A CONTROL DESIGN METHOD FOR SWITCHING SYSTEMS
WITH APPLICATION TO PNEUMATIC SERVO SYSTEMS

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ABSTRACT
The motivation for this work is to formulate a control design method for a pneumatic system with discrete input values (resulting from the use of binary or three-position solenoid valves) that addresses the time-delay that is often associated with the flow state of solenoid on/off valves. This method is shown to be applicable to a general class of nonlinear, non-autonomous systems subject to possibly discontinuous governing dynamics. This method is then extended for time-delayed systems and applied to the case of LTI models with a finite number of allowed input values possessing a pure time delay. A pneumatic inertial positioning system employing binary solenoid valves exhibiting a time delay is modeled, and the control design method proposed is applied. An experimental implementation of the resulting control law for accurate position tracking utilizing only position feedback demonstrates the effectiveness of the method.

1. INTRODUCTION
Servo control of pneumatic systems is generally implemented via the proportional control of servovalves, in which pneumatic fluid flow is controlled via proportional constriction (i.e., throttling). This proportional control approach has been studied by several researchers, including Shearer [1], Liu and Bobrow [2], Kunt and Singh [3], Ye et al. [4], and Bobrow and McDonell [5]. The control of such systems is complicated by the need to accurately describe the flow characteristics of a proportional servovalve, which is typically accomplished empirically with a set of pressure/flow relationships.

Switching control, via binary-position solenoid valves, offers an alternative to the servovalve control approach that both circumvents the accurate dynamic characterization of throttling a fluid flow, and avoids the high cost of proportional servovalves. Though previous research has demonstrated the viability of servocontrol of pneumatic actuators via solenoid on/off valves [6-10], they have in general been marked by a lack of a rigorous analytical approach with which to design and analyze such a system. Exceptions include the work by van Varseveld and Bone [11], Barth et al. [12, 13] and Paul et al. [14]. The work by van Varseveld and Bone [11] experimentally develops a discrete-time model of a PWM-controlled pneumatic servo system with an autoregressive system identification approach, then applies discrete-time control methods to develop a controller. Barth et al. [12, 13] provide a method to transform the non-analytic description of PWM-based control of pneumatic systems into an analytic model, which in turn enables the use of conventional analytical control approaches such as frequency domain design and sliding mode control. The work by Paul et al. [14] formulates a controller that directly switches the valves by evaluating the derivative of the chosen Lyapunov function. The work by Paul et al. [14] is similar to the method presented here, in that it is a special case. The work by Paul et al. is extended here to include a control design method for a more general class of system models including nonlinear, non-autonomous switching models which need not be affine in the control variable, as well as for the case of time-delayed LTI systems. Paul et al. showed experimental results for position step commands, whereas sinusoidal tracking is demonstrated here.

The motivation for this work is to formulate a control design method for a pneumatic system with discrete input values (resulting from the use of binary or three-position solenoid valves) that addresses the time-delay that is often associated with the flow state of solenoid on/off valves. The resulting control design method, as applied to pneumatic systems, avoids the use of costly proportional servovalves and
pressure sensors (the two most expensive components of a pneumatic servo system) while achieving accurate tracking. Though this work is motivated by pneumatic servo control, a general control design methodology for switching systems with a finite set of discrete input values is presented that provides a stability guarantee. This control design method is applicable to general nonlinear, non-autonomous, discontinuous system models (as a result of potentially both discontinuous finite valued inputs, as well as discontinuous finite valued governing dynamics) that have no time-delay associated with the control variables. Based on this, a control design method is then presented for LTI models that have a finite set of allowable control inputs with a pure time delay. Although not presented here, the method for LTI switching systems with time delay can be extended to the more general case of nonlinear, non-autonomous, discontinuous system models with a time delay by using an appropriate nonlinear model-based predictor. To illustrate the control design method, the tracking control of single degree-of-freedom pneumatic inertial positioning system is formulated and experimentally implemented.

2. CONTROL DESIGN FOR SWITCHING SYSTEMS

Assume the plant dynamics are given in the following form,

\[
\frac{d^n}{dt^n} x = f(x) + b(x)u
\]  

(1)

where the state vector is \(x^T = [x, \dot{x}, \ldots, x^{(n-1)}]\), and the exponent term enclosed in parentheses \(\bullet^{(n)}\) is a shorthand notation for the \(n^{th}\) derivative with respect to time. Define the following continuous positive definite Lyapunov function:

\[
V = \frac{1}{2} s^2
\]  

(2)

Choose the typical integral sliding surface as,

\[
s = \left( \frac{d}{dt} + \lambda \right)^n \int_0^t \dot{s} dt = \lambda^n \int_0^t \dot{s} dt + \sum_{r=0}^{n-1} \binom{n}{r} \lambda^r \frac{d^{n-r-1}}{dt^{n-r-1}} e
\]  

(3)

where \(e = x - x_d\). Note that \(s\) is a measurable quantity if \(\frac{d^i}{dt^i} x_d\) for \(i = 0, 1, \ldots, n\) is known. Stability in the Lyapunov sense is guaranteed and the error is driven to zero when the following condition is enforced:

\[
\dot{V} = s \dot{s} \leq 0
\]  

(4)

Substituting in the plant dynamics, Equation (1), \(s\) may be expressed as the summation of a candidate term dependent upon the selection of the input, and a measurable non-candidate term independent of the input:

\[
\dot{s} = b(x)u + f(x) - \frac{d^n}{dt^n} x_d + \sum_{r=1}^{n} \binom{n}{r} \lambda^r \frac{d^{n-r}}{dt^{n-r}} e
\]  

(5)

If the input to the plant is only allowed to be a finite set of \(p\) discrete values,

\[
u \in \{u_1, u_2, \ldots, u_p\}
\]  

(6)

each input value \(u_i\) will have an associated \(\dot{V}_i\) term:

\[\dot{V}_i = s(e^{(n-1)}, \ldots, e) \dot{s}(u_i, x, x_{d}^{(e)}, e^{(n-1)}, \ldots, e)\]

for \(i = 1, 2, \ldots, p\)

(7)

The stability and error convergence condition given by Equation (4) may be enforced by selecting an input value \(u_i\) with an associated \(\dot{V}_i \leq 0\). Given a finite set of allowable input values, each \(\dot{V}_i\) term can be computed on-line in real-time and \(u\) may be selected according to the following control law:

\[
u = u_i \text{ such that } \dot{V}_i = \max_{j=1,2,\ldots,p} \dot{V}_j \text{ and } \dot{V}_i \leq 0 \text{ for } i = 1, 2, \ldots, p
\]  

(8)

That is, the \(u_i\) corresponding to the least negative \(\dot{V}_i\) is selected. In one sense, this control law can be viewed as traditional sliding mode control without the equivalent control term. In a more general sense, it is a method that directly enforces the stability and error convergence condition by selecting trial inputs and evaluating the expected \(\dot{V}\). In this sense, as will be seen below, the method is applicable to more general dynamic systems which are not affine in the control variable due to the fact that the control is not being “solved for”. Implementation of such a control law would most likely be impractical for systems of order greater than three due to measurement noise associated with the higher derivative terms contained in Equation (7). That being said, most pneumatic systems can be adequately modeled with a third order system, and therefore avail themselves to such a control technique (if being controlled by discrete-position valves).

The method presented above can be further generalized whereby multiple valves with multiple discrete positions may be incorporated into the control law. This can be accomplished by forming all possible permutations of the discrete positions of the valves and assigning each permutation a candidate term \(\dot{V}_i\),...
as dictated by the plant dynamics corresponding to those valve positions. The selection of a correct permutation of discrete input values is then provided by extending the control law given by Equation (8) to the multiple input case. In the case where the plant dynamics are different for each permutation, the plant can more generally be given by,

\[
\frac{d^n}{dt^n} x = f_j(x, u), \quad \forall u \in \{u_i : i = 1, 2, \ldots, p\} \tag{1a}
\]

where \( \dot{s} \) is then given more generally as:

\[
\dot{s} = f_j(x, u) - \frac{d^n}{dt^n} x_d + \sum_{r=1}^{n} \binom{n}{r} \dot{x}^r \frac{d^{n-r}}{dt^{n-r}} e \tag{5a}
\]

The permutations of the discrete-valued multiple inputs form the following set, of which the selected input is a member:

\[
u \in \{u_1, u_2, \ldots, u_p\} \tag{6a}
\]

Each input vector \( u_i \) has an associated \( \dot{V}_i \) term:

\[
\dot{V}_i = s(e^{(n-1)}, \ldots, e) \dot{s}(u_i, x, x_d^{(n)}, e^{(n-1)}, \ldots, e) \tag{7a}
\]

for \( i \in \{1, 2, \ldots, p\} \)

The control \( u \) may be selected according to the control law:

\[
u = u_i \quad \text{such that } \dot{V}_i = \max_{j=1,2,\ldots,p} \left( \dot{V}_j \right) \quad \text{and } \dot{V}_i \leq 0 \tag{8a}
\]

for \( i \in \{1, 2, \ldots, p\} \)

The dynamic system given by Equation (1a) is a switching system not only in the sense that the inputs may switch discontinuously between discrete values, but also that the system dynamics \( f_j(x, u) \) may switch discontinuously as well. Also note that although \( x^{(n)} \) is discontinuous, it is assumed to be finite thereby making \( x \) and therefore \( V \) continuous, as required for the stability analysis.

### 3. CONTROL DESIGN FOR LTI SWITCHING SYSTEMS WITH TIME DELAY

For pneumatic systems whereby control is achieved with discrete position valves (such as binary or three-position, three-way solenoid valves), a time delay in the response of the valve is often observed as an inherent property of the configuration of such valves. To apply the above control design procedure to systems with a pure time delay, a model-based predictor is required to estimate future values of \( \dot{V}_i \) for the selection of the present input value. Consider a dynamic system of the form,

\[
\frac{d^n}{dt^n} x(t) = f_j(x(t), u(t - T_D)), \quad \forall u \in \{u_i : i = 1, 2, \ldots, p\} \tag{9}
\]

where \( T_D \) is the pure delay in the system. The estimated derivative of the Lyapunov function can be put in the following form,

\[
\dot{V}_i(t + T_D) = \dot{s}(t + T_D) \dot{s}(t, t + T_D) \tag{10}
\]

where estimates \( \dot{s} \) and \( \dot{s} \) are given more explicitly as:

\[
\dot{s}(t + T_D) = \dot{s}(e^{(n-1)}(t + T_D), \ldots, e_t(t + T_D)) \tag{11}
\]

\[
\dot{s}(t, t + T_D) = \dot{s}(u_j(t), x(t) + e_t(t + T_D), e^{(n-1)}(t + T_D), \ldots, e_t(t + T_D)) \tag{12}
\]

Therefore, in order to select the current input \( u(t) \) at time \( t \) that will result in \( \dot{V}_i(t + T_D) \leq 0 \) at the future time \( t + T_D \), it is necessary to estimate the future states \( \dot{x}(t + T_D) \), and necessary to have available the future desired states \( x_d(t + T_D) \).

Although estimates of future states may be made for the general nonlinear system(s) of Equation (9), consider here the SISO case where the system dynamics are given by the following LTI model:

\[
\dot{x}(t) = Ax(t) + Bu(t - T_D) \tag{13}
\]

To obtain a model-based prediction of future states \( \dot{x}(t + T_D) \) starting from the present measurable state \( \dot{x}(t) \), the time-domain solution of Equation (13) with an initial condition \( x(t) \) can be written as:

\[
\dot{x}(t + T_D) = e^{A_{T_D}} x(t) + \int_0^{T_D} e^{A(t - \tau)} Bu(t - T_D) d\tau \tag{14}
\]

It is important to note that the prediction given by Equation (14) involves only past values of the input, measurable quantities \( x(t) \), and \( e^{A_{T_D}} \) which can be computed off-line for a known time delay. The present control \( u(t) \) is selected by Equations (10-12), Equation (14), and the following control law:
\[ \mathbf{u}(t) = \mathbf{u}_i(t) \]

such that

\[ \dot{V}_i(t + T_D) = \max_{j=0,1,2,...,p} \left( \dot{V}_i(t + T_D) \right) \]

and \( \dot{V}_i(t + T_D) \leq 0 \) for \( i = \{1,2,\ldots,p\} \)

Although not presented here, the above method for LTI switching systems with time delay can be extended to the more general case of nonlinear, non-autonomous, discontinuous system models (Equation (9)) with time delay by using an appropriate nonlinear model-based predictor.

4. MODELING A PNEUMATIC POSITIONING SYSTEM CONTROLLED VIA SWITCHING VALVES

Consider the single degree-of-freedom pneumatic actuation system shown in Figure 1. This inertial positioning system with discrete input values will be used to exemplify the control design procedure proposed. A lumped-parameter model that treats all unmodeled forces, such as coulomb and viscous friction and external disturbance forces, as a single disturbance term is given as,

\[ M \ddot{x} = P_A A_b - P_d A_d + F_{\text{disturbance}} \]

where \( P_A \) and \( P_b \) are the pressures (gage) inside chambers A and B of the pneumatic actuator respectively and \( A_d \) and \( A_b \) are the areas of the piston seen by each chamber. The pressure in each chamber will be controlled by a two-position valve that serves to connect the chamber to either a high pressure supply or atmospheric pressure.

\[ \dot{P} = \frac{\gamma mRT - \gamma P \dot{V}}{V} \]

where \( \gamma \) is the ratio of specific heats. A more general polytropic model is also of the form of Equation (17) where \( \gamma \) becomes the polytropic exponent. Pursuant to modeling a switching system, it will only be necessary to model the pressure dynamics of each chamber in response to connecting the chamber to either the high-pressure supply or exhausting it to atmosphere. These valving operations are reflected in the mass flow rate term \( \dot{m} \) of Equation (17). The mass flow of a gas is often modeled as two flow regimes dependent on the upstream and downstream pressures. Denoting \( P_{\text{high}} \) and \( P_{\text{low}} \) as the upstream and downstream pressures respectively, subsonic and supersonic (choked) flow conditions result in the following commonly accepted models (stated for air):

\[
\dot{m} = c A_0 \left( \frac{2}{RT_{\text{high}}} \right)^{1/2} \left[ P_{\text{low}} (P_{\text{high}} - P_{\text{low}}) \right]^{1/2} \\
\text{for } P_{\text{high}} < 1.89 P_{\text{low}} \quad (18a)
\]

\[
\dot{m} = c A_0 P_{\text{high}} \left( \frac{1}{RT_{\text{high}}} \right)^{1/2} \left[ \gamma \left( \frac{2}{\gamma + 1} \right)^{2/\gamma + 1} \right]^{1/2} \\
\text{for } P_{\text{high}} > 1.89 P_{\text{low}} \quad (18b)
\]

Since the orifice size \( A_0 \) is dependent on the dynamics of the valve opening or closing and the discharge coefficient \( c \) must be measured experimentally, Equations (18a, 18b) are in practice difficult to use. Additionally, including the dynamics of \( A_0 \) would undesirably increase the order of the model.

To obtain a reduced order model of the mass flow, there are certain features that can be exploited by operating the valves in a switching mode. The first feature that can be taken advantage of is that the orifice size \( A_0 \), or rather the dynamics of the orifice size, can be assumed to be independent of operating conditions. This is reflective of two assumptions: 1) in a switching mode of operation the valve is always being commanded to be completely opened or completely closed, and 2) the opening and closing response of the valve is not appreciably influenced by the pressures on either side of the valve for the pressure ranges of interest. A second feature that simplifies the modeling is that the driving pressure is either the supply pressure or atmospheric pressure. Assuming that these pressures are constant, these then become merely parameters of the model. Therefore, for repeatable valve operation and

![Figure 1. Schematic diagram of a pneumatic inertial positioning system actuated with a double-acting pneumatic cylinder and controlled with two binary (2-position) 3-way pilot-assisted solenoid valves.](image-url)
conditions of fixed supply and atmospheric pressures and external temperature, the term $\gamma mRT/V$ of Equation (17) is a function solely of chamber pressure, chamber volume and a discrete-valued input term of either supply or exhaust pressure. As a result, Equation (17) can be approximately modeled for each chamber as,

$$\dot{P}_{A,B} = \frac{u_{A,B} - P_{A,B}}{\tau_{A,B}(x,u_{A,B})} + \frac{(P_{A,B} + P_{atm})\gamma A_{A,B}x}{c_{A,B} + A_{A,B}x}$$  \hspace{1cm} (19)$$

where $\tau_{A,B}$ is shown as an explicit function of the input pressure $u_{A,B}$ (supply pressure or atmospheric pressure) and piston position $x$ (linearly proportional to chamber volume $V$). Implicit is the functional dependence on those parametric variables mentioned above. The constants $c_{A,B}$ represent the volumes of the chambers at the midpoint position of the actuator ($x = 0$). Assume that the first term of Equation (19) may be modeled sufficiently accurately with a constant value for $\tau_{A,B}$. Further assume that for typical pneumatic actuators operated with typical supply pressures, the second term of Equation (19) can be neglected. The pressure dynamics may therefore be approximated by the following linear model:

$$\dot{P}_{A,B} = \frac{u_{A,B} - P_{A,B}}{\tau_{A,B}}$$  \hspace{1cm} (20)$$

The model given by Equation (20) can be experimentally determined for a particular valve and a particular pneumatic actuator. Specifically, $\tau_{A,B}$ can be estimated from data by matching the modeled and measured dynamic pressure response of each chamber of the actuator as it is pressurized and exhausted at the midpoint piston position.

To model the effects of the two valves operating in a switching mode, all permutations of the discrete inputs to the two valves must be enumerated. The permutations, or modes, of operation are the following (in gage pressures):

- **Mode 1**: $u_A = P_{\text{supply}}$ and $u_B = 0$
- **Mode 2**: $u_A = 0$ and $u_B = 0$
- **Mode 3**: $u_A = 0$ and $u_B = P_{\text{supply}}$
- **Mode 4**: $u_A = P_{\text{supply}}$ and $u_B = P_{\text{supply}}$

For the particular system being exemplified here, it will be assumed that Mode 4 is not used since it seems to have little utility with regard to this system. Given that the valve has some time delay $T_D$ associated with its response, and given Equation (16), Equation (20), and Modes 1, 2, and 3 as listed above, the plant model for the pneumatic system under consideration here can be written as,

$$\ddot{x}(t) = \frac{1}{M}\left[P_{\text{supply}}u(t-T_D) - M\dot{x}(t)\right],$$  \hspace{1cm} (21)$$

where:

$$u_1 = -A_A, \quad u_2 = 0, \quad u_3 = A_B$$  \hspace{1cm} (22)$$

The control design procedure for time-delayed LTI switching systems given by Equations (10-15) is now applicable to the pneumatic switching system described by Equations (21, 22).

5. EXPERIMENTAL RESULTS

The preceding analysis and control approach was experimentally implemented on a single degree-of-freedom pneumatically actuated servo system as shown in Figure 1. This setup included a 1.905 cm (3/4 inch) inner diameter, 10 cm (3.9 inch) stroke double-acting pneumatic cylinder (Bimba 044-DXP) equipped with two pilot-assisted, binary, 3-way, solenoid-activated valves (SMC VQ1200H-5B), a 10 kg brass block on a track with linear bearings (Thompson 1CB08FAOL10), a linear potentiometer (Midori LP-100F) for position feedback, and three pressure transducers (Omega PX202) for model validation and to monitor the pressures, but not used for control. Model parameters were estimated or measured to be: $M = 10$ kg, $A_A = 2.5$ cm$^2$, $A_B = 2.8$ cm$^2$, $c_A = 12.6$ cm/s, $c_B = 14.2$ cm/s, $\gamma = 1.4$ and a supply pressure of 586 kPa (85 psig).

For the system and components experimentally implemented here, the value of $\tau$ in Equations (20, 21) and the time delay $T_D$ in Equation (21) was estimated from data by matching the modeled and measured pressure dynamics of each chamber of the actuator as it was pressurized and exhausted while held at the midpoint piston position $x = 0.0$ cm. Figure 2 compares the results of this modeling to the measured response. This resulted in estimated values of $\tau = \tau_A = \tau_B = 11$ ms and $T_D = 20$ ms. The values of $\tau$ and $T_D$ were selected in conjunction, rather than individually, so that the modeled response gave a good overall match to the measured response.

The control law implemented was that given by Equation (15) where each $\dot{V}_i(t + T_D)$ was evaluated according to Equations (10), (11), and (12), with $\dot{x}(t + T_D)$ estimated in real-time by Equation (14) and the state-space representation of the system model given by Equation (21). If at any point in time a control input satisfying the control law did not exist due to
FIGURE 2. DATA USED TO EMPIRICALLY OBTAIN $\tau = 11$ ms and $T_D = 20$ ms. Shown above is the measured pressure response (solid line) and modeled response (dashed line).

FIGURE 3. EXPERIMENTAL CLOSED-LOOP TRACKING OF THE LOAD MASS POSITION AT 0.5 Hz, 1 Hz, 2 Hz, and 3 Hz. The commanded position is shown as dashed and the measured position is shown as solid.
modeling errors and/or noise, the control input with the smallest positive associated term was selected. The integral sliding surface was selected as Equation (3) using a value of $\lambda = 20 \cdot 2\pi$. A differentiating filter was selected to be of the form,

$$D(s) = \frac{s}{a^2 s^2 + 2as + 1} \quad (23)$$

with an 80 Hz cutoff frequency to obtain the signals $\dot{x}$, $\ddot{x}$, $\dddot{x}$, from $x$ and $\dddot{x}$. The tracking performance for this controller is shown in Figure 3 for 0.5, 1, 2, and 3 Hz.

6. CONCLUSIONS

A method was presented for the control design of switching systems with a finite number of discrete input values. This included general nonlinear, non-autonomous system models with discontinuous governing dynamics without time delay, as well as LTI systems with time delay. The method provides a direct evaluation of the stability and error convergence condition and thereby offers both stability and tracking guarantees. Furthermore, since this method directly evaluates each control candidate, it does not require obtaining a closed-form expression for the control law and is therefore applicable to a very general class of dynamic systems. The proposed control design method for time-delayed LTI switching systems was applied experimentally to a pneumatic inertial positioning system controlled with pilot-assisted solenoid valves. The resulting control law required only position feedback information (i.e. no pressure sensors were required). Sinusoidal tracking of this system demonstrated the effectiveness and simplicity of the control design method.

REFERENCES


