MBA 405B

Handout for Lecture 5

Unpredictability and Mixed Strategies

Consider the Holmes-Moriarty game. Recall the Moriarty arrives at Paddington Station while trying to escape from Holmes and must choose to get on either a southbound or northbound train. Holmes arrives second, and must make the same choice not knowing what Moriarty chose. If they get on the same train, Moriarty is caught and Holmes wins the game. If they get on different trains, Moriarty gets away and he wins. Thus:

Holmes-Moriarty Game		Moriarty	
		Southbound	Northbound
Holmes	Southbound	-1 1	1 -1
	Northbound	1 -1	-1 1

The key thing to observe is that there is no Nash equilibrium. Moriarty's best response is always to get on the opposite train as Holmes, while Holmes best response is to get on the same trains as Moriarty.

What about the following alterative: Suppose that Moriarty gets to the train station and follows the strategy of flipping a coin to decide which train to get on, and Holmes does the same. The question is: are these strategies Nash?

Consider this from the standpoint of Holmes. How much does he get by choosing a South or Northbound train:

Expected Southbound payoff to Holmes: $\frac{1}{2} \times 1 + \frac{1}{2} \times -1 = 0$
Expected Northbound payoff to Holmes: $\frac{1}{2} \times -1 + \frac{1}{2} \times 1 = 0$

Thus, Holmes is totally indifferent. The expected payoff is the same.

Key point 1: Since going North is just as good as going south, flipping a coin between the two choices is also just as good as choosing North or South with certainty. In fact, *any probability mixture* between North and South is equally good.

Now assume that Holmes flips a coin. Is flipping a coin a best response by Moriarty? How much does he get by choosing a South or Northbound train?

Expected Southbound payoff to Moriarty: $\frac{1}{2} \times -1 + \frac{1}{2} \times 1 = 0$
Expected Northbound payoff to Moriarty: $\frac{1}{2} \times 1 + \frac{1}{2} \times -1 = 0$

Thus, Moriarty is also totally indifferent. The expected payoff is the same. Therefore, flipping a coin is also just as good.

Suppose instead that Moriarty chooses a different randomization, say 25% South, 75% North. This would be a best response to Holmes' 50/50 mix as we have already noted. However, the 50/50 mix would no longer be a best response by Holmes to Moriarty.

Expected South-bound payoff to Holmes: $\frac{1}{4} \times 1 + \frac{3}{4} \times -1 = -\frac{1}{2}$
Expected North-bound payoff to Holmes: $\frac{1}{4} \times -1 + \frac{3}{4} \times 1 = \frac{1}{2}$

As you can see, going North is a best response for Holmes. This should be obvious. If Moriarty is more likely to be on the Northbound train, this is where Holmes should go.

Key point 2: For a pair of mixed strategies to be a Nash equilibrium, Each agent much be totally indifferent over all of the strategies to which he attaches positive probility (and must prefer these to those that he plays with zero probability).

For example, consider the Rock-Paper-Scissors game.

R-P-S Game	Brittany			
Madonna		Rock	Paper	Scissors
	Rock	0 0	1 -1	-1 1
	Paper	-1 1	0 0	1 -1
	Scissors	1 -1	-1 1	0 0

This game also does not have any equilibrium in *pure strategies*. Let's use the key point raised above to try and find *mixed strategy* equilibrium. Given Brittany's mix, Madonna has to be indifferent over each of her choices. Thus, the expected payoffs from each choice have to be the same. Let P_r, P_p and P_s be the probably with which Brittany plays each strategy and K be some constant. Then:

Rock:

$$P_r \times 0 + P_p \times -1 + P_s \times 1 = K$$

Paper:

$$P_r \times 1 + P_p \times 0 + P_s \times -1 = K$$

Scissors:

$$P_r \times -1 + P_p \times 1 + P_s \times 0 = K$$

Note that $1 = P_r + P_p + P_s$. This gives us four equations and four unknowns. If you do the algebra, you will find that the solution is $P_r = P_p = P_s = 1/3$ and $K = 0$.

It is also easy to verify that a best response to Madonna playing each strategy with probably 1/3 is Brittany playing each strategy with probability 1/3 and inversely. If either puts more weight on any particular strategy, the other should respond with the most advantageous pure strategy.

Notice that equal probabilities are not always best. For example, consider the following tennis problem. The server can serve to the forehand or backhand of the receiver, and the receiver can anticipate and move to receive a serve either with the forehand or backhand. The percentages are the odds of a successful return, which benefits the received and damages the server. This is an example of a zero-sum game, by the way.

Tennis Serving Game		Server	
		Aim to Forehand	Aim to Backhand
Receiver	Move to Forehand	90%	20%
	Move to Backhand	30%	60%

What is the equilibrium? Let PS_f, PS_b be the probably that the serves chooses forehand and backhand, and similarly for the receiver with PR_f, PR_b.

The Server must be indifferent over both choices; thus, the odds of a return must be the same for a forehand and backhand serve:

$$PR_f \times .9 + (1- PR_f) \times .3 = PR_f \times .2 + (1- PR_f) \times .6$$

This implies:

$PR_f (.9 - .3 - .2 + .6) = .6 - .3$

$PR_f = .3$

Similarly for the receiver:

$PS_f \times .9 + (1- PS_f) \times .2 = PS_f \times .3 + (1- PS_f) \times .6$

This implies:

$PS_f (.9 - .2 - .3 + .6) = .6 - .2$

$PS_f = .4$

Thus, the mixed strategy: $(PS_f, PS_b) = (.4, .6)$ and $(PR_f, PR_b) = (.3, .7)$ is a Nash equilibrium.

Now, suppose that the receiver improves his backhand return so that the return rate goes up from 60 to 65 percent:

Tennis Serving Game		Server	
		Aim to Forehand	Aim to Backhand
Receiver	Move to Forehand	90%	20%
	Move to Backhand	30%	65%

What is the new equilibrium?

$PR_f \times .9 + (1- PR_f) \times .3 = PR_f \times .2 + (1- PR_f) \times .65$

This implies:

$PR_f (.9 - .3 - .2 + .65) = .65 - .3$

$1.05\ PR_f = .35$

$PR_f = .33$

Similarly for the receiver:

$PS_f \times .9 + (1- PS_f) \times .2 = PS_f \times .3 + (1- PS_f) \times .65$

This implies:

$PS_f (.9 - .2 - .3 + .65) = .65 - .2$

$1.05PS_f = .45$

$PS_f = .43$

Thus, the mixed strategy: $(PS_f, PS_b) = (.43, .57)$ and $(PR_f, PR_b) = (.33, .67)$ is a Nash equilibrium.

Here is the odd thing: the receiver's backhand improved, yet, in equilibrium, the backhand return is used less often. This is not so surprising when you think of it. Suppose that you have a really good legal department in your company. Do you think that people will sue you very often? Do you think that Mike Tyson has to punch many people out in bars? The fact that you could win a game may discourage your opponents from every challenging you.

Two important points:

- Each player has to choose the right randomization in mixed strategy equilibrium. Suppose that the server chooses the correct equilibrium mixture, but you, the receiver, just go to the backhand. You might be perfectly happy with this strategy, but the best response of the server to this is to hit to forehand each time. Thus, equilibrium happens only when both agents randomize.

- Each server must be individually random. Suppose you systematically hit four forehands followed by six backhands. This does achieve the right mix of serves on average, but it also allows your opponent to anticipate your move and beat you.

A good question is: how can you act randomly? This is not so easy in practice. People often forget, for example, that a heads is just as likely to follow a heads as a tails. Thus, a random series of coin flips will have runs of several heads and also several tails in a row on occasion. When people try to be random, they alternate too often.

The best way to do this is to use some kind of outside random number generator, like the second hand on your watch, or the number on the next license plate that you see, or pointing to a random phone number in a telephone book. For random strategies to be successful, they must truly be random.

A side point: it is not literally necessary that one's opponent randomize, only that you be uncertain about what he will do. For example, in the Holmes-Moriarty game, it may be that Moriarty does not truly flip a coin. However, Holmes does not what kind of criminal Moriarty is: a criminal who likes to go South or a criminal who likes to go North. Thus, even though Moriarty does not flip a coin, from Holmes' standpoint it is as if he does.

Unique situations

Suppose you are faced with a one-time strategic play. For example, suppose you are a general, and you have to decide what strategy to use to fight a battle. The book talks at length about D-Day: should the Allies land at Calais or Normandy?

One might think that the stakes are too high. Surely you should not risk the lives of your men and you should therefore simply land at Calais since it is closest to Dover. But then the Germans will anticipate this, and counter your move. Thus, you may place your men at greater risk if you follow the "safe" course. In other words, unless you treat the unique situation exactly as you would a repeated one, you give your opponent an advantage.

This means that you will lose sometimes. You will make a daring military move, and be crushed, or follow as high-risk business strategy that leads to disaster. Losing is part of the game. If you never lose, you are not playing the game right.

Have you ever missed a plane? If not you are arriving at the airport too early.

Bodyguard of lies: In a unique situation, it is very common to try to mislead your opponent. The Allies made all sorts of efforts to make the Germans believe that the landing would be at Calais. The problem is that this deception should not really work.

In the Movie, *The Princess Bride*, two characters are sitting together and one of them put poison in one of the glasses before them. The other character does not know which. He went through the following logic: My guess is that you put poison in my glass, thus I should drink your glass, but knowing this, you should have put the poison in your glass, but, I know this, and thus, I will drink my glass, but you would have figured this out and put the poison in my glass, thus, I should drink your glass . . . . .This circular logic has no end.

Thus, deception only works if it is truly deceptive. You should ignore any signal your opponent sends you. It is just as likely that he is bluffing as telling the truth. The signal contains no reliable information. The only thing you should listen to is information you think is credible for some reason, for example information obtained through espionage. Beware, however, as your opponent may be feeding you false information through your trusted sources.

Other uses for uncertainty:

Consider the problem facing the IRS. Auditing is very costly, yet if it does not audit tax returns everyone will cheat. How can it get compliance without spending too much on enforcement? Suppose that if you cheat you save $1000 on taxes. Then if the IRS audits every return and forces you to pay what you owe, you should be just indifferent between telling the truth and lying. Suppose instead the IRS randomly audits 10% of the returns, and the punishment for lying is have to pay your back taxes plus a fine of 10 times the amount your underpaid. What is the expected gain from lying on your taxes:

$.9 \times 1000 + .1 \times -10{,}000 = -100$

You should tell the truth. Since the IRS only has to audit 10% of the returns, this is much cheaper for the government and gains full compliance. Thought questions:

- What if the IRS only audits 1% of returns?
- Why jail instead of fines in some cases?
- Is there a limit to this strategy of random enforcement?

Note that this also applies to crime in general. However, the chance of getting caught is to some degree endogenously determined by potential criminals. Can you explain why this is true and how this might lead to two different equilibria?

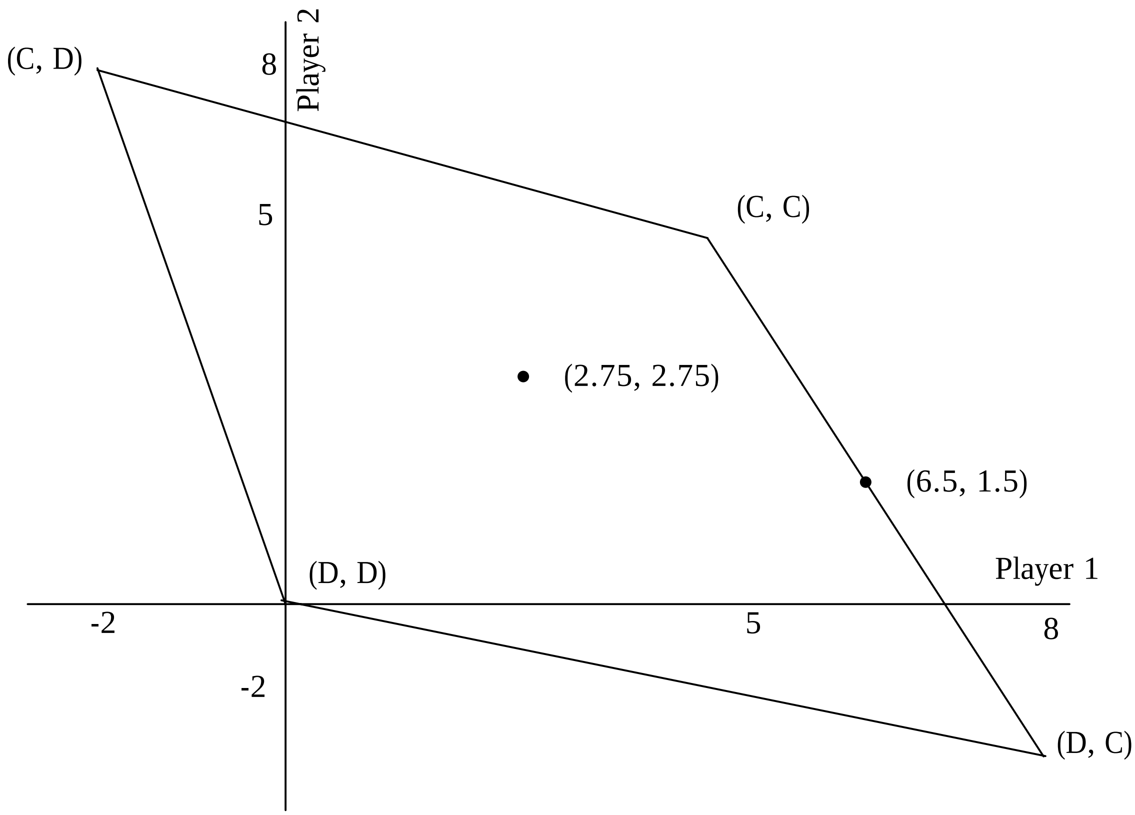
Random strategies, partial cooperation, and the Folk Theorem.

Consider the standard PD game below.

PD game		Player 2: C	Player 2: D
Player 1	C	5 / 5	-2 / 8
	D	8 / -2	0 / 0

Suppose Player 1 were to cooperate half the time on a random basis, and player 2 cooperated all the time. Note that the expected pay to player 1 would be $.5\times5 + .5\times8 = 6.5$ while the expected payoff to player 2 would be $.5\times5 + .5\times-2 = 1.5$.

We can show this in another kind of picture. In the graph below we have drawn the payoffs of the pure four pure strategies. Notice that the payoff of the mixed strategy given above takes us exactly halfway along the line between (C, C) and (D, C). By the choosing mixed strategies, we can also get expected payoffs that are anywhere along the lines between the pure strategy payoffs, and indeed, anywhere inside the diamond.



For example, suppose that each agent randomly cooperates half the time. It is easy to verify that you spend one quarter the time at each pure strategy payoff in this case. Thus the expected payoff is: $.25\times 5 + .25\times -2 + .25\times 8 + .25\times 0 = 2.75$. This suggests two things:

- **Limited Cooperation and the Folk Theorem**. What we have just concluded is that any point in the diamond is a payoff associated with some pair of mixed strategies what involve less than pure cooperation. When we discussed the Folk Theorem before, we talked about supporting equilibrium with side-payments. By the same argument, we can also support any kind of limited cooperation (mixed strategies) provided agents are sufficiently patient (interest rate is small enough) and agents receive more than their disagreement point payoff (0,0). For example, I can say: if you cooperate half the time so will I, but if you don't I will defect forever. A problem with this might be verifying the mixture that agents are using.

- **Coordination**: Here, there may be an advantage from coordinating the mixed strategies using some kind of public randomization device like: if the market goes up, you cooperate and I defect, and if the market goes down, you defect and I cooperate. If we suppose the market goes up and down half the time then half of the time we get the (C, D) payoff, and the other half of the time we get the (D, C) payoff. Thus, our expected payoff is: $.5\times 8 + .5\times -2 = 3$. Notice that this payoff is greater than the 2.75 we get if we don't coordinate our moves even though we still cooperate and defect half the time.