The implications of equal value of life and prioritarianism for the evaluation of population health*

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April 14, 2014

Abstract

We analyze the implications of several principles related to the concept of equal (and prioritarian) entitlement to continued life. These principles, when modeled as axioms for the evaluation of health distributions, and combined with some other structural axioms, provide several characterization results of focal population health evaluation functions.

JEL numbers: D63, I10.

Keywords: equal value of life, priority, population health, axioms

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*Paper prepared for the Vanderbilt Rational Choice & Philosophy Conference; May 2014. Acknowledgement will be added later.

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1 Introduction

One of the central postulates of most egalitarian doctrines is the idea that every life has equal value. The idea is at the core of a wide array of philosophical, political and social discussions. It constitutes one of the central slogans of the Bill & Melinda Gates foundation, the largest private foundation in the world, and it has received strong endorsements by numerous public figures.

The concept of equal value of life has also been scrutinized in academic research. One of its strongest defenders is John Harris, who, in a series of contributions that date back to the 80’s and 90’s, argued ethical concerns for a fundamental entitlement to continued life to which all individuals are entitled to the same extent (e.g., Harris, 1985; 1987; 1996; 1997). Harris’ arguments led to the conclusion that, even if some lives are not lived at perfect health, lives are in fact equally valuable, as long as they are valued by those living those lives. Such conclusion has also been endorsed within the health economics community (e.g., Arnesen and Nord, 1999; Nord, 2001).

Another argument usually considered to defend equal value of life is the recurrent argument within political philosophy that welfare interpersonal comparisons are incommensurate, and, therefore, it is wrong to discriminate on the basis of health state. Nevertheless, such argument has been contested (e.g., Singer et al., 1995; McKie et al., 1997) and debated (e.g., Grimley Evans, 1997; Williams, 1997).

We provide in this paper a new perspective on the concept of equal value of life, in connection with the evaluation of population health. To do so, we consider the new axiomatic approach to the evaluation of population health, recently introduced by Hougaard, Moreno-Ternero and Østerdal (2013). In such approach, the health of an individual in the population is defined according to the two standard dimensions (quality of life and quantity of life), but one of them (quality of life) receives a special treatment, as it is assumed that it might not have a standard mathematical structure. The approach has the advantage that it does not make assumptions about individual preferences over length and quality of life. In doing so, we depart from the strand of the (health economics) literature in which the analysis relies on individual preferences on health and quality of life (e.g., Østerdal, 2005), and also from the popular strand of the literature in which the analysis relies on a generic individual health utility concept (e.g., Bleichrodt, 1995, 1997; Bleichrodt et al., 2004; Doctor et al., 2009), which have faced recurrent

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1See also Moreno-Ternero and Østerdal (2014).
criticisms to their conceptual foundation and elicitation procedures (e.g. Dolan 2000).

We formalize equal value of life as an axiom of social preferences for population health evaluations in the model described above. We show that the combination of such axiom with two other structural axioms (known as time monotonicity at perfect health, and the social zero condition) characterize the population health evaluation function that ranks distributions according to the unweighted aggregation (across agents in the population) of lifetimes in the distribution. Such function does not include any concern whatsoever for the quality of life at which individuals in the population enjoy those lifetimes, which is in contrast with some focal forms of evaluation for health distributions, such as the so-called Quality Adjusted Life Years (QALYs) and Healthy Years Equivalent (HYEs).² It turns out that suitable weakenings of equal value of life (restricting its scope to individuals with certain identical characteristics) in combination with a list of structural axioms (encompassing the previous two mentioned above) characterize the aggregate QALY and HYE population health evaluation functions, as shown by Hougaard, Moreno-Ternero and Østerdal (2013).

The result described above is somewhat connected to the main results in Hasman and Østerdal (2004) where, in the more usual context in the health economics literature, assuming the existence of individual preferences on quality and quantity of life, a general incompatibility between the equal value of life principle and the weak Pareto principle is established.

The idea of equal value of life, as introduced above, prevents any form of discrimination against the disabled, when it comes to allocate extra life years. We shall also be concerned in this paper with the idea that a positive discrimination in favor of the disabled might also be allowed in such allocation process. We refer to such idea as disability priority after Parfit (1997).³ More precisely, we formalize as an axiom of population health evaluation functions the idea that extra life years should not be valued more when they are awarded to agents with perfect quality of life, but not necessarily entail that they are equally valued when awarded to agents enjoying different health states (quality of life). Thus, the new (prioritarian) axiom is a weakening of the axiom of equal value of life. It turns out, nonetheless, that the combination of this new axiom with some other structural axioms also characterizes the population health evaluation function

²See e.g., Pliskin, Shepard and Weinstein (1980) and Mehrez and Gafni (1989).
³For a different analysis of the ethics of priority in resource allocation, the reader is referred to Moreno-Ternero and Roemer (2006, 2012). The conflict between priority and the veil of ignorance is exposed in Moreno-Ternero and Roemer (2008). A recent comprehensive endorsement of the prioritarian evaluation of outcomes and policies is provided by Adler (2012).
referring to the unweighted aggregation (across agents in the population) of lifetimes in the
distribution. A weakening of the disability priority axiom allows a characterization of a more
general family of population health evaluation functions in which the lifetimes in a distribution
are previously submitted to an arbitrary (albeit increasing and continuous) function.

Based on the results mentioned above, one might wonder whether all population health
evaluation functions including a concern for morbidity are also excluded when principles related
to prioritarian value of life are imposed. It turns out that is not the case as the family of
population health evaluation functions arising upon aggregating individual HYEs, after being
submitted to a concave (and increasing) function, can also be characterized resorting to an
axiom connected to the principle of priority.

The rest of the paper is organized as follows. In Section 2, we introduce the model and the
structural axioms we consider for our analysis. In Section 3, we introduce the axiom of equal
value of life and explore its implications. In Section 4, we move to extend the analysis to the
case of prioritarian (rather than equal) value of life. We conclude in Section 5. For a smooth
passage, we defer the proofs and provide them in an appendix.

2 The preliminaries

Imagine a policy maker who has to compare distributions of health for a population of fixed
size \( n \geq 3 \).\(^4\) Let us identify the population (society) with the set \( N = \{1, \ldots, n\} \). The health of
each individual in the population is described by a duplet indicating the level achieved in two
parameters: quality of life and quantity of life. Assume that there exists a set of possible health
states, \( A \), defined generally enough to encompass all possible health states for everybody in
the population. We emphasize that \( A \) is an abstract set without any particular mathematical
structure. Quantity of life is simply described by a set of nonnegative real numbers, \( T \subset \mathbb{R} \).
In what follows, we assume that \( T = [0, +\infty) \). Formally, let \( h_i = (a_i, t_i) \in A \times T \) denote the
health duplet of individual \( i \). A population health distribution (or, simply, a health profile)
\( h = [h_1, \ldots, h_n] = [(a_1, t_1), \ldots, (a_n, t_n)] \) specifies the health duplet of each individual in society.
We denote the set of all possible health profiles by \( H \).\(^5\) Even though we do not impose a specific
mathematical structure on the set \( A \), we assume that it contains a specific element, \( a_* \), which
we refer to as perfect health and which is univocally identified, as a “superior” state, by the

\(^4\)The content of this section follows Hougaard, Moreno-Ternero and Østerdal (2013).
\(^5\)For ease of exposition, we establish the notational convention that \( h_S \equiv (h_i)_{i \in S} \), for each \( S \subset N \).
policy maker.

The policy maker’s preferences (or social preferences) over health profiles are expressed by a preference relation \( \succeq \), to be read as “at least as preferred as”. As usual, \( \succ \) denotes strict preference and \( \sim \) denotes indifference. We assume that the relation \( \succeq \) is a weak order.\(^6\)

A population health evaluation function (PHEF) is a real-valued function \( P : H \to \mathbb{R} \). We say that \( P \) represents \( \succeq \) if
\[
P(h) \geq P(h') \iff h \succeq h',
\]
for each \( h, h' \in H \). Note that if \( P \) represents \( \succeq \) then any strictly increasing transformation of \( P \) would also do so.

### 2.1 Basic structural axioms

We now list a set of basic axioms for social preferences that we shall consider in this paper.\(^7\)

**Anonymity** (in short, ANON) says that the evaluation of the population health should depend only on the list of quality-quantity duplets, not on who holds them.

**ANON**: \( h \sim h_\pi \) for each \( h \in H \), and each \( \pi \in \Pi^N \).

**Separability** (in short, SEP) says that if the distribution of health in a population changes only for a subgroup of agents in the population, the relative evaluation of the two distributions should only depend on that subgroup.

**SEP**: \([h_S, h_{N\setminus S}] \succeq [h'_S, h'_{N\setminus S}] \iff [h_S, h'_{N\setminus S}] \succeq [h'_S, h'_{N\setminus S}]\), for each \( S \subseteq N \), and \( h, h' \in H \).

**Continuity** (in short, CONT) says that, for fixed distributions of health states, small changes in lifetimes should not lead to large changes in the evaluation of the population health distribution.

**CONT**: Let \( h, h' \in H \), and \( h^{(k)} \) be a sequence in \( H \) such that, for each \( i \in N \), \( h_i^{(k)} = (a_i, t_i) \rightarrow (a_i, t_i) = h_i \). If \( h^{(k)} \succeq h' \) for each \( k \) then \( h \succeq h' \), and if \( h' \succeq h^{(k)} \) for each \( k \) then \( h' \succeq h \).

**Perfect health superiority** (in short, PHS) says that replacing the health status of an agent by that of perfect health, ceteris paribus, cannot worsen the evaluation of the population health.

**PHS**: \([ (a_*, t_1), h_{N\setminus\{i\}} ] \succeq h \), for each \( h = [h_1, \ldots, h_n] \in H \) and \( i \in N \).

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\(^6\)More precisely, we assume that \( \succeq \) is complete (for each health profiles \( h, h' \), either \( h \succeq h' \), or \( h' \succeq h \), or both) and transitive (if \( h \succeq h' \) and \( h' \succeq h'' \) then \( h \succeq h'' \)).

\(^7\)The reader is referred to Hougaard, Moreno-Ternero and Østerdal (2013) for further discussion of the axioms.
Time monotonicity at perfect health (in short, TMPH) says that if each agent is at perfect health, increasing the time dimension is strictly better for society.

**TMPH:** If \( t_i \geq t'_i \), for each \( i \in N \), with at least one strict inequality, then \( [(a_s, t_1), \ldots, (a_s, t_n)] \succ [(a_s, t'_1), \ldots, (a_s, t'_n)] \).

Positive lifetime desirability (in short, PLD) says that society improves if any agent moves from zero lifetime to positive lifetime (for a given health state).

**PLD:** \( h \succ [h_{N \setminus \{i\}} (a_i, 0)] \), for each \( h = [h_1, \ldots, h_n] \in H \) and \( i \in N \).

Finally, the social zero condition (in short, ZERO) says that if an agent gets zero lifetime, then her health state does not influence the social desirability of the health distribution. Formally,

**ZERO:** For each \( h \in H \) and \( i \in N \) such that \( t_i = 0 \), and \( a'_i \in A \), \( h \sim [h_{N \setminus \{i\}} (a'_i, 0)] \).

In what follows, we refer to the set of axioms introduced above as the basic structural axioms (in short, BASIC).

### 3 Equal value of life

We start this section adding to the previous list of basic structural axioms the axiom modeling the notion of equal value of life, discussed above. In words, **Equal Value of Life** says that a certain amount of additional life years to individual \( i \) is socially seen as just as good as the same amount of additional life years to individual \( j \), regardless of health states.\(^8\)

**EVL:** For each \( h \in H \), \( c > 0 \), and \( i, j \in N \),

\[
[(a_i, t_i + c), (a_j, t_j), h_{N \setminus \{i,j\}}] \sim [(a_i, t_i), (a_j, t_j + c), h_{N \setminus \{i,j\}}] .
\]

Our first result exhibits the strength of the notion of equal value of life. More precisely, Theorem 1 shows that it suffices to combine it with only two of the structural axioms described above to characterize the so-called aggregate lifetime PHEF, which evaluates population health distributions by means of the aggregate lifetime the distribution yields. Formally,

\[
P^t[h_1, \ldots, h_n] = P^t[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^n t_i .
\]  

\(^8\)Hasman and Østerdal (2004) define a similar axiom in their model.
Thus, $P_t$ reflects the traditional view for the evaluation of the impact of health care only in terms of its effect on mortality. As the next result shows, this PHEF is characterized by the combination of the axioms of equal value of life, time monotonicity at perfect health, and the social zero condition.

**Theorem 1** The following statements are equivalent:

1. $\succeq$ is represented by a PHEF satisfying (1).

2. $\succeq$ satisfies EVL, TMPH, and ZERO.

We also state the next trivial corollary, which will be useful for the ensuing discussion.

**Corollary 1** The following statements are equivalent:

1. $\succeq$ is represented by a PHEF satisfying (1).

2. $\succeq$ satisfies BASIC and EVL.

An obvious limitation of $P_t$ is its total neglect of morbidity as a relevant factor for the evaluation of population health. The most widely employed way of combining the quality of life and quantity of life derived from a particular health care intervention is by means of QALYs. The following PHEF, which we call (aggregate) QALY, evaluates population health distributions by means of the unweighted aggregation of individual QALYs in society, or, in other words, by the weighted (through health levels) aggregate lifetime the distribution yields. Formally,

$$P_q[h_1, \ldots, h_n] = P_q[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} q(a_i) t_i,$$

(2)

where $q : A \to [0, 1]$ is a function satisfying $0 < q(a_i) \leq q(a_*) = 1$, for each $a_i \in A$.

It turns out, as stated in the next result (whose proof can be found in Hougaard, Moreno-Ternero and Østerdal, 2013), that this family is characterized when the basic set of structural axioms is added to the weakening of equal value of life obtained when restricting its scope to agents enjoying equal health. Formally,

**TICH**: For each $h \in H$, $c > 0$, and $i, j \in N$, such that $a_i = a_j = a$,

$$[(a, t_i + c), (a, t_j), h_{N\{i,j\}}] \sim [(a, t_i), (a, t_j + c), h_{N\{i,j\}}].$$

**Proposition 1** (Hougaard, Moreno-Ternero and Østerdal, 2013) The following statements are equivalent:
1. $\succeq$ is represented by a PHEF satisfying (2).

2. $\succeq$ satisfies BASIC and TICH.

An alternative way of combining the quality of life and quantity of life derived from a particular health care intervention is by means of healthy years equivalents (HYEs). The next PHEF, which we call (aggregate) HYE, evaluates population health distributions by means of the aggregation of individuals’ HYEs. Formally,

$$P^h[h_1, \ldots, h_n] = P^h[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} f(a_i, t_i), \quad (3)$$

where $f : A \times T \to T$ is a function indicating the HYEs for each individual, i.e., for each $h = [h_1, \ldots, h_n] = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$, and each $i \in N$,

$h \sim [(a_*, f(a_i, t_i))]_{i \in N}$.

It is worth mentioning that the (aggregate) QALY PHEF can therefore be seen as a specific instance of the (aggregate) HYE PHEF, in which $f(a_i, t_i) = q(a_i)t_i$, for each $(a_i, t_i) \in A \times T$.

As shown by the next result (whose proof can also be found in Hougaard, Moreno-Ternero and Østerdal, 2013) the HYE PHEF is characterized by adding to the basic structural axioms the further weakening of equal value of life to the case in which all agents enjoy perfect health.

**TIPH**: For each $h \in H$, $c > 0$, and $i, j \in N$, such that $a_i = a_j = a_*$,

$$[(a_*, t_i + c), (a_*, t_j), h_{N \setminus \{i,j\}}] \sim [(a_*, t_i), (a_*, t_j + c), h_{N \setminus \{i,j\}}].$$

**Proposition 2** (Hougaard, Moreno-Ternero and Østerdal, 2013) The following statements are equivalent:

1. $\succeq$ is represented by a PHEF satisfying (3).

2. $\succeq$ satisfies BASIC and TIPH.

One might find compelling reasons to argue that Theorem 1 (and, for that matter, Corollary 1) can be seen as a prior stage to an impossibility result. As mentioned above, there is an obvious limitation of the population health evaluation function characterized therein ($P^t$), as it neglects morbidity as a relevant factor for the evaluation of population health. In some sense, the last two results could be seen as “ways out” from that impossibility. They both show that focal
population health evaluation functions (including a concern for quality of life), are singled out, provided one is willing to weaken the scope of the principle of equal value of life (while still endorsing the basic structural axioms described above). This is a somewhat frequent route in economics, as it has recently been highlighted by Weymark (2013). For instance, this is the route taken by Fleurbaey and Maniquet (2011) to partly motivate their theory of fairness and social welfare. More precisely, they show that, in standard problems of resource allocation in economic environments, requiring that a transfer of resources from one agent to another agent, who is assigned less of all commodities, be a social improvement, turns out to be incompatible with standard versions of the Pareto principle. Then, they opt out from this dilemma redefining resource equality requirements (mostly restricting the scope of transfer axioms) to make them compatible with the Pareto principle and, ultimately, characterizing social ordering functions (according to their parlance).

4 Prioritarian value of life

We propose in this section an alternative to the previous analysis around the concept of equal value of life. For that matter, we formalize a prioritarian view for the entitlement to continued life.

We begin formalizing the axiom of disability priority, which says that a certain amount of additional life years to an individual not enjoying perfect health is socially seen at least as good as the same amount of additional life years to an individual enjoying perfect health. Formally,

\[ (a^*, t_i), (a_j, t_j + c), h_{N \setminus \{i,j\}} \succeq [(a^* + c), (a_j, t_j), h_{N \setminus \{i,j\}}]. \]

A weakening of the previous notion refers to the fact in which the principle is only applied to the status quo of zero lifetimes. Formally,

\[ (a^*, 0), (a_j, c), h_{N \setminus \{i,j\}} \succeq [(a^*, c), (a_j, 0), h_{N \setminus \{i,j\}}]. \]

It turns out that adding the previous axiom to the set of basic structural axioms, we characterize a general form of lifetime aggregation in which lifetimes are submitted to an arbitrary increasing function. More precisely, we define the generalized aggregate lifetime PHEF as the
PHEF that evaluates population health distributions by means of the aggregate value obtained when all the lifetimes the distribution yields are submitted to an increasing function. Formally,

\[ P^{gt}[h_1, \ldots, h_n] = P^{gt}[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} g(t_i), \quad (4) \]

where \( g : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a strictly increasing and continuous function.

**Theorem 2** The following statements are equivalent:

1. \( \succsim \) is represented by a PHEF satisfying (4).
2. \( \succsim \) satisfies BASIC and WDP.

If the full-fledged version of the disability priority axiom is considered, instead of its weakening, we recover the canonical lifetime aggregation PHEF, as shown in the next result.

**Theorem 3** The following statements are equivalent:

1. \( \succsim \) is represented by a PHEF satisfying (1).
2. \( \succsim \) satisfies BASIC and DP.

An obvious implication from the previous theorem is that, even though equal value of life is, in principle, a stronger axiom than disability priority, both become equivalent under the presence of the basic structural axioms.

An alternative axiom, which is also somewhat connected to the idea of prioritarianism, albeit not logically related to the previous ones, is **priority to lower lifetimes at perfect health**. It says that, among agents at perfect health, we prioritize those with lower lifetimes when it comes to allocate extra additional life years. Formally,

**PLPH:** For each \( c > 0, h \in H \), and \( i, j \in N \), such that \( a_i = a_j = a_* \), and \( t_i > t_j \),

\[ [(a_*, t_i), (a_*, t_j + c), h_{N \setminus \{i,j\}}] \succsim [(a_*, t_i + c), (a_*, t_j), h_{N \setminus \{i,j\}}] . \]

Let us define the **concave aggregate HYE** PHEF, which evaluates population health distributions by means of the aggregate value the distribution yields, after submitting each individual HYE to a concave function. Formally,

\[ P^{\hat{g}}[h_1, \ldots, h_n] = P^{\hat{g}}[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} \hat{g}(f(a_i, t_i)), \quad (5) \]
where $\hat{g}: \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, concave, and continuous function, and $f: A \times T \to T$ is a function indicating the HYEs for each individual, as described in (3). We have the following result:

**Theorem 4** The following statements are equivalent:

1. $\succeq$ is represented by a PHEF satisfying (5).
2. $\succeq$ satisfies BASIC and PLPH.

## 5 Discussion

We have explored in this paper the implications of the principle of equal value of life, which conveys an equal entitlement to additional life years, in the context of the evaluation of health distributions. Our main result shows the strength of that principle as its combination with two weak structural axioms (one stating the appeal of enjoying more life years at perfect health; another indicating the irrelevance of the health status when there is no expected lifetime to enjoy it) leads to evaluating health distributions by the aggregate lifetime they offer, dismissing any concern whatsoever for the morbidity associated to health distributions.

We have then explored several ways out from that strong result. One amounted to weaken the scope of the principle to individuals sharing some characteristics. In doing so, we are able to recover more general population health evaluation functions including, not only a concern for mortality, but also a concern for morbidity. Instances are the functions arising after aggregating quality adjusted life years, or healthy years equivalent, rather than just life years.

Another related alternative we have explored is what we called the principle of prioritarian value of life, in which disabled individuals are prioritized in the allocation of additional life years. Several axioms formalizing that principle have been considered. Some turn out to exhibit equally strong implications as equal value of life, at least under the presence of structural axioms. Others have weaker implications leading to characterize more general population health evaluation functions, such as general aggregation of lifetimes, or concave aggregation of healthy years equivalent.

To conclude, it is worth mentioning that our work has been set in a context without uncertainty. In other words, and following Broome (1993), we consider a formulation of the population health evaluation problem which contains no explicit element of risk, and in which we obtain characterizations of population health evaluation functions without assumptions on
the policy maker’s (or individuals’) risk attitudes. It is left for further research to extend our analysis in that direction. In that sense, it is worth mentioning that Adler, Hammit and Treich (2014) have recently examined how different welfarist frameworks evaluate the social value of mortality risk reduction. More precisely, they have discussed whether cost-benefit analysis and various social welfare functions (including several forms of prioritarianism) satisfy the equal value of risk reduction.
Appendix: Proof of the results

Proof of Theorem 1

We focus on its non-trivial implication, i.e., $2 \rightarrow 1$. Formally, assume $\succeq$ satisfies EVL, ZERO and TMPH. Let $P$ be a PHEF representing $\succeq$ and let $h = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$. By iterated application of EVL, and the transitivity of $\succeq$,

$$h \sim [(a_1, t_1 + \ldots + t_n), (a_k, 0)_{k \neq 1}].$$

By iterated application of ZERO, and the transitivity of $\succeq$,

$$[(a_1, t_1 + \ldots + t_n), (a_k, 0)_{k \neq 1}] \sim [(a_1, t_1 + \ldots + t_n), (a_1, 0)].$$

By EVL,

$$[(a_1, t_1 + \ldots + t_n), (a_1, 0)] \sim [(a_1, 0), (a_1, t_1 + \ldots + t_n), (a_1, 0)].$$

By ZERO,

$$[(a_1, 0), (a_1, t_1 + \ldots + t_n), (a_1, 0)] \sim [(a_1, 0), (a_1, t_1 + \ldots + t_n), (a_1, 0)].$$

Finally, by EVL,

$$[(a_1, 0), (a_1, t_1 + \ldots + t_n), (a_1, 0)] \sim [(a_1, 0), (a_1, t_1 + \ldots + t_n), (a_1, 0)].$$

Altogether, by the transitivity of $\succeq$, we obtain,

$$h \sim [(a_1, t_1 + \ldots + t_n), (a_1, 0)],$$

from which we conclude that $\succeq$ depends only on $t_1 + \ldots + t_n$.

Let now $h' = [(a_1', t_1'), \ldots, (a_n', t_n')] \in H$. By the above argument,

$$h' \sim [(a_1', t_1' + \ldots + t_n'), (a_1, 0)].$$

Thus, by TMPH,

$$h' \sim [(a_1, t_1' + \ldots + t_n'), (a_1, 0)] \succeq [(a_1, t_1 + \ldots + t_n), (a_1, 0)] \sim h.$$

if and only if

$$\sum_{i=1}^{n} t_i' \geq \sum_{i=1}^{n} t_i.$$

Thus, the transitivity of $\succeq$ concludes. \hfill \square
Proof of Theorem 2

We focus on the non-trivial implication, i.e., $2 \rightarrow 1$. Formally, assume $\succeq$ satisfies BASIC and WDP. Then, by Theorem 1 in Hougaard, Moreno-Ternero and Østerdal (2013), $\succeq$ can be represented by a separable PHEF, i.e.,

$$
P^s[h_1, \ldots, h_n] = P^s[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} g(f(a_i, t_i)),
$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a strictly increasing and continuous function, and $f : A \times T \rightarrow T$ is a function indicating the HYEs for each individual, i.e.,

- $f$ is continuous with respect to its second variable,
- $0 \leq f(a_i, t_i) \leq t_i$, for each $(a_i, t_i) \in A \times T$, and
- For each $h = [h_1, \ldots, h_n] = [(a_1, t_1), \ldots, (a_n, t_n)] \in H$,

$$
h \sim [(a_\ast, f(a_i, t_i))_{i \in N}].
$$

Let $c > 0$, $h \in H$, and $j \in N$ be such that $a_j \neq a_\ast$. By WDP,

$$
[(a_\ast, 0), (a_j, c), h_{N\setminus(i,j)}] \succeq [(a_\ast, c), (a_j, 0), h_{N\setminus(i,j)}].
$$

Equivalently,

$$
g(f(a_\ast, 0)) + g(f(a_j, c)) \geq g(f(a_\ast, c)) + g(f(a_j, 0)),
$$

i.e.,

$$
g(f(a_j, c)) \geq g(c),
$$

which, in combination with the condition $0 \leq f(a_j, c) \leq c$ (expressed in the definition of $f$ stated above), and the fact that $g$ is an increasing function, leads to the fact that

$$
f(a_j, c) = c,
$$

for each $c > 0$ and $a_j \in A$. By definition, $f(a_j, 0) = 0$, for each $a_j \in A$. Altogether, we obtain that $\succeq$ can be represented by the following PHEF:

$$
P^g[h_1, \ldots, h_n] = P^g[(a_1, t_1), \ldots, (a_n, t_n)] = \sum_{i=1}^{n} g(t_i),
$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a strictly increasing and continuous function, as desired. □
Proof of Theorem 3
We focus on the non-trivial implication, i.e., $2 \rightarrow 1$. Formally, assume $\succeq$ satisfies BASIC and DP. Then, by Theorem 2, $\succeq$ can be represented by a generalized lifetime aggregation PHEF. Now, let $h \in H$, and $i, j \in N$, such that $a_j \neq a_* = a_i$. Then, by DP,

$$[(a_*, t_i), (a_j, t_j + c), h_{N \setminus \{i, j\}}] \succeq [(a_*, t_i + c), (a_j, t_j), h_{N \setminus \{i, j\}}],$$

which translates into

$$g(t_i) + g(t_j + c) \geq g(t_i + c) + g(t_j),$$

for each $t_i, t_j \in T$, and $c > 0$. It then follows that, for each $c > 0$, $g(x + c) - g(x)$ is constant with respect to $x$, which implies that $g$ is an affine function, as desired. \hfill \square

Proof of Theorem 4
We focus on the non-trivial implication, i.e., $2 \rightarrow 1$. Formally, assume $\succeq$ satisfies BASIC and PLPH. Then, as mentioned in the proof of Theorem 2, $\succeq$ can be represented by a separable PHEF, as in (6). Now, let $c > 0$, $h \in H$, and $i, j \in N$, such that $a_i = a_j = a_*$, and $t_i > t_j$. By PLPH,

$$[(a_*, t_i), (a_*, t_j + c), h_{N \setminus \{i, j\}}] \succeq [(a_*, t_i + c), (a_*), h_{N \setminus \{i, j\}}].$$

Equivalently,

$$g(f(a_*, t_i)) + g(f(a_*, t_j + c)) \geq g(f(a_*, t_i + c)) + g(f(a_*, t_j)), $$

i.e.,

$$g(t_j + c) - g(t_j) \geq g(t_i + c) - g(t_i), \quad \text{for each } c > 0, \text{ and } t_i > t_j.$$

As $g$ is continuous, in order to conclude the proof from here, it suffices to show that $g$ is midpoint concave, i.e., for each, $x, y \in T$, $g(x) + g(y) \leq 2g\left(\frac{x+y}{2}\right)$. To show this, let $x, y \in T$ and, without loss of generality, assume that $x < y$. Then, if we set $t_j = x$, $t_i = \frac{x+y}{2}$, and $c = \frac{y-x}{2}$ in the condition above, it follows that

$$g\left(\frac{x+y}{2}\right) - g(x) \geq g(y) - g\left(\frac{x+y}{2}\right).$$

Or, equivalently,

$$2g\left(\frac{x+y}{2}\right) \geq g(x) + g(y),$$

as desired. \hfill \square
References


