selling in the first market, at the price p_1, to betting that the second market will open. If the second market opens the seller will sell the unit at the higher price p_{2t}. If it does not open the unit will be stored and sold in the next period. To calculate the value of inventories note that when the second market does not open $I_{t+1} = k_{2t} + k_{3t}$ and the price in the next period's first market is: $\alpha(k_{2t} + k_{3t})$. The current value of a unit stored in this case is therefore: $\beta\alpha(k_{2t} + k_{3t})$. Condition (14.59) thus says that the seller is indifferent between selling a unit in the first market to allocating it to the second market.

The fifth equation determines current production: marginal cost = the price in the first market. To derive this condition, note that in equilibrium the firm cannot make money by increasing production and selling the additional amount in the first market. And it cannot make money by increasing production and selling the additional amount in the second market. Thus $C' = p_{1t} = (1/2)p_{2t} + (1/2)\beta\alpha(k_{2t} + k_{3t})$, as implied by conditions (14.59) and (14.60).

The last equation is a resource constraint. It says that the firm must allocate the available supply to the three markets.

The effect of storage on price dispersion: In the absence of storage, the relative price p_2/p_1 is 2. When storage is allowed the relative price p_2/p_1 is less than 2: Goods that are not sold have some value as inventories and therefore a smaller relative price is required to compensate for the risk of not making a sale. We now show this claim formally.

> *Claim 1:* $1 \leq p_2/p_1 \leq 2$.
>
> *Proof:* Since (14.59) implies that the value of inventories when only one market opens is: $\beta\alpha(k_{2t} + k_{3t}) = 2p_{1t} - p_{2t} \geq 0$, we get: $p_2/p_1 \leq 2$. To show that $p_2/p_1 \geq 1$, note that (14.58) and the fact that $\alpha(\,)$ is decreasing, implies: $p_{2t} \geq \beta\alpha(k_{3t}) > \beta\alpha(k_{2t} + k_{3t})$. Since according to (14.59) p_1 is a weighted average of p_2 and $\beta\alpha(k_{2t} + k_{3t})$ it follows that $p_1 < p_2$. □

Allowing for costly storage: Assume that (the present value of the) storage cost is ρ dollars per unit stored. When free disposal is possible the firm will store only if: $\beta\alpha(I_{t+1}) - \rho \geq 0$. The value of inventories is therefore: $\max\{\beta\alpha(I_{t+1}) - \rho, 0\}$ and conditions (14.58) and (14.59)

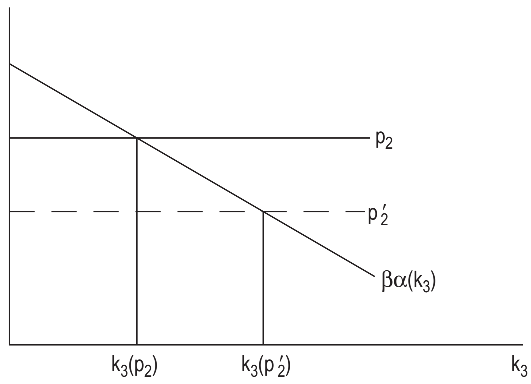

Figure 14.15 Pure speculation

are now:

$$p_{2t} \geq \beta\alpha(k_{3t}) - \rho \text{ with equality if } k_{3t} > 0; \tag{14.58$'$}$$

$$p_{1t} = \tfrac{1}{2}p_{2t} + \tfrac{1}{2}\max\{\beta\alpha(k_{2t} + k_{3t}) - \rho, 0\}. \tag{14.59$'$}$$

Storage costs do not add much because we can choose β to reflect both the interest costs and the storage costs.

14.5.2 Solving for a temporary equilibrium

We first compute the quantity demanded for a given p_2. We use (14.58) to solve for k_3 denoting the solution by $k_3(p_2)$. Figure 14.15 illustrates the solution. The demand for k_3 is strictly positive if p_2 is below the intersection of the $\beta\alpha(k_{3t})$ curve with the vertical axis. When we reduce p_2 the quantity demanded, $k_3(p_2)$, increases because the function $\alpha(\,)$ is decreasing.

We proceed by substituting the quantities $k_2 = \Delta/p_2$ and $k_3(p_2)$ in (14.59) to solve for the price in the first market p_1 (assuming $\rho = 0$):

$$p_1(p_2) = \tfrac{1}{2}p_2 + \tfrac{1}{2}\beta\alpha[\Delta/p_2 + k_3(p_2)]. \tag{14.62}$$

Since $\alpha(\,)$ is decreasing and $k_3(p_2)$ is decreasing we can show the following Claim.

Claim 7: The function $p_1(p_2)$ is increasing.

The demand in the first market is:

$$k_1(p_2) = N/p_1(p_2), \tag{14.63}$$

Claim 7 implies that $k_1(p_2)$ is decreasing.

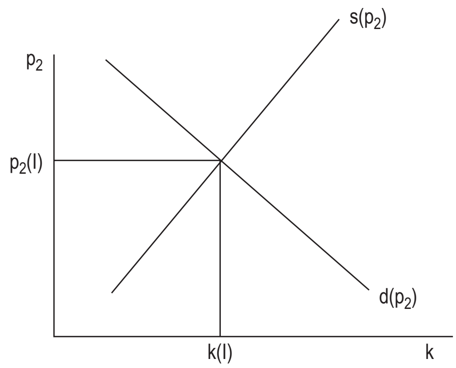

Figure 14.16 Partial equilibrium solution

We repeat this process for each price p_2 and get the demand schedule:

$$d(p_2) = k_3(p_2) + \Delta/p_2 + N/p_1(p_2). \tag{14.64}$$

Since all $k_s(\,)$ are decreasing, the demand function $d(\,)$ is decreasing as in figure 14.16.

We now use (14.60) to calculate current production:

$$x(p_2) = \tfrac{1}{2}p_1(p_2). \tag{14.65}$$

Supply is given by:

$$s(p_2) = x(p_2) + I_t. \tag{14.66}$$

Since $p_1(p_2)$ is an increasing function, the supply is upward sloping as in figure 14.16.

The intersection of supply and demand yields a solution to: $d(p_2) = s(p_2)$. This solution is the temporary equilibrium level of p_2 and is denoted by $p_2(I)$. We can now solve for the other partial equilibrium magnitudes: $p_1(I) = p_1[p_2(I)], x(I) = x[p_2(I)], k_1(I) = k_1[p_2(I)]$, $k_2(I) = k_2[p_2(I)], k_3(I) = k_3[p_2(I)], k(I) = k_1(I) + k_2(I) + k_3(I)$.

We have thus solved for a partial equilibrium that assumes a given model of expectations $\alpha(\,)$ and a given level of the beginning of period inventories, I. We now vary I and get partial equilibrium functions: $[p_1(I), p_2(I), x(I), k_1(I), k_2(I), k_3(I)]$. We now turn to characterize these partial equilibrium functions.

Claim 8: The partial equilibrium functions $[p_1(I), p_2(I), x(I)]$ are monotonically decreasing and the partial equilibrium functions $[k_1(I), k_2(I), k_3(I)]$ are monotonically increasing.

We use figure 14.17 to show this claim. An increase in the beginning of period inventories shifts the supply curve to the right by the change in inventories $\Delta I = I' - I > 0$. This reduces the price in the second market from $p_2(I)$ to $p_2(I')$. By claim 7, the first market price goes down

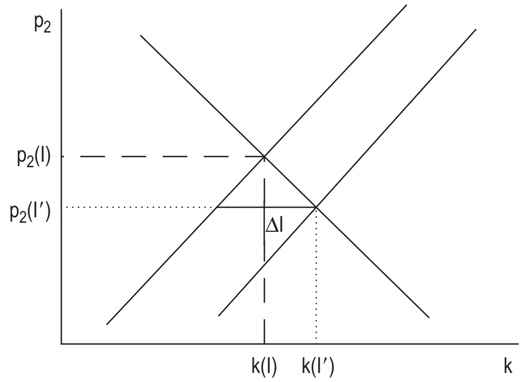

Figure 14.17 An increase in the beginning of period inventories

and since $C'(x) = p_1$, current output goes down. The supply to market s, $k_s(I) = k_s[p_2(I)]$ is decreasing in I because it is increasing in p_2.

We also have:

Claim 9: A unit increase in the beginning of period inventories reduces output by less than a unit.

It is clear from figure 14.17 that $\Delta k = k(I') - k(I)$ is positive. Since $\Delta k = \Delta x + \Delta I$ it follows that: $|\Delta x| < \Delta I$.

Claim 10: An increase in the beginning of period inventories increases the expected value of next period's inventories. (Thus, $E(I_{t+1}|I_t)$ is increasing in I_t.)

Since $k_s(I)$ are increasing functions, the expected next period inventories, $k_3(I) + (1/_2)k_2(I)$, is an increasing function.

Claim 10 suggests that a demand shock may affect output in future periods because it changes inventories. The following claim suggests that the effect dies out gradually.

Claim 11: An increase in the beginning of period inventories by one-unit increases next period expected inventories by less than a unit.

To see this claim note that an increase in I_t by one unit increases $k = k_1 + k_2 + k_3$ by less than a unit (see figure 14.17). Since claim 8 says that k_1 increases it follows that $k_2 + k_3$ increases by less than a unit and therefore: $(1/2)\Delta k_2 + \Delta k_3 < 1$.

14.5.3 Full equilibrium

We have assumed that the next period's first market price is given by the function $\alpha(I_{t+1})$. We then derived the price in the current period first market: $p_1(I_t;\alpha)$. A consistent description requires rational expectations: $\alpha(I) = p_1(I;\alpha)$. We therefore define:

The vector of functions $[p_1(I), p_2(I), x(I), k_1(I), k_2(I), k_3(I), \alpha(I)]$ is an equilibrium if it satisfies (14.56)–(14.61) and $p_1(I) = \alpha(I)$ for all I.

To solve for the equilibrium functions, we proceed as follows. We first pick $\alpha(\ )$ and I_t arbitrarily and solve for a temporary equilibrium and for $p_1(I_t;\alpha)$. We then repeat this procedure for alternative choices of I_t (keeping the same α). This yields points in the (p_1, I) space. We use these points to approximate for the partial equilibrium function $p_1(I)$. If this function is the same as $\alpha(\ )$ we are done. Otherwise, we use the approximated function $p_1(\ )$ as $\alpha(\ )$ and start the loop again.

Since a full equilibrium is also a partial equilibrium all the properties of the partial equilibrium functions hold in a full equilibrium if indeed the function $p_1(\ ) = \alpha(\ )$ is decreasing. It is shown in Bental and Eden (1993) that under certain conditions a full equilibrium with decreasing α does exist.

14.5.4 Efficiency

We consider an economy with two goods (X and Y) and $N + \Delta + 1$ infinitely lived agents. As in section 14.1.2 there are three possible types of agents: sellers (S), definite buyers (DB) and possible buyers (PB). The state is an i.i.d random variable with two possible realizations that occur with equal probabilities: $s = 1$ and $s = 2$.

All agents get income (endowment) each period in the form of a large amount, $\bar{y}$, of the numeiraire commodity Y. The seller (and only the seller) can use part of their endowment of Y to produce X. It costs x^2 units of Y to produce x units of X.

The seller's single period utility function is: $u(x, y) = y$.
The definite buyers' utility function is: $u(x, y) = U(x) + y$.
The possible buyers' utility function is: $u(x, y) = \{y \text{ if } s = 1 \text{ and } U(x) + y \text{ if } s = 2\}$.

Thus the seller does not like X, definite buyers like X and possible buyers like X only in state 2.

The sum of expected utility is:[11]

$$\sum_{t=1}^{\infty} \beta^t [y_t + NU(k_{1t}/N) + (1/2)\Delta U(k_{2t}/\Delta)], \quad (14.67)$$

where k_{1t}/N is the quantity of X per DB, k_{2t}/Δ is the quantity of X per PB in state 2 and y_t is the total consumption of good Y.

The amount of inventories carried to the next period is:

$$I^1_{t+1} = x_t + I_t - k_{t1} \text{ if } s = 1 \quad \text{and} \quad I^2_{t+1} = x_t + I_t - k_{1t} - k_{2t} \text{ if } s = 2. \quad (14.68)$$

The total consumption of Y is:

$$y_t = \bar{y} - (x_t)^2. \tag{14.69}$$

The social planner maximizes (14.67) subject to (14.68), (14.69), an initial condition and non-negativity constraints.

Using $k_3 = x + I - k_1 - k_2$ for the aggregate purely speculative inventories we have: $I^1 = k_3 + k_2$ for next period's inventories if $s = 1$ and $I^2 = k_3$ for next period's inventories if $s = 2$. We can now describe the social planner's problem by the following Bellman equation.

$$V(I) = \max_{\{k_1\ k_2\ k_3\ x\}} NU(k_1/N) + \tfrac{1}{2}\Delta U(k_2/\Delta) + \bar{y} - x^2 + \tfrac{1}{2}\beta V(k_2 + k_3) + \tfrac{1}{2}\beta V(k_3)$$

$$\text{s.t.} \quad k_1 + k_2 + k_3 = I + x \quad k_1,\ k_2,\ k_3 \geq 0. \tag{14.70}$$

It is shown in Appendix 14B, that the first order conditions for a strictly positive solution to (14.70) are:

$$U'(k_2/\Delta) \geq \beta V'(k_3) \quad \text{with equality if } k_3 > 0; \tag{14.71}$$

$$U'(k_1/N) = \tfrac{1}{2}U'(k_2/\Delta) + \tfrac{1}{2}\beta V'(k_2 + k_3); \tag{14.72}$$

$$U'(k_1/N) = 2x. \tag{14.73}$$

We can also use the envelope Theorem to get:

$$V'(I) = 2x. \tag{14.74}$$

Proposition 5: The UST allocation maximizes the sum of expected utilities (14.67).

To show this proposition, note that a buyer who faces the price p chooses the quantity x by solving max $U(x) - px$. The first order condition for this problem is: $U'(x) = p$ and therefore in a UST equilibrium, $U'(k_1/N) = p_1$ and $U'(k_2/\Delta) = p_2$. Substituting this and $V' = \alpha$ in (14.71)–(14.73) leads to the equilibrium conditions (14.56)–(14.61).

PROBLEMS WITH ANSWERS

1 Show that an increase in the beginning of period inventories by a unit, leads to

(a) an increase in $k = k_1 + k_2 + k_3$ by less than a unit;
(b) an increase in $k_2 + k_3$ by less than a unit;
(c) an increase in k_3 by less than a unit.

Answer

(a) With the help of figure 14.17 we show that: $\Delta k < 1$.
(b) Since claim 4 says that all $k_s(I)$ are monotonically increasing, it follows that $\Delta k_1 > 0$. This and (a) imply $\Delta(k_2 + k_3) < 1$.
(c) Follows directly from claim 9.

2 In future markets people look at the difference (spread) between the current price and the price for delivery next period. Interpreting the current price as the price in the first market this is: $w_t = E(P_{1t+1}) - P_{1t}$. What can you say about the relationship between the spread (w_t) and the beginning of period inventories?

Hint: Assume full equilibrium and that the derivative $P_1'(I) = P_1'$ does not depend on I. Use your answer to question 1.

Answer

We define, $w(I) = (1/2)P_1[k_2(I) + k_3(I)] + (1/2)P_1[k_3(I)] - P_1(I)$. If I goes up by a unit, $k_2 + k_3$ goes up by less than a unit. Therefore the expected next period price $(1/2)P_1[k_2(I) + k_3(I)] + (1/2)P_1[k_3(I)]$ goes down, in absolute value, by less than the decline in the current period price $P_1(I)$ and the spread goes up.

This argument can be made by taking the derivative of w(I) and using the result that $k_2' + k_3' < 1$. This yields:

$$w'(I) = (1/2)P_1'(k_2' + k_3') + (1/2)P_1'k_3' - P_1'$$
$$= P_1'\{(1/2)(k_2' + k_3') + (1/2)k_3' - 1] > 0.$$

3 Analyze the effect of an increase in Δ on the *temporary* equilibrium levels of: $p_2(I)$ and $p_1(I)$.

Answer

The amount of speculative inventories that solves (14.58) for a given p_2, $k_3(p_2)$, is not affected by the change in Δ. Since $\alpha[\Delta/p_2 + k_3(p_2)]$ is decreasing in Δ, (14.62) implies that $p_1(p_2)$ is lower for any given p_2 . Therefore, (14.65) implies that production $x(p_2)$ will be lower for any given p_2 and as a result the supply schedule $s(p_2)$ will shift to the left as in figure 14.18.

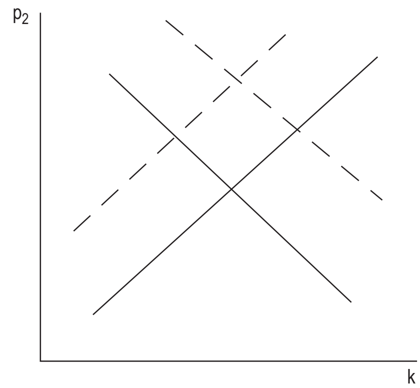

Figure 14.18 An increase in Δ

For any given p_2, an increase in Δ does not change the demand for purely speculative inventories $k_3(p_2)$. The demand in the second market Δ/p_2 goes up, and the demand in the first market $N/p_1(p_2)$ also goes up because $p_1(p_2)$ goes down. It follows that the demand curve (14.64) shifts to the right. The effect on the temporary equilibrium level $p_2(I)$ is unambiguous: It goes up.

Since $p_1(p_2)$ is down it is not clear what happens to the temporary equilibrium level $p_1(I)$.

4 Does your answer to the previous question imply that an increase in Δ will lead to an increase in the full equilibrium level of $p_2(I)$?

5 Consider the case: $\alpha(I) = 1/10I$, $I = 2.67$, $N = \Delta = 1$, $\beta = 0.9$ and $\rho = 0$.

Someone solved for *temporary* equilibrium and got: $p_2 = 1$. Check whether the solution is correct. In your answer solve for the temporary equilibrium magnitudes of k_3, k_2, p_1, k_1 and x. (I suggest to follow the above order). Is it a full equilibrium?

Answer

Since $p_2 = 1$, equation (14.58) implies that a strictly positive k_3 must satisfy: $1 = 0.9(1/10k_3)$. This equation yields: $k_3 = 0.09$. Substituting $p_2 = 1$ in (14.57) yields: $k_2 = 1$. The level of inventories if only one market opens is therefore: $k_2 + k_3 = 1.09$. The price in the next period's first market if only one market opens is: $\alpha(1.09) = 1/10.9 = 0.092$. Substituting this in (14.59) leads to: $p_1 = 0.54$. Substituting $p_1 = 0.54$ in (14.56) leads to: $k_1 = 1.85$. Total demand is therefore: $d(1) = 0.09 + 1 + 1.85 = 2.94$.

Substituting $p_1 = 0.54$ in (14.60) leads to: $x = 0.27$. Thus $k = x + I = 2.94$ which is equal to total demand. We have shown that $(p_1 = 0.54, p_2 = 1, k_3 = 0.09, k_2 = 1, x = 0.27)$ is a temporary equilibrium and therefore the suggested solution is correct.

This is not however a full equilibrium because $\alpha(2.67) = 0.03 \neq 0.54$.

6 Assume that α is decreasing and $k_3 = 0$.

(a) Show that an increase in storage cost (ρ) leads to an increase in the level of temporary equilibrium price in the second market (p_2).

(b) In a full equilibrium the function α changes with ρ. Assume that as a result of an increase in ρ the function $\alpha(I)$ changed to $\hat{\alpha}(I)$ where $\hat{\alpha}(I) < \alpha(I)$ for all I. What happens to the price p_2 in a full equilibrium as a result of the increase in ρ?

(c) Show that when storage cost is sufficiently high the ratio of prices (p_2/p_1) reaches a maximum level of 2.

Answer

(a) For any given p_2, the demand in the second market does not change as a result of an increase in ρ (it remains $k_2 = \Delta/p_2$). The price in the first market, $p_1(p_2) = p_2/2 + [\beta\alpha(\Delta/p_2) - \rho]/2$, is lower and therefore supply shifts to the left as in figure 14.19. Demand in the first market $N/p_1(p_2)$ goes up because $p_1(p_2)$ goes down. As a result the demand curve shifts to the right. Therefore the temporary equilibrium level of p_2 goes up as a result of the increase in storage cost.

(b) The effect of the change in α also works to increase the equilibrium level of p_2. Now, $p_1(p_2) = p_2/2 + [\beta\hat{\alpha}(\Delta/p_2) - \rho]/2$, is lower than before and this pushes the supply further to the left and the demand further to the right.

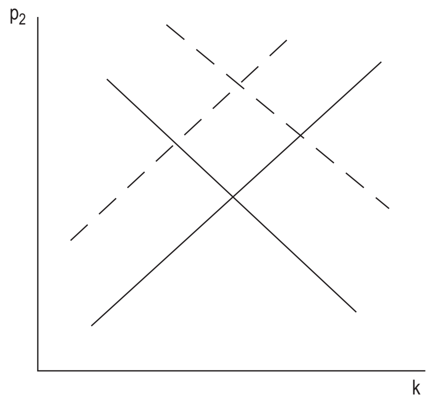

Figure 14.19 An increase in storage cost

(c) Equilibrium condition implies: $2 = p_{2t}/p_{1t} + \max[(\beta\alpha(k_{2t} + k_{3t}) - \rho), 0]/p_{1t}$. When ρ is large, storage is prohibitively expensive and $\max[(\beta\alpha(k_{2t} + k_{3t}) - \rho), 0] = 0$.

7 The result in question 6 requires $k_3 = 0$. Use the planner's problem (14.70) to discuss the effect of changes in ρ in general.

APPENDIX 14A THE FIRM'S PROBLEM

To formulate the firm's problem, let us distinguish between the level of inventories that the firm has at the beginning of the period (i) and the average per-firm level of inventories I. The price-taking firm expects that prices depend on the average per-firm level of inventories and are given by the functions: $p_s(I)$. Using this notation, the Bellman equation is:

$$\begin{aligned} V(i; I) = \max_{\{k_s, x\}} \; & p_1(I)k_1 + \tfrac{1}{2}p_2(I)k_2 - C(x) \\ & + \tfrac{1}{2}\beta V(k_2 + k_3; I^1) + \tfrac{1}{2}\beta V(k_3;\ I^2) \\ \text{s.t.} \quad & k_1 + k_2 + k_3 = i + x \quad \text{and non-negativity constraints.} \end{aligned} \tag{A14.1}$$

Here V(i, I) is the expected utility of a firm that starts with i units of inventories when the average per-firm level of inventories is I and I^s denotes the expected average inventories next period given that s markets open this period.

To solve (A14.1) we set the lagrangian:

$$\begin{aligned} L = p_1(I)k_1 + \tfrac{1}{2}p_2(I)k_2 - C(x) + \tfrac{1}{2}\beta V(k_2 + k_3; I^1) \\ + \tfrac{1}{2}\beta V(k_3; I^2) + \lambda(i + x - k_1 - k_2 - k_3). \end{aligned} \tag{A14.2}$$

We now derive the following first order conditions that a solution $k_1 > 0,\ k_2 > 0,\ k_3 \geq 0,\ x > 0$ must satisfy. These are:

$$\partial L/\partial k_1 = p_1(I) - \lambda = 0; \tag{A14.3}$$

$$\partial L/\partial k_2 = \tfrac{1}{2}p_2(I) + \tfrac{1}{2}\beta V'(k_2 + k_3; I^1) - \lambda = 0; \tag{A14.4}$$

$$\partial L/\partial k_3 = \tfrac{1}{2}\beta V(k_2 + k_3; I^1) + \tfrac{1}{2}\beta V'(k_3; I^2) - \lambda \leq 0 \tag{A14.5}$$

$$\text{with equality if } k_3 > 0;$$

$$\partial L/\partial x = -C'(x) + \lambda = 0. \tag{A14.6}$$

From (A14.4) we get $(1/2)\beta V'(k_2 + k_3; I^1) - \lambda = -(1/2)p_2(I)$. Substituting this in (A14.5) leads to:

$$\beta V'(k_3; I^2) \leq p_2(I) \quad \text{with equality if } k_3 > 0. \tag{A14.7}$$

Substituting (A14.3) into (A14.4) leads to:

$$p_1(I) = \tfrac{1}{2}p_2(I) + \tfrac{1}{2}\beta V'(k_2 + k_3; I^1). \tag{A14.8}$$

Substituting (A14.3) into (A14.6) leads to:

$$C'(x) = p_1(I). \tag{A14.9}$$

The first order conditions (A14.7)–(A14.9) are the arbitrage conditions (14.48)–(14.50) in the text where V' replaces α for the value of inventories. We now observe that at the optimum it does not matter how an additional unit of inventories is used and therefore the firm cannot do better than selling it in the next period's first market.

APPENDIX 14B THE PLANNER'S PROBLEM

To solve the maximization problem (14.60) we set the lagrangian:

$$\begin{aligned} L = {} & NU(k_1/N) + \tfrac{1}{2}\Delta U(k_2/\Delta) + \bar{y} - x^2 \\ & + \beta\left[\tfrac{1}{2}V(k_2 + k_3) + \tfrac{1}{2}V(k_3)\right] + \lambda(I + x - k_1 - k_2 - k_3) \end{aligned} \tag{B14.1}$$

We now derive the first order conditions for a solution $k_1,\ k_2,\ x > 0;\ k_3 \geq 0$. These are:

$$\partial L/\partial k_1 = U'(k_1/N) - \lambda = 0; \tag{B14.2}$$

$$\partial L/\partial k_2 = \tfrac{1}{2}U'(k_2/\Delta) + \beta\tfrac{1}{2}V'(k_2 + k_3) - \lambda = 0; \tag{B14.3}$$

$$\partial L/\partial k_3 = \beta\left[\tfrac{1}{2}V'(k_2 + k_3) + \tfrac{1}{2}V'(k_3)\right] - \lambda \leq 0 \tag{B14.4}$$

$$\text{with equality if } k_3 > 0.$$

$$\partial L/\partial x = -2x + \lambda = 0. \tag{B14.5}$$

From (B14.3) we get: $-(1/2)U'(k_2/\Delta) = \beta(1/2)V'(k_2 + k_3) - \lambda$. Substituting this in (B14.4) leads to:

$$V'(k_3) \leq U'(k_2/\Delta) \quad \text{with equality if } k_3 > 0. \tag{B14.6}$$

Substituting (B14.2) in (B14.3) leads to:

$$\tfrac{1}{2}U'(k_2/\Delta) + \beta\tfrac{1}{2}V'(k_2 + k_3) = U'(k_1/N). \quad \text{(B14.7)}$$

Substituting (B14.2) in (B14.5) leads to:

$$U'(k_1/N) = 2x. \quad \text{(B14.8)}$$

Conditions (B14.6)–(B14.8) are the same as (14.61)–(14.63) in the text.

NOTES

1 The CPI is the outcome of a survey about quoted price.

2 Dividing (14.11) by $(1 + N + \Delta)$ yields the expected utility of the representative agent at $t = 0$.

3 Jeff Campbell suggested the first question.

4 Wilson (1988) has shown that when the individual demand function is not differentiable (a step function) a monopolist may want to charge more than one price even if there is no uncertainty about the number of buyers that will arrive. Thus a monopoly may want to charge more than one price per group (market). We assume that the individual demand function, D(p) is differentiable and therefore the constraint of one price per market is not binding.

5 The first order condition for this problem is $C'(x) = \pi$ which is the same as the first order condition (14.26) for the monopoly's problem.

6 When individual demand is downward sloping, relatively more capacity is wasted in the UST model and therefore productivity should be relatively more procyclical. See Claim 1 in 14.1.1.

7 Here y is the amount paid for X. The utility could also be stated as: $U(x) - y + \bar{y}$ if he wants to consume and $\bar{y}$ otherwise, where $\bar{y}$ is the initial endowment of Y.

8 To derive these first order conditions we set the lagrangian: $L = \sum_{s=1}^{S} \zeta_s q_s[U(x_s) - y_s] + \lambda\left(\sum_{s=1}^{S} \zeta_s q_s y_s - \sum_{s=1}^{S} p_s\zeta_s q_s x_s\right)$. We then take the derivatives:

$$\partial L/\partial x_s = \zeta_s q_s U'(x_s) - \lambda p_s \zeta_s q_s = 0 \text{ and}$$
$$\partial L/\partial y_s = -\zeta_s q_s + \lambda\zeta_s q_s = 0.$$

9 The probability Π_s is the probability of state s from the actual sellers' point of view (conditional on being chosen as an actual seller). This distinction will become important later.

10 To show this claim we apply Bayes' rule and compute the probability that exactly $i > s$ markets will open given that market s opens. This is: Π_i/q_s. The probability that market i will open given that market s opens is: $\sum_{k=i}^{S} \Pi_k/q_s$. In equilibrium the unconditional expected revenue (from a unit supplied to market i) is $P_i \sum_{k=i}^{S} \Pi_k = \lambda$ and the conditional expected revenue (from a unit supplied to market i given that market s opens) is: $P_i \sum_{k=i}^{S} \Pi_k/q_s = \lambda/q_s$. Since in equilibrium $P_s = \lambda/q_s$ the opening of market s does not provide an incentive for the firm to move units from market s to market i or vice versa. Since the conditional expected revenue is λ/q_s for all $i > s$, the firm does not have an incentive to move units allocated to markets $s + 1, \ldots, S$. Thus, not surprisingly the initial plan is time consistent.

11 Note that we can divide the sum (14.79) by $(N+\Delta+1)$ to get the expected utility of the representative consumer before he knows his type.