Anonymous Pricing in Public Goods Economies†

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June 1997

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† The following was written by the authors via e-mail. One author, on sabbatical in Spain, did not have her voluminous files with her so many of the comments on the literature are written from memory of the original sources. Thus, there may well be errors. These are not intended to mis-represent or slight any relevant works. We invite and would appreciate comment on any aspect of the paper. The authors are indebted to Vicky Barham, Birgit Grodal and Karl Vind for interesting and helpful discussions. Published version appears as: John Conely and Myrna Wooders, "Anonymous Pricing in Public Goods Economies", in Topics in Public Economics, D. Pines, E. Sadka, and I. Zilcha, editors, Cambridge University Press, 1997pp. 89-120.

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Abstract

One of the most important ideas in public economics is Tiebout's hypothesis that if public goods were "local" then markets would be able to overcome the free rider problem. In this paper we discuss the different approaches to formally stating this idea as a decentralization theorem. Special attention is devoted to structure of the price systems required for decentralization. We argue that unless prices are anonymous in the sense that they cannot discriminate between agents on the basis of unobservable characteristics (tastes, for example), they are not decentralizing in the same way as Walrasian prices in private goods economies. We consider the theorems available for three basic local public models: anonymous crowding, differentiated crowding, and a new model called crowding types.

Journal of Economic Literature Classification Numbers: H41, H72.

1. Introduction

One of the persistent problems in public economics is how to achieve efficient outcomes through market mechanisms in the presence of public goods. The Lindahl equilibrium as formalized by Samuelson (1954) is not in itself a satisfactory solution. Decentralizing efficient allocations through this equilibrium notion requires that prices faced by an agent depend upon his preferences mapping. As a consequence, self-interested agents may prefer not to reveal their true preferences in the hope of getting lower prices. Solving this "free rider" problem typically requires appealing nonmarket mechanisms. See Jackson and Moulin (1994) for a recent survey of such mechanisms (as well as an interesting new mechanism).

In his seminal paper, Tiebout (1956) proposed a solution. He observed that many types of public goods are "local" rather than "pure". Tiebout suggested that when public goods are local and there are many jurisdictions, competition between jurisdictions for members would lead to market-type efficiency. In effect, agents reveal their preferences by their choice of jurisdiction. As a consequence, the free-rider problem would disappear and the equilibrium outcome would be efficient.

While Tiebout's approach was very informal, a large literature has subsequently developed which treats his ideas with more precision. One of the earliest contributors was Eitan Berglas, who, in two influential papers (Berglas 1974, 1976) introduced a model of differential crowding and raised the major question of the nature of economic equilibrium. Subsequent workers in the field owe him a debt of gratitude for directing attention to this rich and interesting area of research, and simulating numerous articles. We will discuss the work of Berglas in more detail below.

The purpose of this paper is to discuss the different approaches to formalizing Tiebout's hypothesis. There are many possible interpretations. One could reasonably view Tiebout's hypothesis as being equivalent to a First Welfare Theorem, a Second Welfare Theorem, an existence theorem, a core/equilibrium convergence theorem or a core/equilibrium equivalence theorem. Before we consider any of these possibilities, however, we must understand what is meant by equilibrium. In particular, we must decide on the nature of the decentralizing price system. This is the focus of current paper. One of our main points is that unless prices are anonymous in the sense that they cannot discriminate between agents on the basis of unobservable characteristics (tastes, for example), they are not decentralizing in the same way that we are accustomed to in private goods markets. We consider the theorems available for three basic local public models: anonymous crowding, differential crowding, and a new model called crowding types.

The plan of the paper is as follows. In section 2, we will formally define and motivate the major modeling approaches, In section 3, we define the core and consider problems of existence. In section 4, we discuss different notions of market equilibrium paying special attention to the associated price systems. In section 5, we discuss different notions of small group effectiveness. In section 6, we will connect local public goods economies to the literature on market games. Section 7 concludes.

2. Models of Local Public Goods Economies

There is no formal distinction between local public goods economies and club economies. In both cases, the object of study is goods which are subject to crowding. We imagine something like a swimming pool which satisfies neither the pure rivalry in consumption of private goods, nor the nonrivalry in consumption of pure public goods. Historically, authors have had different motivations in mind when they chose to place their research in one of these two categories.

Authors who write on club economies usually have in mind a private membership club, a country club for example. They are concerned the question of the extent to which the market can provide institutions that substitute for government provision of such goods. Most papers consider the problem from the standpoint of one profit maximizing and price taking club. The general equilibrium problem of how to allocate all agents in the economy to clubs does not necessarily arise in this context. Perhaps the most important difference is that since club membership is not particularly associated with location, there is no reason to restrict each agent to joining at most one club. The question of variable usage of club facilities, and how this affects crowding and pricing is also natural in this context. Agents express their demands in a direct way by joining such clubs.

Studies of local public goods economies typically are motivated by locational models. We imagine optimizing jurisdictions who competitively offer bundles of public goods and associated tax prices. Agents express their demands indirectly by "voting with their feet" and moving to the locality with the best mix of taxes and public goods. The restriction that agents can live in only one location and so must join exactly one of these local public goods "clubs" in natural in this context. The focus of such models is usually on the general equilibrium question of how the entire population allocates themselves to jurisdictions in response to market signals. Natural extensions of this class of models include questions associated with property ownership. The problems of how variable land consumption choice and capitalization of the present value of the public good consumption affect the nature and efficiency of the equilibrium are very interesting.

We focus of local public good economies in this paper. We hasten to add that this is in the spirit of choosing a certain approach and motivation to examining the Tiebout hypothesis rather than rejecting part of the literature. General equilibrium club models can and have be stated. By the same token, there is a relatively new literature on hierarchical local public good economies in which an agent might be a member of several overlapping jurisdictions, a state and a county, for example.[1] Within the local public goods literature, there are three basic models: anonymous crowding, differentiated crowding, and crowding types.[2] We treat each in turn below.

[1] Also see Shubik and Wooders (1986).

[2] Calling crowding by numbers "anonymous crowding" is somewhat misleading since in the differentiated crowding model, in which different types have different crowding effects, crowding is also anonymous. That is, it is not the names of agents that matter to other agents, but rather their types. Within a given type, agents are perfect substitutes and crowd anonymously. It would be more accurate, but also more awkward to call anonymous crowding "nondifferentiated crowding".

We consider an economy with one private good and M public goods in this paper. The motivation for assuming one private good is mostly technical. We will discuss the difficulties in generalizing to many private goods when we define the core. We assume that agents can be members of only one jurisdiction at a time.

2.1 Anonymous and Differentiated Crowding Models

Anonymous crowding is a special case of differentiated crowding, although a special case that captures many real economic situations. This makes it possible to state a formal model which includes both anonymous and differential crowding.

In this model, there are I agents, denoted $i \in \{1, \ldots I\} \equiv \mathcal{I}$, each with a preference mapping indexed by $t \in \{1, \ldots T\} \equiv \mathcal{T}$, and an associated endowment of private good $\omega_t \in \Re_+$. The total population of agents is denoted by $N = (N_1, \ldots, N_t, \ldots, N_T)$ where N_t is interpreted as the number of agents of type t in the entire economy. A *jurisdiction* is a group of agents who collectively produce and consume a common level of public good. A jurisdiction is represented by a vector $n^k = (n_1^k, \ldots, n_t^k, \ldots, n_T^k)$, where n_t^k is interpreted as the number of agents of type t in the jurisdiction k. The set of all feasible jurisdictions is denoted by $\mathcal{N}$.

A *partition* $n = \{n^1, \ldots, n^K\}$ of the population is a collection of jurisdictions satisfying $\sum_k n^k = N$.[3] Let $\theta : \mathcal{I} \to \mathcal{T}$ be a function that indicates the type of a given individual. Thus, if agent i is of type t, then $\theta(i) = t$. With a slight abuse of notation, if individual i is a member of jurisdiction n^k, we shall write $i \in n^k$. We will also write $n^k \in n$ when a jurisdiction of type n^k is in the partition n.

For simplicity we will assume that the preferences of agent of type t can represented by a continuous utility function,

$$u_t : \Re \times \Re^M \times \mathcal{N} \to \Re.$$

[3] The total number of jurisdictions K is determined endogenously in the model.

Thus, $u_t(x, y^k, n^k)$ is the utility an agent of type t receives from consuming an amount x of private good and the bundle y^k of public goods, while in a jurisdiction with composition n^k. We note, however, that continuous preferences are not necessarily needed for market decentralization of efficient allocations in Tiebout economies.

We now turn to the production side. The cost in terms of private good of producing y^k public good for a jurisdiction with composition n^k is given by the cost function

$$f : \Re_+^M \times \mathcal{N} \rightarrow \Re_+.$$

Thus $f(y^k, n^k)$ is the amount of private good necessary to produce the bundle y^k public goods in a jurisdiction with composition n^k. We could also represent this by a production set, but a cost function is easier to work with when there is only one private good.

Crowding effects are allowed in both production and consumption. There is no restriction in general that these crowding effects be positive, negative, convex, or even monotonic. To make such restrictions would exclude many important economic applications. For example, agents may crowd each other positively at a party over some range, and then negatively as it becomes too crowded to dance. In production, we might find that two Spanish speaking carpenters can build a house just as fast as two English speakers, but one Spanish and one English speaker working together would take much longer. Thus, none of the ordinary assumptions on preferences or production are appropriate in the context of crowding.

Both anonymous and differentiated crowding models have been widely studied in the literature. In the anonymous crowding case, agents are affected only by the total number of people in the jurisdiction they join. The identities or tastes of their neighbors make no difference to them. An example of this is a highway. The only thing that affects other agents is the total number of people on the road. The internal rate of time preference and the musical tastes of the other agents, for example, are irrelevant. This approach has a great deal of appeal, but it seems to disallow many real world situations. In many cases, agents are not perfect substitutes for one another. For example, agents are crowded differently by men and women at a dance. One gender may generate positive externalizes while the other may simply crowd. To capture this, we can let production costs and utility functions depend on the entire profile of agent types instead of simply the total number of agents in a jurisdiction.

Formally, anonymous crowding means that two conditions called *Anonymous Crowding in Consumption* (ACC) and *Anonymous Crowding in Production* (ACP) are satisfied

(ACC) for all $n^k, n^{\hat{k}} \in \mathcal{N}$, if $\sum_t n_t^k = \sum_t n_t^{\hat{k}}$ then for all $x \in \Re_+$ $y \in \Re_+^M$ and all $t \in \mathcal{T}$ it holds that $u_t(x, y, n^k) = u_t(x, y, n^{\hat{k}})$.

(ACP) for all $n^k, n^{\hat{k}} \in \mathcal{N}$, if $\sum_t n_t^k = \sum_t n_t^{\hat{k}}$ then for all $y \in \Re_+^M$ it holds that $f(y, n^k) = f(y, n^{\hat{k}})$.

Typically, the utility and production functions for the anonymous crowding case are written in a reduced form: $u_t(x, y, \sum_t n_t^k)$ and $f(y, \sum_t n_t^k)$.

2.2 The Crowding Type Model

There is an important sense in which the differentiated crowding model is an unsatisfactory generalization of the anonymous crowding model. While is certainly reasonable that different types of agents should crowd each other differently. It is far from clear the tastes of one agent should directly affect the welfare of another. Consider the labor complementary model. The skills that an agent brings to a jurisdiction should affect the cost of producing public good, but why should his preferences over consumption bundles? A plumber who likes big cars contributes just a much to production as one who likes compact cars. In the standard differentiated crowding model, an agent's tastes and his crowding effects are perfectly correlated. There is no evident reason for this.

An alternative generalization is to explicitly endow agents with crowding characteristics which are formally distinct from his preferences. This is called a "Crowding types" model. Agents still are endowed with one of T different sorts of tastes or preference maps, but in addition, agents are identified as having one of C different sorts of crowding characteristics. The crowding type of agents is denoted by $c \in \{1, \ldots, C\} \equiv \mathcal{C}$.[4] No correlation between c and t is assumed. Imagine, for example, a dance in which men and women crowd each other differently. Some individuals like country music and some like jazz. There are men and women with each type of preference. The tastes of individuals are private information, but their crowding characteristics are publicly observable.

The rest of the crowding types model is the natural extension of the differentiated crowding model stated above. The population of agents is denoted by $N = (N_{11}, \ldots, N_{ct}, \ldots, N_{CT})$, where N_{ct} is interpreted as the total number of agents with crowding type c and taste type t in the economy. A jurisdiction is represented by a vector $n^k = (n^k_{11}, \ldots, n^k_{ct}, \ldots, n^k_{CT})$, where n^k_{ct} is interpreted as the number of agents with crowding type c and taste type t in the jurisdiction k. We will denote by $\mathcal{N}_c$ the set of feasible jurisdictions that contain at least on agent of crowding type $c \in \mathcal{C}$. Formally:

$$\mathcal{N}_c \equiv \{n^k \in \mathcal{N} \mid \ \exists\, t \in \mathcal{T} \text{ such that } n^k_{ct} > 0\}.$$

We will say that two jurisdictions, n^k and $n^{\hat{k}}$, have the *same crowding profile* if for all $c \in \mathcal{C}$, $\sum_t n^k_{ct} = \sum_t n^{\hat{k}}_{ct}$. That is, two jurisdictions have the same crowding profile if the number of agents of any given crowding type is the same in both jurisdictions. Let $\theta : \mathcal{I} \to \mathcal{C} \times \mathcal{T}$ be a function that indicates the type of a given individual. Thus, if agent i is of crowding type c and taste type t, then $\theta(i) = (c, t)$.

The notion of a crowding type is meant to capture all the characteristics of an agent that enter into to the constraints or objectives of any other agents. Tastes are irrelevant in this respect. We state this formally in the two assumptions, *Taste Anonymity in consumption* (TAC), and *Taste Anonymity in Production* (TAP).

(TAC) for all $n^k, n^{\hat{k}} \in \mathcal{N}$, if for all $c \in \mathcal{C}$ it holds that $\sum_t n^k_{ct} = \sum_t n^{\hat{k}}_{ct}$ then for all $t \in \mathcal{T}$, $x \in \Re$ and $y \in \Re^M_+$, it holds that $u_t(x, y, n^k) = u_t(x, y, n^{\hat{k}})$.

[4] Note that each crowding type may denote a point in a finite (or infinite) divisional vector space.

(TAP) for all $n^k, n^{\hat{k}} \in \mathcal{N}$, if for all $c \in \mathcal{C}$ it holds that $\sum_t n^k_{ct} = \sum_t n^{\hat{k}}_{ct}$ then for all $y \in \Re^M_+$ it holds that $f(y, n^k) = f(y, n^{\hat{k}})$.

Observe that if we set $C = 1$, this is exactly the anonymous crowding model. Also, if we set $C = T$ and $N_{c,t} = 0$ for all $c \neq t$, then we have the differentiated crowding model. In words, the crowding types model is equivalent to the standard differentiated crowding model when crowding types and taste types are perfectly correlated in the population.[5] In this case, each taste type crowds in its own independent way. Because of this, we will use the notation of crowding types model below the explicate the results in the literature.

A *feasible state of the economy* (X, Y, n) is a partition n of the population, an allocation $X = (x_1, \dots, x_I)$ of private goods, and public good production plans $Y = (y^1, \dots y^K)$ such that

$$\sum_k \sum_{c,t} n^k_{ct} \omega_t - \sum_i x_i - \sum_k f(y^k, n^k) \geq 0.$$

We denote the set of feasible states by F. We will also say that (x^k, y^k) is a *feasible allocation for a jurisdiction* n^k if

$$\sum_{c,t} n^k_{ct} \omega_t - \sum_{i \in n^k} x^k_i - f(y^k, n^k) \geq 0.$$

3. The Core

In the one private good case, the definition of the core is straight forward. A jurisdiction $n^{\hat{k}} \in \mathcal{N}$ producing a feasible allocation $(x^{\hat{k}}, y^{\hat{k}})$ *improves upon* a feasible

[5] Of course there is not requirement that the same types t have either the same crowding type or the same endowment. Making explicit, however, the separation of crowding type and taste type enables us to obtain new and interesting results.

state $(X, Y, n) \in F$ if for all $i \in n^{\hat{k}}$ such that for some $c \in \mathcal{C}$, $\theta(i) = (c, t)$:

$$u_t(x_i^{\hat{k}}, y^{\hat{k}}, n^{\hat{k}}) \geq u_t(x_i^k, y^k, n^k),$$

and for some $i \in n^{\hat{k}}$ such that for some $c \in \mathcal{C}$, $\theta(i) = (c, t)$:

$$u_t(x_i^{\hat{k}}, y^{\hat{k}}, n^{\hat{k}}) \geq u_t(x_i^k, y^k, n^k),$$

where agent $i \in n^k \in n$ in the original feasible state. A feasible state $(X, Y, n) \in F$ is in the *core* of the economy if it cannot be improved upon by any jurisdiction.

Note that when a jurisdiction improves upon a state, it does so without trading private goods outside the jurisdiction. This restriction is unimportant in the one private good case since no gains from trade are possible. We might interpret a such model an abstraction in which prices for all private goods are taken as given and the one private good explicitly in the model is "money".

If we put many private goods in the model on the other hand, gains from trading are possible. If we nevertheless choose to define the core in a way which does not allow private goods trading between jurisdictions, then most theorems can be directly generalized from the one private good case. If instead we take the more reasonable view that while public goods are produced within jurisdiction, private goods are traded economy wide, however, things become more complicated. When a coalition of agents defects to form a new set of jurisdictions, they not only have to produce independent bundles of public goods, but they lose the private goods trading opportunities with the non-defecting agents. This makes direct generalization of results impossible.

An even more serious problem is that regardless of which definition is used it is often the case that the core is empty. This is true even in large economies with one private good satisfying all the properties that ordinarily guarantee existence, (convexity, monotonicity, etc.) This has been known at least since Pauly (1967), and was discussed at length by Wooders (1978). This and other problems lead Bewley (1982) to conclude that decentralization is essentially impossible unless public goods are really just publicly provided private goods (public services). The basic reason is that typically in local public goods economies there exist jurisdictional structures which are optimal is the sense that they maximize the pre capita payoff of their memberships. When agents of the various types are not present in numbers that exactly fill out these optimal jurisdictions without any left overs, we often get a cycling problem. For example, suppose we have a population of three identical agents with the following characteristic function derived from the underlying economy:[6]

$$\Gamma(n^k) = \begin{cases} 0 & \text{if } \mid n^k \mid \; = 1 \\ 1 & \text{if } \mid n^k \mid \; = 2 \\ 0 & \text{if } \mid n^k \mid \; \geq 3 \end{cases} .$$

Obviously, every feasible state can be blocked. No matter how the two person coalition divides their surplus, the agent who is left out can always make at least one of these agents better off. This same problem appears for every population with an odd number of agents. Although the fraction of left over agents decreases to zero as the economy gets large, there will still be one leftover. This is enough to generate the same type of cycling, and thus nonexistence of the core, no matter how large the economy.

It is important to emphasize that this is not an integer problem. Allowing for "fractional" agents, perhaps part time members of club, does not lead to existence of the core. The core is empty because of an imbalance in the *proportion* of agents of various types which make it impossible to completely exhaust the population while putting all agents in optimal jurisdictions.

These problems, and the fact that there are costs to coalition formation, motivated the study of approximate cores and equilibrium of economies with local public goods and of large games with small effective groups (Wooders (1978a, 1980a, 1983), and many subsequent papers.) Fortunately, approximate cores have a very natural interpretation in the context of local public goods economies. Informally, we simply modify the notion of what it means to improve on a state to require it be possible to make the defecting agents better off while paying a small cost of jurisdictional formation. In other words, agents who contemplate defecting must pay a transaction cost of $\epsilon \geq 0$ each, which may represent moving or setup costs, and must still be better off in the new jurisdiction.

[6] This matching game is a special case the more general class of local public goods economies.

Formally, a jurisdiction $n^{\hat{k}} \in \mathcal{N}$ producing an allocation $(x^{\hat{k}}, y^{\hat{k}})$ $\epsilon-$*improves upon* a feasible state $(X, Y, n) \in F$ if

$$\sum_{c,t} n_{ct}^{\hat{k}} \omega_t - \sum_{i \in n^{\hat{k}}} x_i^{\hat{k}} - f(y^k, n^k) \geq \epsilon \sum_{c,t} n_{ct}^k,$$

for all $i \in n^{\hat{k}}$ such that for some $c \in \mathcal{C}$, $\theta(i) = (c, t)$:

$$u_t(x_i^{\hat{k}}, y^{\hat{k}}, n^{\hat{k}}) \geq u_t(x_i^k, y^k, n^k),$$

and for some $i \in n^{\hat{k}}$ such that for some $c \in \mathcal{C}$, $\theta(i) = (c, t)$:

$$u_t(x_i^{\hat{k}}, y^{\hat{k}}, n^{\hat{k}}) \geq u_t(x_i^k, y^k, n^k),$$

where agent $i \in n^k \in n$ in the original feasible state. A feasible state $(X, Y, n) \in F$ is in the $\epsilon-$*core* of the economy if it cannot be $\epsilon-$improved upon by any jurisdiction.

To see why this solves the existence problem, return to the example above. Consider the $\epsilon-$core for $\epsilon = \frac{1}{6}$. We claim that equal division of the surplus gotten by the two person coalition is in the $\epsilon-$core. To see this, consider the agents who are in the two person coalition. If they defect from the state in which they pay a third of their surplus to the left over agent, they must pay a cost of $\frac{1}{3}$ (two times ϵ) to set up a new (identical) jurisdiction. Clearly, they are just as well off paying this surplus to the left over agent, as they are paying it as a setup cost. The excluded agent, on the other hand, cannot propose an $\epsilon-$improvement. If he forms a jurisdiction with one of other agents, the new jurisdiction has a surplus of $\frac{2}{3}$ to distribute after paying the setup costs, which is just enough to leave these agents as well off as in the original state.

In general, the $\epsilon-$core exists for arbitrarily small ϵ for sufficiently large economies. The intuition is that we can take away a very small amount of private good from each of a large number agents in optimal jurisdictions and use it to compensate the small fraction of agents who are left out of optimal jurisdictions. The other advantage of the $\epsilon-$core is that it is easier to treat the many private good case. Even though defecting coalitions lose the opportunity to trade private goods with the remaining agents, if the defecting coalition is large enough, it can realize almost all of the gains from trade internally. Thus, results that are true for the $\epsilon-$core and $\epsilon-$equilibrium (defined below) for the one private good case, are generally true for the many private goods case as well.

There are alternative definitions of the $\epsilon-$core. For example, we could simply ignore a fraction ϵ of agents, or we could assign agents a probability ϵ of not being able to find a jurisdiction to join. In all cases, the fundamental idea is to somehow deal with this small fraction of left over agents. It is interesting to note, however, that proofs on the nonemptiness of various notions of $\epsilon-$core and existence of approximate equilibrium typically begin by showing nonemptiness of the type of $\epsilon-$core defined formally above and then proceed to show how this implies the nonemptiness of the particular notion in question. [7] Which notion of approximate core is appropriate depends on the economic situation being modeled. In some models studying the noncooperative foundations of cooperation, such as Selten (1981) and Bennett (1991), the natural notion of approximate core may be one in which a small percentage of players is ignored. On the other hand, for a model of an economy consisting of a federation of separate jurisdictions, it may be that the central government, in order to achieve stability, taxes some jurisdictions by a small amount and uses the revenue to make transfers to less advantaged jurisdictions. For these reasons, we will only treat the definition given formally in this survey.

4. Equilibrium and Prices

Two major difference among the many equilibrium notions in local public goods economies are objectives of the city planers and the method used to decide on pub-

[7] For example, the proofs of the nonemptiness of approximate cores in Shubik and Wooders (1983) and Kaneko and Wooders (1982) where a small percentage of players can be ignored use lemmas previously proven in Wooders (1983) on the way towards showing nonemptiness when all players “sacrifice”.

lic goods levels. City planners have be variously modeled as profit maximizing entrepreneurs, population maximizing politicians, and agents of property owners who seek to maximize land values. The decision of how much public good to provide may either be done by dictate of the planner, or by majority rule or similar voting mechanism. For a class of models – those with small effective groups – all three sorts of objectives for city planners give the same outcome. We will only treat of the profit maximizing entrepreneur model. This is because it is closest to the market mechanism, and so seems to most directly address the question of when the market can successfully provide public goods, as Tiebout suggested. This should not be interpreted as a rejection of the other models. Indeed, they may even be more appealing since they may most closely reflect the way public good levels are really decided. However, the purpose of this paper is to study market mechanisms and so we will focus on the entrepreneurial model.

There are several properties that a price system must have in order to decentralize the efficient allocations in the way that is traditional in the private markets. The first is completeness.

A *complete price system* allows each agent to calculate the exact cost of joining every conceivable type of jurisdiction with every feasible level of public goods. If we are to expect that agents fully optimize over the all of the feasible allocations, then, of course, agents must be able to derive a price for all feasible allocations. Such a price system might have an infinite or a finite number of prices. For example, in a model with anonymous crowding, familiar price systems (as in Wooders (1978) for example) require just one per unit price for public goods for each jurisdiction and one participation price (wage or profit share). Such a system has twice as many prices as there are possible jurisdiction structure for the one public good case. Alternatively, a complete price system may specify a price for each jurisdiction structure/public good quantity pair (cf. Scotchmer and Wooders (1987)).

It is important to point out that with respect to the set of possible commodity spaces and price systems, economies with local public goods are really no different from pure private goods economies. The usual price system for a finite dimensional private goods commodity space has only a finite number of per unit prices. We could just as easily, however , represent this economy with an infinite dimensional commodity space where each commodity is an entire market basket with its own associated price. The important question of the "appropriate" commodity space, which seldom (if ever) arises in private goods economies, clearly emerges in local public goods contexts.

The reason that the assumption of a complete commodity space and price system appears to be stronger in a local public goods environment is that a vast array of commodities immediately present themselves to our attention. Even though most of these commodities are not traded in equilibrium, specifically, jurisdictions with associated public good levels which do not appear in the equilibrium jurisdictional structure, full optimization still requires that they be priced. It is not hard to imagine analogs in private goods economies, purple polka-dotted Mercedes, or houses made from recycled tires for example. Or, more seriously, a continuum of possible sizes of Mercedes. The context in which such nontraded private goods arise are differentiated product markets where the choice of product produced is endogenous. It should come as no surprise that planners who are allowed to consider products which are not priced or traded in a either local public, or private goods economies may be able to find allocations which are Pareto superior to the equilibrium allocations. The response in private goods models is to restrict the commodity space to the set of goods that are traded. Since a major question in local public goods models is whether or not the equilibrium Jurisdiction structure will be efficient, it is necessary to extend the commodity space and price to include all feasible products even if they are not traded in equilibrium.

The second major requirement of the prices system is that it be anonymous. An *anonymous price system* is one which does not discriminate between agents on the basis of private information. This is, of course, the fundamental difficulty with Lindahl equilibrium in the a pure public goods economy. Small group effectiveness ensures that all participant of the same type in the same jurisdiction pay the same Lindahl price, but Lindahl prices typically will differ between agents with different preference mappings. In some circumstances, it is possible to elicit Lindahl prices in models with nondifferentiated or anonymous crowding (cf. Barham and Wooders (1994)), or when crowding occurs only in production and there are constant returns to scale (Conley and Wooders (1994b). Unless the model and equilibrium concept have the feature that agents voluntarily select jurisdictions to equate their marginal rates of substitution to stated prices (not dependant on preferences) then Lindahl prices are not as appealing as Walrasian prices.

The essence of Tiebout's hypothesis is that, when public goods are locally provided, agents will find that it is optimal to reveal their preferences by moving to the their most preferred jurisdictions. This is just as in private goods markets where prices are anonymous and equally available to all, and agents reveal their preferences by choosing their most preferred consumption bundle. Thus, showing the existence of an anonymous set of prices that decentralize efficient allocations is tantamount to proving the Tiebout hypothesis.

The last property of price systems we will discuss is the dimension of the price space. There are two major approaches to pricing. We call these admission prices and Lindahl prices. Informally, admissions prices give single price for each type of agent for every jurisdiction, for every possible level of public good. These may be thought of as functions which map each jurisdiction and public goods level into a price of admission for agents. A Lindahl price system, on the other hand, lists two prices for each type of agent for each jurisdiction. The first is a participation price. This is like an entrance fee and, if positive, may be motivated as a wage or profit share. If negative, it can be interpreted as a prices to enjoy the externalities provided by others, or as a Pigouvian tax to compensate others for the negative effects imposed on others by the agent. The second is a per unit price of each public good.

Admission prices have a certain amount of appeal since they provide a description of the lump sum taxes suggested by Tiebout. If, given prices for private goods, firms can freely enter (or consumer groups can "opts-out"), but cannot profitably do so, then the resulting state of the economy can described by a complete set of admission prices or, equivalently, a complete Lindahl price system note Wooders (1989, 1993) are based on Wooders (1980b, 1981) and the nonemptiness of approximate cores of general economies with local public goods in Wooders (1988). Related results on the admissions price equilibrium in differentiated crowding models are contained in Scotchmer and Wooders (1986), and subsequent incomplete revisions of that paper. These papers are discussed at more length below.// as shown by Wooders (1980a) for the case of anonymous crowding, and by Wooders (1980b, 1989, and 1993) for differentiated crowding. A disadvantage of admission prices is that they require more centralization of decision making at the production level. For example, if a jurisdiction offers several public goods, then their productions must be coordinated. Moreover, an admission price equilibrium requires an infinite number of prices.

Lindahl price systems have the advantage that in large economies they convey the same information as admission prices, but in a more economical way. In particular, only a finite number of prices is required. The Lindahl equilibrium also has the advantage that it allows much more flexibility – an agent may pay a property tax, but may also pay a per hour cost for renting the municipal tennis courts or instead chose only to use the community swimming pool. This reflect the commonly observed pricing system of a property tax plus various user charges.

The extant literature showing the equivalence of admission pricing and Lindahl pricing (Scotchmer and Wooders (1986), Wooders (1993, 1994c)) may raise the question of whether or not there is any motivation for admission pricing since Lindahl pricing does the same job more economically. As shown by Conley and Wooders (1994b), however, important differences between they two types of price systems emerge when we introduce crowding types. We elaborate on this below.

In the following, we treat the admissions price system first. For each crowding type $c \in \mathcal{C}$ we have a price function α_c giving an admission price for every jurisdiction it is possible for an agent of this type to join, for every possible public good level. Agents are able to contemplate joining any jurisdiction that contains at least one member of their crowding type. For example, no matter how much Wynton Marsalis may wish it, it is impossible for him to join an all girl band. Once he joins, it is no longer an all girl band since it includes at least one boy. Thus, we should provide admissions prices for bands that include at least one male, but it makes no sense to provide an admissions price to Wynton Marsalis for female only bands. Formally:

$$\alpha_c : \Re^M_+ \times \mathcal{N}_c \to \Re.$$

An *admissions price system*, is simply the collection of price systems for each crowding type.

Notice that we allow different prices for different crowding types, but not for different taste types. This is because we are only interested in anonymous prices. Unfortunately, this is not quite enough for full anonymity. Observe that α gives an admissions price for every jurisdiction n^k, and that included in this description is the *taste profile* of the jurisdiction. Since we assume that tastes are not observable a system has *Fully Anonymous Prices* (FAP) only if jurisdictions with the same crowding profile are priced identically. Formally:

(FAP) for all $n^k, n^{\hat{k}} \in \mathcal{N}$, if for all $c \in \mathcal{C}$ it holds that $\sum_t n^k_{ct} = \sum_t n^{\hat{k}}_{ct}$ then for all $y \in \Re^M_+$ it holds that $\alpha(y, n^k) = \alpha(y, n^{\hat{k}})$.

Note that in the anonymous crowding model, there is only one crowding type and so only one price function. In the differential crowding model, anonymity is lost since taste and crowding are perfectly correlated, and so providing each crowding type with a separate price function is identical to requiring that each taste type have a separate price system. We are now able to define our first equilibrium notion. An *admissions price equilibrium* is a feasible state $(X, Y, n) \in F$ and a price system α such that

1. for all $n^k \in n$, all individuals $i \in n^k$ such that $\theta(i) = (c, t)$, all alternative jurisdictions $n^{\hat{k}} \in \mathcal{N}_c$, and for all levels of public good production $y^{\hat{k}} \in \Re^M_+$:

$$u_t(\omega_t - \alpha_c(y^k, n^k), y^k, n^k)) \geq u_t(\omega_t - \alpha_c(y^{\hat{k}}, n^{\hat{k}}), y^{\hat{k}}, n^{\hat{k}})),$$

2. for all potential jurisdictions $n^k \in \mathcal{N}$ and all $y^k \in \Re^M_+$,

$$\sum_{c,t} n^k_{ct} \alpha_c(y^k, n^k) - f(y^k, n^k) \leq 0.$$

3. for all $n^k \in n$,

$$\sum_{c,t} n^k_{ct} \alpha_c(y^k, n^k) - f(y^k, n^k) = 0.$$

Condition (1) says that all agents maximize utility given the price system. Note that the price schedule available to an agent depends only on his crowding type. Condition (2) says that given the price system, no firm can make positive profits by entering the market and offering to provide any sort of jurisdiction. Condition (3) says that all equilibrium jurisdictions make zero profit, and so cover their costs.

The alternative price system that is widely discussed in the literature is Lindahl price equilibrium. In this type of price system for each crowding type $c \in \mathcal{C}$ we have a price function λ_c which gives an participation price for every jurisdiction it is possible for an agent of this type to join, and a per unit price for each public good type. Formally:

$$\lambda_c : \mathcal{N}_c \to \Re \times \Re^M.$$

An *Lindahl price system*, is simply the collection of price systems for each crowding type. It is convenient to decompose this formally into the component prices. Thus $\lambda_c(n^k) \equiv (p_c(n^k), q_c(n^k))$, where $p_c : \mathcal{N}_c \to \Re$ is the participation price, and $q_c : \mathcal{N}_c \to \Re^M$ is the set of per unit prices for public goods.

The corresponding equilibrium notion is the following. A *Lindahl price equilibrium* is a feasible state $(X, Y, n) \in F$ and a price system λ such that

1. for all $n^k \in n$, all individuals $i \in n^k$ such that $\theta(i) = (c, t)$, all alternative jurisdictions $n^{\hat{k}} \in \mathcal{N}_c$, and for all levels of public good production $y^{\hat{k}} \in \Re^M_+$:

$$u_t(\omega_t - p_c(n^k) - q(n^k)y^k, y^k, n^k) \geq u_t(\omega_t - p_c(n^{\hat{k}}) + q_c(n^{\hat{k}})y^{\hat{k}}, y^{\hat{k}}, n^{\hat{k}}),$$

2. for all potential jurisdictions $n^k \in \mathcal{N}$ and all $y^k \in \Re^M_+$,

$$\sum_{c,t} n^k_{ct} p_c(n^k) + \sum_{c,t} n^k_{ct} q_c(n^k) y^k - f(y^k, n^k) \leq 0.$$

3. for all $n^k \in n$,

$$\sum_{c,t} n^k_{ct} p_c(n^k) + \sum_{c,t} n^k_{ct} q_c(n^k) y^k - f(y^k, n^k) = 0.$$

Notice that the admission price system bears a strong resemblance to the valuation equilibrium for pure public goods economies. For a given jurisdiction, the admissions price system is just a general nonlinear function that assigns part of the cost of public good to each agent. The valuation equilibrium does the same thing except there is never more than one jurisdiction in the core of a pure public goods economy. The Lindahl price system is a natural generalization to local public goods economies. For a given jurisdiction, the cost of public good to an agent is linear in the quantity he demands. A missing piece in this literature is the generalization of the cost share equilibrium to local public goods economies.[8] Here, the cost of public goods to agents is linear is the cost of providing the public good.

An advantage of admissions price equilibrium is that given the nonlinear structure of the price functions, there is no need to assume convexity, continuity or monotonicity or either the utility or cost functions. Lindahl decentralizations require these assumptions.

It is possible to define other variants of these equilibrium concepts. One sort of variant makes the prices for players independent of the jurisdiction. The disadvantage of such a pricing system is that it makes existence of equilibrium less likely. (It is a general rule – the more restrictions that we place on an equilibrium concept, the more stringent the conditions required on the economy to obtain existence.)

A uniform price system q for agents of crowding type $c \in \mathcal{C}$ gives an admission price for each crowding type of agent and each level of public good.

$$q_c : \Re_+ \to \Re, \ c = 1, ..., C.$$

A *uniform Tiebout price system,* is simply the collection of price systems for each crowding type for each level of public good.

[8] We thank Robert Gilles for this observation

A *uniform Tiebout equilibrium* is a feasible state $(X, Y, n) \in F$ and a uniform price system q such that

1. for all $n^k \in n$, all individuals $i \in n^k$ such that $\theta(i) = (c, t)$, all alternative jurisdictions $n^{\hat{k}} \in \mathcal{N}_c$, and for all levels of public good production $y^{\hat{k}} \in \Re_+$:

$$\omega_t - q_c(y^k) + h_t(y^k, n^k) \geq \omega_t - q_c(y^{\hat{k}}) + h_t(y^{\hat{k}}, n^{\hat{k}}),$$

2. for all potential jurisdictions $n^k \in \mathcal{N}$ and all $y^k \in \Re_+$,

$$\sum_{c,t} n^k_{ct} q_c(y^k) - f(y^k) \leq 0.$$

3. for all $n^k \in n$,

$$\sum_{c,t} n^k_{ct} q_c(y^k) - f(y^k) = 0.$$

The conditions of the definition of the uniform Tiebout equilibrium have all the same interpretations as in the definition of the Tiebout equilibrium, except that the prices, based on crowding types, do not depend on the jurisdiction.

It is clear that a uniform Tiebout equilibrium state of the economy is in the core since a uniform Tiebout equilibrium is a Tiebout equilibrium. With further restrictions on the economic model, it holds that the uniform Tiebout equilibrium coincides with the core (Conley and Wooders, in progress).

A similar modification can be made in the definition of the Lindahl equilibrium. We leave this to the reader.

The equilibrium allocations are generally in the core. This implies that if the core is empty then equilibrium does not exist. This motivated the study of the $\epsilon-$core. We need a corresponding notion of $\epsilon-$equilibrium, which will generally exist for large economies, since the $\epsilon-$core generally exists. The intuition is very similar to the $\epsilon-$core. We modify the definitions of equilibrium to require that agents pay a jurisdiction formation cost of ϵ when they consider jurisdictions other than the one they occupy in equilibrium. Formally, an $\epsilon-$*admissions price equilibrium* is a feasible state $(X, Y, n) \in F$ and a price system α such that

1. for all $n^k \in n$, all individuals $i \in n^k$ such that $\theta(i) = (c,t)$, all alternative jurisdictions $n^{\hat{k}} \in \mathcal{N}_c$, and for all levels of public good production $y^{\hat{k}} \in \Re_+^M$:

$$u_t(\omega_t - \alpha_c(y^k, n^k), y^k, n^k)) \geq u_t(\omega_t - \alpha_c(y^{\hat{k}} - \epsilon, n^{\hat{k}}), y^{\hat{k}}, n^{\hat{k}})),$$

2. for all potential jurisdictions $n^k \in \mathcal{N}$ and all $y^k \in \Re_+^M$,

$$\sum_{c,t} n_{ct}^k \alpha_c(y^k, n^k) - f(y^k, n^k) \leq 0.$$

3. for all $n^k \in n$,

$$\sum_{c,t} n_{ct}^k \alpha_c(y^k, n^k) - f(y^k, n^k) = 0.$$

In the same spirit, an $\epsilon-$*Lindahl price equilibrium* is a feasible state $(X, Y, n) \in F$ and a price system λ such that

1. for all $n^k \in n$, all individuals $i \in n^k$ such that $\theta(i) = (c,t)$, all alternative jurisdictions $n^{\hat{k}} \in \mathcal{N}_c$, and for all levels of public good production $y^{\hat{k}} \in \Re_+^M$:

$$u_t(\omega_t - p_c(n^k) - q(n^k)y^k, y^k, n^k) \geq u_t(\omega_t - p_c(n^{\hat{k}}) + q_c(n^{\hat{k}})y^{\hat{k}} - \epsilon, y^{\hat{k}}, n^{\hat{k}}),$$

2. for all potential jurisdictions $n^k \in \mathcal{N}$ and all $y^k \in \Re_+^M$,

$$\sum_{c,t} n_{ct}^k p_c(n^k) + \sum_{c,t} n_{ct}^k q_c(n^k) y^k - f(y^k, n^k) \leq 0.$$

3. for all $n^k \in n$,

$$\sum_{c,t} n_{ct}^k p_c(n^k) + \sum_{c,t} n_{ct}^k q_c(n^k) y^k - f(y^k, n^k) = 0.$$

Of course, it is possible to define other notions of $\epsilon-$equilibrium in the same spirit as the alternative notions of the $\epsilon-$core.

5. Tiebout's Assumption Six

In Tiebout's original paper, he laid out seven informal assumptions which he believed were sufficient for market decentralization of efficient allocations in a local public goods economy. The sixth of these was "For every pattern of community services . . . there is an optimal community size." In other words, the economies associated with sharing the cost of producing public goods, are eventually overwhelmed by the costs of crowding. This is more a definition of what a local public goods economy is than an assumption on such economies. If there is not an optimal jurisdiction size, then there is no need for jurisdictions to form, and thus no possibility of competition between jurisdictions that lead market type outcomes. In both the pure public and pure private goods case, the optimal jurisdiction size is equal to the entire population. The presence of an optimal group size, however, leads to the "the integer problem" discussed earlier. Thus, it was generally accepted that it was not possible to show existence economic equilibrium concept satisfying the requirements of Tiebout. (See, for example, Pauly (1970), Bewley (1981), and Atkinson and Stigliz ()). It has now been shown in a series of papers that the assumption of small optimal or near optimal group size is virtually sufficient, by itself, to obtain nonemptiness of approximate cores and existence of approximate equilibrium in large economies.[9]

In the following we will state precisely, for economies with quasi-linear utilities, some conditions ensuring existence of optimal or near-optimal bounded group sizes, and indicate the extension of these conditions to economies and to games without side payments. First, we note that in the case of anonymous or non-differentiated crowding, although there are technical differences between different sorts of assumptions of (strict) small group effectiveness, all such assumptions require that, for each type t, there is a group size that maximizes per capita utility of all the players of that type, given equal cost sharing. For all sufficiently large economies, the core is nonempty if and only if the participants can be partitioned into optimal sized groups where all members of any given jurisdiction have the same demands. Thus, if there are "many" consumers of each type, agents can be partitioned into type-optimal jurisdictions with only a few "left-overs" and, for sufficiently large economies, approximate cores are

[9] Wooders (1979a, b, 1983, 1994b) and a number of other papers.

non-empty and "close" to price-taking equilibrium outcomes.[10] With differentiated crowding the situation is more complex since "optimal" groups depend on relative scarcities of participants, as exposited in a number of papers.[11]

We first introduce the notion of a pregame. Let T be a integer and let Z_+^T denote the $T-$fold Cartesian product of the non-negative integers. Thus, an element of Z_+^T is a profile, listing a number of players of each of a finite number of types. Let Ψ be a function from Z_+^T to the non-negative real numbers R_+. Then the pair (T, Ψ) is a *pregame.* Note that a pregame differs from a game in that there is no fixed population; the pregame states that if the total population is $N = (N_1, ..., N_T)$ and $s \in Z_+^T$ with $s \leq N$ then the payoff or total monetary worth of the coalition s is $\Psi(s)$.

The mildest form of small group effectiveness is boundedness of per capita payoffs. In the case of quasi-linear utilities, this is the assumption that the supremum of average payoffs is finite.[12] Specifically, for economies representable as games with side payments the condition of *per capita boundedness* is that there is a constant C such that for all coalitions S it holds that

$$\frac{\Psi(s)}{|s|} < C.$$

In the case of general games without side payments the assumption is that when the economy grows in size, the set of equal-treatment payoffs, a subset of R^T, is bounded.[13]

A number of papers on economies with local public goods have required an assumption of per capita boundedness. Specifically, Wooders (1981, 1989, 1993) assume that there *exist* states of the economy in the $\epsilon-$ core for all replications of the economy. A state of the economy is in the ϵ-core for all replications if the replication of the state

[10] The above result were obtained in Wooders 1976, 1978, 1980a). For related results, also see Scotchmer and Wooders (1987) and Barham and Wooders (1994).

[11] cf., Wooders (1979a), Wooders and Zame (1984,1987), Shubik and Wooders (1986), and especially the Discussion Paper versions, and also Scotchmer and Wooders (1986).

[12] Wooders (1979b, 1983, 1994b) and other papers. For the convenience of the reader who may be interested in histories of results, a summary and statement of the non-emptiness result of Wooders (1979b) is contained in Kannai (1992).

[13] Wooders 1983,1991c.

of the economy is in the $\epsilon-$core of the corresponding replicated economy. (In the replicated state of the economy the "clones" of an agent in the initial state receive the same allocation as that agent received in the initial state.) This assumption implies boundedness of per capita payoffs since, eventually no matter how large the economy. Thus, this mild assumption implies that the cores converge to the equilibrium outcomes.

To obtain asymptotic results of nonemptiness of approximate cores, existence of equilibrium, and equivalence of cores to competitive outcomes of representing markets for games/economies with a finite number of player types and a finite number of types of commodities, per capita boundedness is virtually the only assumption required. In addition, it is required that the percentage of players (or commodities) of each type is bounded away from zero.[14]

If the finiteness of types is relaxed or the "thickness", ensuring that there are many players (and commodities, if relevant) then the assumption that small groups can realize all or almost all gains to collective activities suffices to ensure that asymptotically equilibria exist and core equivalence holds (in the same models for which exact equivalence obtains under the assumption of strict small group effectiveness.) For economies representable by pregames, the assumption of *small group effectiveness* [15] is

Given $\epsilon > 0$ there is an integer $\eta(\epsilon)$ such that for all profiles N there is a partition n of N such that

1. $\sum_t n_t^k < \eta(\epsilon)$ for each n^k in n and
2. $\max_{n'} \sum_{n^k \in n'} \Psi(n'^k) - \sum_{n^k \in n} \Psi(n^k) \leq \epsilon \sum_t n_t$.[16]

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[14] Wooders 1994a, Section 4, 1994a). Wooders and Zame (1984) provide an example illustrating the need for this "thickness" assumption to ensure nonemptiness of approximate cores satisfying just the condition of per capita boundedness.

[15] The concept of small group effectiveness was introduced in Wooders (1992a) and further studied in Wooders (1992b,1993,1994a,b). Following Wooders (1979b, 1983), Engl and Scotchmer assume per capita boundedness but also they essentially assume thickness (in the form of their results) so they too require small group effectiveness. (Earlier versions of Engl and Scotchmer assumed more restrictive conditions.)

[16] It would be possible to not require superadditive but instead require a feasibility condition on payoffs that $x(N) := \sum_{i \in N} x_i \leq \max \sum_k V(n^k)$ where the maximum is taken over all partitions of N. Obviously, this will not affect the substance of our results.

These conditions, informally, state that given any positive real number ϵ there is a group size $\eta(\epsilon)$ such that within ϵ per capita of all gains to collective activities can be realized by groups bounded in size of membership by $\eta(\epsilon)$. [17] The following example illustrates that without "thickness", small group effectiveness is not asymptotically equivalent to per capita boundedness.

Example: Let (T, Ψ) be a pregame with two types of players, that is, $|T| = 2$. Suppose that players of type 1 know the secret of happiness, and thus, for any coalition $s = (s_1, s_2)$, if $s_1 > 0$ then $\Psi(s) = s_1 + s_2$ but if $s_1 = 0$, then $\Psi(S) = 0$. In such a economy, per capita payoffs are bounded but small groups are *not* effective. In particular, if we bound group sizes by B then for any profile s with $s_1 = 1$ it holds that at $\Psi(s) = s_1 + s_2$ but $\max \sum_k s^k$ where (s^k) is a partition of s is equal to B; the one player of type one can only belong to one coalition. Small group effectiveness is thus equivalent to per capita boundedness with "thickness" (and when per capita boundedness "works"), but small group effectiveness is well suited to handle a broader variety of situations, such as ones with a compact metric space of player types.

It is a remarkable fact that the topology on the spaces of player types can also be relaxed; this is done in Wooders (1993b) and Wooders, Zhong, and Chen (1994). In this case, however, it is necessary to assume both small group effectiveness and per capita boundedness. The following examples from Wooders (1993b) illustrate that these two assumptions are no longer interchangeable.

Example: Let (N^m, v) be a superadditive game with $2m$ players where every pair of players can realize a payoff of $2m$ and $v(N^m) = m^2$. Clearly the game (N^m, v) has 2-player effective groups. But, since the per capita payoff equals m, the per capita payoff

[17] Another closely related condition is that all or almost all gains to improvement can be realized by groups bounded in absolute size, relaxing the minimum efficient scale assumption discussed above. The approximate version of his condition was introduced in Wooders and Zame (1987). A closely related condition is used in Engl and Scotchmer (1993), where it is called "approximate exhaustion of blocking opportunities". Asymptotically, these condition is equivalent to small group effectiveness, defined above (see Wooders (1994a) for a precise statement and a proof.) The condition can quite approximately be called small group effectiveness for improvement.

becomes infinite as the games become large.

Example: Let (N^m, v) be a superadditive game with m players where $v(N^m) = m$ and $v(S) = 0$ for all $S \neq N$. Then the games $\{(N, v)\}$ all have one type, many players of each type, and the same per capita bound of C. Yet the games $\{(N, v)\}$ do not have $\epsilon-$effective B -bounded groups for any $\epsilon > 0$ and $B > 0$.

In conclusion, we remark that the strongest form of the strict small group effectiveness is that all gains to collective activities can be realized by groups bounded in absolute size. It is a remarkable fact that with "thickness", we can well approximate games satisfying the extremely mild condition of per capita boundedness by games satisfying the condition of strict small group effectiveness[18] Of course we can also approximate large games satisfying per capita boundedness by games satisfying less restrictive conditions of small group effectiveness, such as by ones that exhaust gains to scale. Whichever assumption of strict small group effectiveness one chooses (as our choice in this paper) is merely a matter of technical convenience.

6. Results

In this section we give a very brief survey the literature in the context of the considerations given above.

6.1 Anonymous or non-differentiated crowding

A number of non-game-theoretic papers in the literature considered equilibrium and Pareto optimum of economies with anonymous crowding. We remark in particular, McGuire (), Boadway (1980), Berglas and Pines (1981). These papers considered the characterization of optimal equilibrium outcomes.

[18] Wooders (1994a,b).

The formulation of the Tiebout Hypothesis as the convergence of cores to anonymous price-taking equilibrium outcomes was initiated in Wooders (1976,1978,1980). The equilibrium price system introduced in Wooders (1976,1978) has one per unit price for public goods for each jurisdiction (both potential jurisdictions and actual jurisdictions) and a participation price/wage/profit share for each jurisdiction. The profit share is required only in the case of non-constant returns to scale. In this case, the zero-profit condition of free entry equilibrium and potential competition between jurisdictions requires that any surplus arising from marginal cost pricing is re-distributed to the members of a jurisdiction. (Barham and Wooders (1994) provide further discussion of this aspect of the equilibrium concept.) If the set of agents can be partitioned into "type optimal" groups and there are sufficiently many participants of each type then the core is non-empty and coincides with the equilibrium outcome s.Moreover, all states of the economy in the core have the equal treatment property. Wooders (1976,1978) made a number-theoretic assumption to ensure that agents can be partitioned into type optimal groups and that the core is nonempty. Wooders (1989) showed that approximate cores converge to equilibrium outcomes.

As we noted, another approach to a Tiebout Theorem may be to demonstrate that in economies with local public goods, First and Second Welfare Theorems hold. Since the equilibrium states are in the core, the First Welfare Theorem holds for the model of Wooders (1976,1978). Scotchmer and Wooders (1987) provide a Second Welfare Theorem for a closely related model (allowing variable intensity of consumption). Barham and Wooders extend the Second Welfare Theorem to more than two types of participants.

Wooders (1978) illustrated by an example that in the core and in equilibrium agents of different types might "mix" in the same jurisdiction but this would occur only if the agents all had the same "demands" for public goods and crowding. Scotchmer and Wooders (1987) allowed variable intensity of the public good, and provided another example showing that agents of different types may mix in equilibrium if they have the same demands. Scotchmer and Wooders (1987) also state a Second Welfare Theorem.

The price system of Wooders (1976,1978) is closer to an admission price system than to a Lindahl price system– it is trivial to aggregate the per unit prices and participation prices into admission prices. Barham and Wooders introduce the concept of Lindahl equilibrium with participation prices into the anonymous crowding framework and show that even if entrepreneurs may charge different Lindahl prices within the same jurisdiction, the only jurisdictions that will succeed in attracting residents are those where all participants of the same jurisdiction pay the same price. Barham and Wooders (1994) also extend the Second Welfare Theorem of Scotchmer and Wooders (1987).

Other recent studies characterize the equilibrium outcomes in economies with anonymous crowding........Berglas and Pines (1981) showed that transportation costs could affect the desirability of homogeneous jurisdictions.

Remark. With the purpose of informing the reader and stimulating debate, we note that Scotchmer, in correspondence with a number of individuals and in various drafts of papers, states that the equilibrium concept of Wooders (1978) is not price taking and the equilibrium concept of Barham and Wooders (1994), which the authors view as that of Wooders (1978), is instead equivalent to the equilibrium of Scotchmer and Wooders (1986) or (1987). Scotchmer has also expressed the opinion that convergence of cores to price taking equilibrium outcomes was initiated in Scotchmer and Wooders (1987), and thus she apparently reject eh earlier result. finally, Scotchmer is of the opinion that the equal treatment property of the core in economies with local public goods was initiated in her joint work with Wooders (and thus apparently not in Wooders (1976, 1978, Theorem 3 (i)).

6.2 Differentiated Crowding

Models of differentiated crowding were studied in McGuire () and Berglas (,). These authors discussed the problem of allocating individuals to jurisdictions from the viewpoint of a social planner. The model of differentiated crowding that lead to the framework with crowding types uses in this paper was introduced in Wooders (1981). That paper, and several subsequent papers considered convergence of the core and approximate cores to equilibrium outcomes in the presence of several public goods and several private goods. Before discussing this work further, however, we discuss the some unpublished research of Scotchmer and Wooders (1986).

Scotchmer and Wooders (1986) introduce the analogues of the equilibrium concepts presented earlier for the differentiated crowding case, where the prices depend on the "type" of the participant. They use a one public, one private good model and assume a form of strict small group effectiveness. It is shown that the core, the Lindahl, and the admission equilibrium outcomes coincide. It is clear their results all depend on the nonemptiness of the core. This is itself not a serious problem since, as shown in Wooders (1979a,1983), Shubik and Wooders (1983a,b) and other papers, the conditions on the model ensure that if agents can be appropriately partitioned, then the core is non-empty and large economies of the sort studied in Scotchmer and Wooders (1986) have non-empty approximate cores. The restrictiveness of the model to one private and one public good and to strict small group effectiveness appear to present more major problems.

The model developed in Wooders (1981,1988,1989,1993) is significantly less restrictive than that in Scotchmer and Wooders (1986). In particular, in Wooders (1981) it is shown if all participants take prices for private goods as given and markets for public goods are "contestable" – firms can enter and provide public goods or consumers can "opt out" and provide the public goods for themselves – then the core converges to the set of equilibrium outcomes. Moreover, in the proof of Wooders (1981) it is essentially shown that the core converges to set of the Lindahl equilibrium outcomes as defined in Scotchmer and Wooders (1986). The convergence theorems of Wooders (1981,1989) involves the notion of coalition formation costs. In Wooders (1993) the fact that Wooders (1981,1989) essentially shows convergence to Lindahl equilibrium outcomes is presented as a Theorem. Moreover, it is shown that even in these general circumstances, equivalence of the Lindahl equilibrium, the admissions equilibrium, and the core obtains.

It is an immediate consequence of Wooders (1983, Theorem 3) that, in the environments of the above papers, if small groups are strictly effective then the core has the equal-treatment property. If in addition, utilities are quasi-linear, the equal treatment, and asymptotic equal treatment results of Wooders (1979a,b,1994a) immediately apply.

Some recent related work includes McGuire (1994), Brueckner (1994), ...

Remark. In correspondence with several researchers, Scotchmer has claimed that the equal treatment property of the core of economies with public goods was first shown in Scotchmer and Wooders (1986,1987). Scotchmer also sees the proof of convergence of Wooders (1993) as originating in joint work of Wooders with Scotchmer rather than in Wooders (1981). The authors of this paper agree that Scotchmer and Wooders (1986) does contain an equal treatment result and the proof of all the above convergence theorems are related.

6.3 Crowding Types

The crowding type model is introduced in Conley and Wooders (1994a). We study admission price equilibrium for a transferable utility economy. We show that the first welfare theorem is true but that the second welfare theorem is not. This is mainly because the core may not exist in general. If the economy satisfies SSGE, then the core is equal treatment and is equivalent to the set of admissions price equilibrium allocations.

In subsequent research, we find that the equivalence of the Lindahl equilibria and core obtains only in a restricted class of economies. This is mainly because it is not possible in every case to anonymously decompose admission price equilibrium into per unit prices for public goods. This is somewhat surprising, since no such difficulty is encountered in either the anonymous or differential crowding case. If Lindahl prices exist, then it is immediate that they imply a set of admissions prices that also decentralize the core. Also, when they exist, the Lindahl equilibrium allocations are contained in the core.

The homogeneity properties of the core in the crowding type model are also something of a surprise. Mixing of types within jurisdictions is optimal in general in the differential crowding case because some agents may be complementary. Optimal symphony orchestras contain more than just violinists, for example. There is a basic tension between segregating according to type in order to eliminate conflict over what public goods bundle should be produced, and mixing in order to take advantage of the beneficial types of crowding. In the crowding types model, there is no such tension. It is possible to have taste homogeneous jurisdictions that take advantage of the full array of different skills. In the example above, we would expect that a symphony which agreed on the best number of concerts to give each year would be able to provide it's members with more per capita utility than one in which members had different opinions.

Unfortunately, this turns out not to be true. Consider the following simple matching problem. Suppose there are two crowding types, Smokers and Nonsmokers, denoted S and N, respectively. Also suppose there are two taste types, Lovers and Haters of second hand smoke, denoted L and H, respectively. Assume agents of all four possible types, denoted $SL, SH, NL,$ and NH, appear in equal proportion in the population. The utility functions are the following:

$$U^H(\{S,S\}) = 0, \quad U^L(\{S,S\}) = 10,$$

$$U^H(\{S,N\}) = 5, \quad U^L(\{S,N\}) = 5,$$

$$U^H(\{N,N\}) = 10, \quad U^L(\{N,N\}) = 0,$$

and the utility received from being in every other possible type of jurisdiction is zero. This implies the following characteristic function for the associated game:

$$\Gamma(\{SL,SL\}) = \Gamma(\{NH,NH\}) = 20$$

$$\Gamma(\{SH,NL\}) = \Gamma(\{SL,SH\}) = \Gamma(\{SL,NL\}) = 10$$

$$\Gamma(\{SL,NH\}) = \Gamma(\{SH,NH\}) = \Gamma(\{NL,NH\}) = 10$$

$$\Gamma(\{SH, SH\}) = \Gamma(\{NL, NL\}) = 0,$$

and zero for every other jurisdiction type. Observe that any state which can be improved upon can be improved upon by two person jurisdictions. Thus, if at least two of every agent type appear in the population, SSGE is satisfied. Now consider the case where the population consists of two of each of the four agent types. We claim that one particular core state consists of one jurisdiction each with compositions: $\{SL, SL\}, \{NH, NH\}$, and two jurisdictions with composition: $\{SH, NL\}$. Clearly, forcing the two heterogeneous jurisdiction to become homogeneous is Pareto dominated. Thus, the core may be taste heterogeneous in general.

7. Market Games with Crowding Types

Some familiarity with the models and results discussed in preceding sections may suggest that there are certain features common to all the models that drive the results. These features are shared by models with coalition production, such as those in Böhm (1974) and Bennett and Wooders (1979) and also by private goods exchange economies, such as Shapley and Shubik (1967). Roughly, the common features of the economies are superadditive and small group effectiveness (all or almost all gains to collective activities can be realized by groups of participants bounded in size of membership). Equivalently, when there are "many" commodities of each type and many players similar to each player, then the common features can be described as simply superadditive and boundedness of per capita payoffs. As shown in increasing generality in a series of papers initiated by Wooders (1979a), with the assumptions of superadditive and small group effectiveness, large economies, including ones with coalition production, clubs, local public goods, and collectively consumed and/or produced goods more generally, share familiar properties of exchange economies with concave utility functions.

Tiebout's intuition was that whenever small groups can realize all or almost all gains to collective consumption in economies with local public goods, then large economies are competitive. The observations above leads to the intuition that, in any economy, when small groups are effective then the economy 'resembles' a competitive market is the intuition behind the theory of large games and economies with effective small groups. This theory builds on the work of Shapley and Shubik (1967) most directly on large economies with private goods[19] and on Shapley and Shubik (1969), which relates markets to 'totally balanced' games. The approach initiated in Wooders (1979a) shows that when small groups are effective then large economies are like competitive markets – there is some set of commodities such that, relative to those commodities there is a (complete) price system satisfying virtually all the properties of a competitive equilibrium in a private goods exchange economy. In this section we review some of the main points of the theory of large games and economies with effective small groups and indicate its extension to models satisfying the anonymity properties introduced in Conley and Wooders (1994a).

Let us first consider an example of a game derived from an economy. We will use the economic model with crowding types of this paper but with the assumption of quasi-linear utilities.

Example We consider exactly the general model introduced above but with the additional assumption that the utility function of each agent is quasi-linear, that is, $u_t(x_i, y^k, n^k) = x_i + h_t(y^k, n^k)$ where $i \in n^k$ and y^k is the quantity of public good produced in jurisdiction n^k. The cost in terms of private good of producing y^k public good for a jurisdiction with membership n^k is given by the function $f(y^k, n^k)$. The maximum total transferable utility available to a jurisdiction with membership n^k is

$$V(n^k) = \max_{y^k} \left(\sum_{c,t} n^k_{ct}\omega_t - f(y^k, n^k) + \sum_{c,t} n^k_{ct} h_t(y^k, n^k) \right).$$

Clearly, a feasible state $(X, Y, n) \in F$ is Pareto efficient if and only if it maximizes

[19] Of course numerous other papers on large private goods economies are relevant, but the path-breaking work that is especially relevant for us in the paper is Shapley and Shubik (1967).

$\sum_k V(n^k)$.

Recall that a feasible state $(X, Y, n) \in F$ is in the *core* of the economy if it cannot be improved upon by any jurisdiction. In the context of games with side payments (N, V) the core is described as a set of utility vectors. A utility vector u is in the *core of the game* (N, V) if there does not exist a coalition S such that

$$V(S) > u(S) := \sum_{I \in s} u_i.$$

This illustrates the derivation of a game from an economy. The following discussion is based on Wooders (1978b).

For a game (N, V) we can define an equilibrium where the equilibrium prices are utility prices. A price vector μ is an *equilibrium price system*if

$$\mu \cdot s \geq V(s) \text{ for all subprofiles } s \text{ of } n \text{ and}$$

$$\mu \cdot n = V(n).$$

We may think of these prices as utility admission prices to groups. The price μ_t for a player of type t states the admission price/wage/profit share required to entice a player to type t to join a group. Note also that prices are linear functions of amounts of players of each type.

We can view a (utility) price system as a complete price system in a market where all participants have the utility function $V(\cdot)$.[20]

Define

$$V^b(n) = \min_q q \cdot n$$

where the minimum is over all vectors q satisfying $q \cdot s \geq V(s)$ for all profiles $s \leq n$. The following Proposition is based on the Bondareva-Shapley result that a game has a nonempty core if and only if is 'balanced.'

[20] In fact, this is precisely what Shapley and Shubik do. To show that a balanced game is equivalent to a market they construct a market from the game. The market constructed is one where the commodities are the player types and where all participants have the utility function (or payoff function V).

Proposition. (Wooders 1979a), Theorem 3) A game (N, V) has an equilibrium if and only if one of the following two conditions is satisfied:

1. $V^b(n) = V(n)$;
2. the game (N, V) has a nonempty core.

A necessary and sufficient condition for the existence of an equilibrium can also be demonstrated in terms of properties of partitions of the total player set into coalitions. This is a generalization of the famous "integer problem" of economies with clubs and/or local public goods.

Proposition. (Wooders 1979a), Theorem 3). A price vector μ is an equilibrium price vector if and only if there is a partition n of the total player set N into coalitions n^ksuch that

1. $\mu \cdot s \geq V(s)$ for all subprofiles s of N and
2. $\mu \cdot n^k = V(n^k)$ for all groups n^k in the partition.

In general, the equilibrium prices coincide with the equal-treatment core of the game. Note that it is not required that small groups are strictly effective.

Proposition. (Wooders (1979a), Theorem 6). Let (N, V) be a game. Then μ is an equilibrium price vector for the game if and only if u is in the equal-treatment core of the game.[21]

7.1 Anonymous Market Games

There is an important question that needs to be addressed for the above results to be applicable to anonymous pricing in public good economies. As discussed in Wooders (1991), for any economy where all participants have quasi-linear utilities, the payoff to

[21] Note that the above result does not depend on any assumption about the effectiveness of small groups. Scotchmer (1994) provides an example of this result in the context of a club model. Also observe (as is rather obvious) prices are linear functions with domain the space of profiles.

a group of participants can be represented as a function of the number of players of each type in the group (where 'type' may be a vector of attributes, including taste type) and the amounts of endowments of commodities owned by the members of the groups. As pointed out, the crucial question is when the payoff to a group of players in an economy as independent of the tastes of the members of that particular group. If we can show that under some set of circumstances the payoff to a group is independent of the utility functions of the members of the group, then all the above results apply and the price system can be shown to have the anonymity property. Moreover, if small groups are effective, all the above results hold – nonemptiness of approximate cores, existence of approximate equilibrium, convergence of cores to equilibrium outcomes, and so on.

Let us suppose now, for simplicity *independence of crowding and taste types,* that is, letting $c(s)$ denote the crowding profile of a profile of players s, it holds that there is a valuation function W defined on subprofiles of N so that if p is an equilibrium price system for the game (N, W) then it holds that for any group s with crowding profile $c(s)$, there is an equilibrium price system μ for the game (N, V) so that, for each type t,

$$\max V(s) - p \cdot c(s) = \mu_t.$$

Independence of crowding and taste types may appear to be a strong assumption. As we show in Conley and Wooders (1994c), with a relatively mild assumption on the economy, called *type continuity,* it holds quite generally. Since its formulation is complicated but the idea is easy, we note that type continuity ensures that small groups of players can have only small effects on per capita payoffs of large groups – type continuity is small group effectiveness stated in terms of the economic variables underlying a game.

For large economies satisfying the type continuity condition all the properties that have been shown for large games apply. These properties include:

1. Approximate cores are nonempty and the approximation can be made arbitrar-

ily close as the economy becomes large. (Wooders (1978,1983), and numerous subsequent papers, including Wooders (1992a,1994b)).

2. Approximate price-taking equilibria exist and equilibrium outcomes are in approximate cores.[22] The equilibrium concept is one where the player crowding types themselves are the commodities. The existence of such approximate equilibria was first shown in Wooders (1978). Related papers include Shubik and Wooders (1986), Wooders (1988,1992b,1994a,b), and Engl and Scotchmer (1993).[23]
3. Core payoffs are monotonic – that is, if the abundance of one type of player increases then the core payoff to that type does not decrease and may well increase. (Scotchmer and Wooders (1986), Wooders (1994b), Engl and Scotchmer (1993).
4. Approximate cores converge to equilibrium outcomes Wooders (1979), Wooders and Zame (1987), Wooders (1991a,1992,1994b), and Engl and Scotchmer (1993).[24]
5. Approximate cores are asymptotically equal-treatment. This was first shown in Wooders (1979), and also in the 1982 Discussion Paper version of Shubik and Wooders (1986); the result is made more accessible in Wooders (1994b). Engl and Scotchmer (1993) present a different formulation (aggregating over groups) of a closely related result.
6. When small groups are strictly effective then the core has the equal treatment property; see Wooders (1979a, 1983,1994b), Scotchmer and Wooders (1986) and other papers.

The above results all hold for economies with quasi-linear utility functions. A

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[22] For exchange economies this result was first shown by Shapley and Shubik (1967).

[23] A full discussion of concepts is beyond the scope of this paper. We note, however, that the hedonic core equivalent (in the case of games with indivisible players) to the prices for player types of Shapley and Shubik (1969) and Wooders (1978) and the equilibrium prices for the commodities in the markets representing games and economies in Wooders (1988,1991a,b,1994a,b). It is a distinct idea from that of the attribute core of Wooders (1992).

[24] Wooders and Zame (1987) used a strong form of small group effectiveness but in Wooders (1991a,1994b) it is shown that the Wooders and Zame argument extends to hold with simply per capita boundedness and "thickness", bounding the percentage of players of each type away from zero. While Engl and Scotchmer considered core convergence in earlier versions of their papers, they first obtained such results with per capita boundedness in 1993. Their results partially extend the results of Wooders (1992a), in that Engl and Scotchmer allow divisible commodities.

number of the results also hold for economies modeled as games with nontransferable utilities, cf. Wooders (1983). [25]

The representing markets studied in prior research are ones where the commodities of the market are the player types and the pricing system prices the players themselves. Alternatively, one can take the payoff function as depending on the player types and on the observable characteristics and endowments of commodities of participants in the economy. The important question is when we can represent the economy by a game (or by another economy) where the utility functions of the players do not enter into the valuation function (the payoff or worth function of the game). In other words, given an economic model with T preference types, described by concave utility functions u_t over commodities and C crowding types (which are special sorts of commodities), when can we represent the economy by a game where the payoffs to a group depend only on the crowding profile of the group and the endowment of the group members of (other) commodities? When such a representation is possible then, with a commodity space consisting of commodities and observable characteristics of participants, there is a first best price taking equilibrium.[26]

7.2 Shapley-Shubik Prices for Player Types, Subsidy-Free Prices, and the Hedonic and Attribute Cores

In the following subsection we discuss the "hedonic core", which is closely related to the equilibrium in utility prices discussed above and subsidy-free pricing and contrast the hedonic core with the attribute core, a distinct concept.

Let $(\mathbf{R}^M_+, A; m)$; be a market. Define the function $\overline{w}$ as follows:

$$\overline{w}\ (x,s)\ = \max \sum_{tq} u^t(x^{tq}; s)$$

[25] Other research and work in progress (Kovolenkov (1994)) indicates that all the above results extend and extend with remarkable generality. Indeed, there is in general no need to impose the requirement that there is a finite set of types of players and commodities. (or even that there is a topology on the set of player types) (cf, Wooders (1993a,b) and Wooders, Zhong, and Chen (1994)) for core convergence).

[26] This question was posed also in Wooders (1991b). More generally, each crowding type could be a point in some vector space.

where the maximum is taken over the set of allocations of commodities $\{x^{tq} : t = 1, ..., T,\ q = 1, ..., m_t\}$ such that

$$\sum_{tq} x^{tq} = x.$$

Now, for any profile $s \leq m$ define $e(s) = \sum_t s_t e^t$; $e(s)$ is the total endowment of a group s with profile s.

We will now define a (market) game in characteristic form by (N, w) where the characteristic function is given by

$$w(s) = \overline{w}\ (e(s), s).$$

Now let W be the concave and continuous utility function defined as in Shapley and Shubik (1967) and with domain $R^M \times R^T$. The equilibrium payoffs coincide with the equal-treatment core of the game (Wooders (1979a); see Appendix A of this paper).

Given W and m, a vector q is in the *hedonic core* if

1. $W(e(m), m) = q \cdot (m, e(m))$ and
2. $W(s, e(s)) \leq q \cdot e(s)$ for all subprofiles s of m.

It is important to note that *only coalitions of participants may improve upon a payoff;* we restrict (2) to subprofiles of m. It is apparent that the hedonic core is very similar to the equilibrium price system of Wooders (1978) and hedonic core payoffs are also the equilibrium prices for players of the canonical representation of a game as a market in Shapley and Shubik (1969). The environment that considered here is somewhat more general since the players may be characterized by a vector of attributes. When the commodities are the player types, the framework is exactly that of Shapley and Shubik (1967), except that divisibility and monotonicity are not required.

Proposition. Let p be an equilibrium price system for an economy with profile of participants m and where a participant of type t has the endowment of 1 unit of the t^{th} commodity (his player type) and where all participants have the utility function W. Then p is in the hedonic core. Moreover, if q is in the hedonic core, then q is an equilibrium price system.

Note that the Shapley-Shubik price system and the hedonic core place constraints on trade. This is apparent when we compare these concepts to the prices for players of Wooders (1992b). We now turn to another equilibrium concept for economies modeled as games in characteristic form (with divisible or indivisible players). For this concept, the commodities/observable attributes of players are themselves the players of the game. For example, a player may put his money into a coalition called a mutual fund, and he may put his leisure time into a tennis club. Unlike the situations modeled by subsidy-free pricing and the hedonic core, players do not put their total endowment into one coalition.

Let $\epsilon \geq 0$ be given and let z be an endowment. Let Λ be a superadditive function mapping vectors of attributes/commodities into R_+. (Recall our footnote above about superadditive.) When we interpret the endowment z as the description of the player set of a game with z_q players of type q, then the endowment determines a game, called an *attribute game*. A vector $p \in \mathbf{R}_+^Q$ is in *the attribute ϵ-core* (given the total endowment z) if

$$p \cdot z' \geq \Lambda(z') - \epsilon \|z'\| \text{ for all } z' \in Z_+^Q, z' \leq z \text{ and}$$
$$p \cdot z \leq \Lambda * (z).$$

The following example illustrates the difference between the Shapley-Shubik notions, subsidy free prices and the hedonic core and the attribute core, where the players are units of commodities. (This example also appears in Wooders (1992b)).

Example: Let $(2, \Lambda)$ be the technology given in Example 1. Let $\{(N^\nu, e^\nu)\}$ be a sequence of economies where $N = \{1, ..., \nu + 1\}$,

$$e^\nu(1) = (2\ \nu, 0) \text{ and}$$
$$e^\nu(i) = (0, 1) \text{ for } i \in N^\nu, i \geq 2.$$

Player 1 is assigned 2ν units of the first attribute and zero units of the second, while all the other players are each assigned one unit of the second attribute.

The (equal-treatment) core of the economy (N^ν, e^ν) is the set

$$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + \nu\ x_2 = \nu\},$$

the attribute core is the set

$$\{(p_1, p_2); p_1 = 0, p_2 = 1\},$$

while the set of hedonic core payoffs (or, equivalently, subsidy-free prices) is

$$\{(p_1, p_2) : p_1 \geq 0, p_2 \geq 0 \text{ and } 2\nu p_1 + \nu p_2 = \nu\}.$$

Note that $(p_1, p_2) = (1/2, 0)$ is a subsidy-free equilibrium price system and $p_1 e^\nu(\{1\}) = \nu$. Also $(p_1, p_2) = (0, 1)$ is an equilibrium price system and for this price system $p_1 e^\nu(\{1\}) = 0$, indicating that the set of subsidy-free equilibrium payoffs coincides with the core of the limiting market. Note also the non-convergence of approximate and exact cores to competitive payoffs. The fact that in the definition of the hedonic core the constraints on the hedonic prices (linear functions on the space of attributes) are coalitional constraints where the coalitions are coalitions of players places constraints on trades.

As shown in Wooders (1992b) when small groups are effective then the attribute core payoffs to participants and the hedonic core payoffs to participants converge to the same limits. This is, of course, a consequence that the Walrasian prices for the markets where all individuals have the utility function given by Λ^* are equal to the attribute core payoffs, contained in the set of hedonic core payoffs, contained in the core. Thus, when small groups are effective, convergence of the core to the Walrasian prices ensures that all three concepts have the same limiting payoffs to players.

Remark. In private discussions with Karl Vind, he has expresses the view that the attribute core is a notion closer notion to Edgeworthian competition than the core (or the hedonic core). According to Vind, Edgeworth seemed to have the view that agents could enter into multiple contracts and make different contracts for different goods with different agents.

Remark. Scotchmer has a very different viewpoint than Vind. She has expressed the opinion, quite strongly, that the attribute core is the same as the hedonic core and was mis-appropriated from the work of Engl and Scotchmer. She has also made a number of claims concerning the equal treatment property of the core with strictly effective small groups and prices for players (in Wooders (1988,1979a)). Again, to stimulate discussion and to avoid any unfair presentation of ideas, we bring these claims to the attention of the reader. Scotchmer claims that

1. The price system of Wooders (1988b), (and implicitly, therefore the price system associated with the market constructed in Wooders 1994a) was taken from joint work with Scotchmer. The pricing system in Wooders (1988) is a price system for a differentiated commodities market as in Mas-Colell. Moreover, in the finite-dimensional case it is exactly the price system of the canonical market in Shapley and Shubik(1969) and indeed it is the Walrasian price system of the market in Shapley and Shubik (1969). Thus, the authors of this paper, while recognizing that the reader may have a different opinion, choose to attribute the equilibrium of such markets to Walras, Shapley, and Shubik rather than Scotchmer and Wooders (1986).
2. The fact that Wooders' (1979,1983) results ensured that large games were approximately market games was noted in Shubik and Wooders (1986) and its 1982 Discussion paper version. We remark that in none of the papers of this author originality is not claimed for the price system.[27] Also, recall our discussion of the price system of Wooders (1979) and Bennett and Wooders (1979).
3. That Wooders (1992b) has mis-appropriated the hedonic core from Engl and Scotchmer. The attribute core is clearly a distinct concept, as illustrated by our

[27] The man purpose of these papers is to explain that large games in general behave like markets.

example. It is the case that, if small groups are effective, then under the assumption of small group effectiveness the core converges to the competitive payoffs of the representing market. (A version of this was first shown in Wooders (1979a)). It is indeed the case that the original version of the Engl-Scotchmer paper predates Wooders (1992b) and of course this is noted in the later paper.

4. That the equal treatment and asymptotic equal treatment results of Wooders (1992b,1994b) are derivative of unpublished research of Scotchmer and Wooders (1986). It is the case, however, that these results initially appeared in Wooders (1979a,1979b) and in fact the asymptotic equal treatment Wooders (1992b,1004b) is exactly copied from the earlier work. (In particular, a version of the asymptotic equal treatment result of these papers also appears in the Cowles Discussion Paper version of Shubik and Wooders (1986), and the result in Wooders (1994b) is simply copied from that source.)

Another issue arises with resect to Engl-Scotchmer (1993) and Wooders (1992b), which should be pointed out. In versions of their paper prior to (1993) Engl and Scotchmer required stronger assumptions than just the mild assumptions of Wooders (1979b,1992b) and several other papers. Thus, their (except for the feature that Engl-Scotchmer allow divisible players/commodities) their core convergence results and asymptotic equal treatment were weaker than those of Wooders (1991a), themselves based on earlier results of Wooders and Zame (1987) and Wooders (1979b).

8. Conclusions

References

Barham, V. and M. Wooders (1994): "First and Second Welfare Theorems for

Economies with Collective Goods," University of Ottawa Working Paper, to appear.

Bennett, E. and Wooders, M.H (1979): "Income Distribution and Firm Formation," *Journal of Comparative Economics,* **3**, pp. 304-317.

Bewley, T (1981): "A Critique of Tiebout's Theory of Local Public Expenditure," *Econometrica,* **49**, pp. 713-740.

Berglas, E (1976): "Distribution of Tastes and Skills and the Provision of Local Public Goods," *Journal of Public Economics,* **6**, pp. 409-423.

Berglas, E. and D. Pines (1981): "Clubs, Local Public Goods, and Transportation Models: A Synthesis," *Journal of Public Economics,* **15**, pp. 141-62.

Boadway, R (1980): "A Note on the Market Provision of Club Goods," *Journal of Public Economics,* **13**, pp. 131-7.

Böhm, V (1974): "The Limit of the Core of an Economy with Production," *International Economic Review,* **15**, pp. 143-148.

Bondareva, O (1962): "Theory of the Core in an n-Person Game," Vestnik, LGU13, 141-142 (in Russian), (Leningrad State University, Leningrad).

Brueckner, J (1994): "Tastes, Skills and Local Public Goods," *Journal of Urban Economics,* **35** pp. 201-20.

Cole, H. and E. Prescott (1994): "Valuation Equilibria with Clubs," Federal Reserve Bank of Minneapolis Staff Report 174.

Conley J (1991): "Convergence Theorems on the Core of a Public Goods Economy: Sufficient Conditions," *Journal of Economic Theory,* **62** pp. 161-85.

Conley, J. and M.H. Wooders (1994a): "Equivalence of Tiebout Equilibrium and the Core in a Model with Crowding Types," University of Illinois Working Paper, revised.

Conley, J. and M.H. Wooders (1994c): "Models with Crowding Types and Market Games," notes in manuscript.

Debreu G. and Scarf, H (1963): "A Limit Theorem on the Core of an Economy," *International Economic Review,* **4**, pp. 235-246.

Edgeworth, F.Y (1881): *Mathematical Psychics.* Kegan Paul: London.

Ellickson, B (1973): "A Generalization of the Pure Theory of Public Goods," *American Economic Review,* **63,** pp. 417-432.

Ellickson, B (1979): "Competitive Equilibrium with Local Public Goods," *Journal of Economic Theory,* **21**, pp. 46-61.

Elul, R (1987): "A Comment on the Existence of Strong $\epsilon-$Cores of Large Games," Weizmann Institute Department of Mathematics manuscript.

Engl, G. and Scotchmer, S (1993): "The Core and Hedonic Core: Equivalence and Comparative Statics," University of Irivne Discussion Paper (with other versions dated 1991 and 1992).

Faulhaber, G.R (1975): "Cross-Subsidization; Pricing in Public Enterprises," *American Economic Review,* **65**, pp. 966-977.

Foley, D (1970): "Lindahl's Solution and the Core of an Economy with Public Goods," *Econometrica,* **38**, pp. 66-72.

Kaneko, M. and Wooders, M.H (1982): "Cores of Partitioning Games," *Mathematical Social Sciences,* **3**, pp. 313-327.

Kaneko, M. and Wooders, M.H (1984): "The Core of a Game with a Continuum of Players and Finite Coalitions; Nonemptiness with Bounded Sizes of Coalitions," IMA Preprint Series No. 126, Institute of Mathematics and Its Applications, University of Minnesota.

Kaneko, M. and Wooders, M.H (1986): "The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and some Results," *Mathematical Social Sciences,* **12**, pp. 105-137.

Manning, J (1992): "Local Public Goods: First Best Allocations and Supporting Prices," Department of Economics, University of Rochester Discussion Paper.

Mas-Colell, A (1975): "A Model of Equilibrium with Differentiated Commodities," *Journal of Mathematical Economics,* **2**, pp. 263-95.

Mas-Colell, A (1979): "A Refinement of the Core Equivalence Theorem," *Economic Letters,* pp. 307-310.

McGuire, M (1974): "Group Segregation and Optimal Jurisdictions," *Journal of Political Economy,* **82**, pp. 112-132.

McGuire, M.C (1991): "Group Composition, Collective Consumption, and Collaborative Production," *American Economic Review,* **81**, pp. 1391-1407.

Moulin, M (1988): *Axioms of Cooperative Decision Making.* Econometric Society Monograph No.15, Cambridge Press, Cambridge.

Pauly, M (1970): "Cores and Clubs," *Public Choice,* **9**, pp. 53-65.

Samuelson, P (1954): "The Pure Theory of Public Expenditures," *Review of Economics and Statistics,* **36**, pp. 387-389.

Scarf, H (1967): "The Core of an n-person Game," *Econometrica,* **35**, pp. 50-69.

Schweizer, U (1983): "Efficient Exchange with a Variable Number of Consumers," *Econometrica,* **51**, pp. 575-585.

Schotter, A. and Schwödiauer, G (1980): "Economics and Game Theory: A Survey," *Journal of Economic Literature,* **58**, pp. 479-527.

Scotchmer, S (1994): "Public Goods and the Invisible Hand," in *Modern Public Finance,* by J. M. Quigley, and E. Smolensky Harvard University Press, Cambridge, Massachusetts, and London, England, pp. 93-125.

Scotchmer, S. and M.H. Wooders (1986): "Optimal and Equilibrium Groups," Harvard Discussion Paper 1251.

Scotchmer, S. and Wooders, M.H (1987): "Competitive Equilibrium and the Core in Economies with Anonymous Crowding," *Journal of Public Economics,* **34**, pp. 159-174.

Scotchmer, S. and Wooders, M.H (1988): "Monotonicity in Games that Exhaust Gains to Scale," IMSSS Technical Report No. 525, Stanford University.

Shapley, L.S. and Shubik, M (1969): "Quasi-Cores in a Monetary Economy with Nonconvex Preferences," *Econometrica,* **34**, pp. 805-827.

Shapley, L.S. and Shubik, M (1969): "On Market Games," *Journal of Economic Theory,* **1**, pp. 9-25.

Shapley, L.S. and Shubik, M (1975): "Competitive Outcomes in the Cores of Market Games," *International Journal of Game Theory,* **4**, pp. 229-237.

Sharkey, W.W. and Telser, L.G (1978): "Supportable Cost Functions for the Multiproduct Firm," *Journal of Economic Theory,* **18**, pp. 23-37.

Shubik, M. and Wooders, M.H (1983): "Approximate Cores of Replica Games and Economies: Part I. Replica Games, Externalities, and Approximate Cores," *Mathematical Social Sciences,* **6**, pp. 27-48.

Shubik, M. and Wooders, M.H (1983): "Approximate Cores of Replica Games and Economies: Part II. Set-up Costs and Firm Formation in Coalition Production Economies," *Mathematical Social Sciences,* **6**, pp. 285-306.

Shubik, M. and Wooders, M.H (1989): "Near-Markets and Market-Games," *Economic Studies Quarterly,* **37**, pp. 289-299.

Silva, E.C.D. and Kahn, C.M (1991): "Exclusion and Moral Hazard; the Case of Identical Demand," *Journal of Public Economics,* forthcoming.

Tauman, Y (1987): "The Aumann-Shapley Prices: A Survey," in *The Shapley Value: Essays in Honor of Lloyd S. Shapley,* by ed. A. Roth Cambridge University, Cambridge, MA.

Tiebout, C (1956): "A Pure Theory of Local Expenditures," *Journal of Political Economy,* **64**, pp. 416-424.

Winter, E. and Wooders, M.H (1990): "On Large Games with Bounded Essential Coalition Sizes," University of Bonn Sonderforschungsbereich 303 Discussion Paper B-149. In progress with *Games and Economic Behavior* .

Wooders M.H (1978): "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good," *Journal of Economic Theory,* **18**, pp. 328-348.

Wooders, M.H (1979a): "A Characterization of Approximate Equilibria and Cores in a Class of Coalition Economies," (A revision of Stony Brook Department of Economics Working Paper No. 184).

Wooders, M.H (1979b): "Asymptotic Cores and Asymptotic Balancedness of Large Replica Games," Stony Brook Working Paper No. 215, Revised July 1980.

Wooders, M.H (1980): "The Tiebout Hypothesis: Near Optimality in Local Public Good Economies," *Econometrica,* **48**, pp. 1467-1486.

Wooders, M.H (1981): "A Limit Theorem on the $\epsilon-$Core of an Economy with Public Goods," National Tax Institute of Japan Paper No. 20.

Wooders, M.H (1983): "The Epsilon Core of a Large Replica Game," *Journal of Mathematical Economics,* **11**, pp. 277-300.

Wooders, M.H (1988): "Stability of Jurisdiction Structures in Economies with a Local Public Good," *Mathematical Social Sciences,* **15**, pp. 24-29.

Wooders, M.H (1988): "Large Games are Market Games 1. Large Finite Games," C.O.R.E. Discussion Paper No. 8842 (Revised June 1989).

Wooders, M.H (1989): "A Tiebout Theorem," *Mathematical Social Sciences,* **18**, pp. 33-55.

Wooders, M.H (1991a): "On Large Games and Competitive Markets 1: Theory," University of Bonn Sonderforschungsbereich 303 Discussion Paper No. B-195 (Revised August 1992).

Wooders, M.H (1991b): "On Large Games and Competitive Markets 2: Applications," University of Bonn Sonderforschungsbereich 303 Discussion Paper No. B-196 (Revised August 1992).

Wooders, M.H (1992a): "Inessentiality of Large Groups and the Approximate Core Property; An Equivalence Theorem," *Economic Theory,* **2**, pp. 129-147.

Wooders, M.H (1992b): "The Attribute Core, Core Convergence, and Small Group Effectiveness; The Effects of Property Rights Assignments on Attribute Games," *Essays in Honor of Martin Shubik,* Dubey, P. and Geanakoplos, J. eds. forthcoming.

Wooders, M.H (1993): "Convergence of the Core to Competitive Outcomes in Economies with Public Goods," University of Toronto, Department of Economics, Discussion Paper No. 9301.

Wooders, M.H (1994a): "Large Games and Economies with Effective Small Groups," in *Game-Theoretic Methods in General Equilibrium Analysis,* by eds. J-F. Mertens and S. Sorin Kluwer Academic Publishers, Dordrecht, Boston, and London.

Wooders, M.H (1994b): "Equivalence of Games and Markets," *Econometrica,* **62**, pp. 1141-1160.

Wooders, M.H (1994c): "Equivalence of Lindahl Equilibrium with Participation Prices and the Core," *Economic Theory,* (to appear).

Wooders, M.H. and Zame, W.R (1984): "Approximate Cores of Large Games," *Econometrica,* **52** pp. 1327-1350.

Wooders, M.H. and Zame, W.R (1987): "Large Games; Fair and Stable Outcomes," *Journal of Economic Theory,* **42**, pp. 59-93.

Zajac, E (1972): "Some Preliminary Thoughts on Subsidization," presented at the Conference on Telecommunications Research, Washington. 7-278.