Generalized Samuelson Conditions and Welfare Theorems†
for Nonsmooth Economies

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Abstract

We give intuitive Samuelson conditions for a very general class of economies. Smoothness and monotonicity are not required. We provide necessary and sufficient conditions for all Pareto efficient allocations, including those on the boundary. We prove that if all agents have a cheaper point, the supporting prices fully decentralize the allocation. We also show first and second welfare theorems as corollaries to the characterization of efficient allocations.

1. Introduction

Samuelson (1954, 1955) gave the first modern study of economies with public goods. One of his main results was to provide a set of calculus-based conditions for Pareto efficiency. These "Samuelson conditions" have since become one of the fundamental tools for understanding public goods economies. However, his work has several important limitations. In particular, he did not deal with the issue of corner allocations, in which at least one good is not consumed at all by at least one agent. Given that this is probably the typical rather than the exceptional case in real life, his omission is not trivial. Unless we can characterize corner allocations as well, we must doubt the practical relevance of the studies based on Samuelson conditions.

Later, economists assumed that the most obvious modification of Samuelson's efficiency conditions would be sufficient to deal with the problem of corner solutions. This modification involves setting the sum of the marginal rates of substitution of the consumers equal to the marginal rate of transformation in production for each public good (relative to the same private good), unless the amount of a public good is zero, in which case the sum of the marginal rates of substitution may be less than the marginal rate of transformation. However, as Campbell and Truchon (1988) point out in an important paper, there are cases where some efficient allocations violate the Samuelson conditions, even as modified. Campbell and Truchon conclude that the Samuelson conditions miss some efficient allocations, and they provide a different specification of the Samuelson conditions which they show to be necessary and sufficient for efficiency in economies with one private good and a finite number of public goods. They assume differentiability of the utility and cost functions, convexity of preferences and cost, and monotonicity of preferences.

Unfortunately, the analysis of Campbell and Truchon is limited by their assumption that there is only one private good, and their need for differentiability. The first assumption restricts the application of their contribution to an essentially partial equilibrium domain. Requiring differentiability also reduces the class of economies to which their analysis may be applied. These observations motivate an approach to the problem using convex analysis, in the standard fashion established by Arrow (1951) for economies with private goods only.

Such an analysis was offered by Foley (1970) in the course of formalizing the general notion of Lindahl equilibrium. However, he requires in his definition that allocations be in the relative interior of the private goods subspace of the consumption set of each agent. Thus, Foley does not deal with corner allocations either. Khan and Vohra (1987) generalize Foley (1970) to allow general preferences and nonconvexities, but they focus on the second welfare theorem and do not elaborate on how to deal with boundary allocations.

In this paper we provide efficiency conditions for economies with a finite number of private and public commodities, without assuming differentiability. We do not require that commodities be goods. This allows us to consider the important real world situation in which a public project (a garbage incinerator, for example) benefits some agents while imposing costs on others. We require only that agents have locally nonsatiated preferences. Our analysis deals with corner and interior allocations in a unified way. Unlike Campbell and Truchon (1988), we do not need to appeal to the Karush-Kuhn-Tucker theorem, and our proofs are simple and geometric in nature. We develop the most general form of the Samuelson conditions in a simple and operational form, and we further show the existence of fully (Lindahl) supporting prices at any Pareto efficient allocation, for all agents who are allowed a cheaper point by the Samuelson prices corresponding to the allocation. As corollaries to these efficiency conditions we show first and second welfare theorems.

2. The Model

We consider an economy with L private commodities and M public commodities, I individual consumers, and F firms. We use the convention $\mathcal{I} \equiv \{1, \ldots, I\}$, and similarly for $\mathcal{L}, \mathcal{M}$ and $\mathcal{F}$. Superscripts are used to represent firms and consumers and subscripts to represent commodities.

Each agent $i \in \mathcal{I}$ is characterized by an endowment $\omega^i \in \Re^L_+$, and a preference relation $\succeq^i$ over the consumption set $C^i \equiv \Re^{L+M}_+$. A typical consumption bundle will be written (x, y) where x is a bundle of private commodities, and y is a bundle of public commodities. We remark that assuming the consumption set to be the nonnegative orthant is not less general than Campbell and Truchon's introduction of a nonnegative lower bound for the consumption of the private good by each agent, since we can always translate the preferences in order to make this lower bound zero. It is also possible to generalize the results in this paper to bounded below, convex consumption sets which may differ across the agents at the cost of complicating the proofs.[1]

We make the following assumptions on $\succeq^i$ for all $i \in \mathcal{I}$.

A1) $\succeq^i$ is complete and transitive.

A2) $\succeq^i$ is continuous (the upper and lower contour sets are closed relative to C^i).

A3) If $(x, y) \succeq^i (\tilde{x}, \tilde{y})$, then for all $\lambda \in [0, 1]$, $\lambda(x, y) + (1 - \lambda)(\tilde{x}, \tilde{y}) \succeq^i (\tilde{x}, \tilde{y})$. (Weak convexity)

A4) For all $(x, y) \in C^i$ and all $\epsilon > 0$ there exists $(\tilde{x}, \tilde{y}) \in C^i$ such that $\| (x, y) - (\tilde{x}, \tilde{y}) \| < \epsilon$ and $(\tilde{x}, \tilde{y}) \succ^i (x, y)$. (Local nonsatiation)[2]

The price space is denoted by

$$\Pi \equiv \{(p, q) \in \Re^{L+IM} \mid (p, q) \neq 0\}.$$

Through this paper we will use the convention that omitted superscripts indicate that the entire vector is being referred to. Thus, $q = (q^1, \ldots, q^I)$ where $q^i \in \Re^M$ is inter-

[1] We thank Tomoichi Shinotsuka for this observation.

[2] As usual, $(x, y) \succ^i (\tilde{x}, \tilde{y})$ if $(x, y) \succeq^i (\tilde{x}, \tilde{y})$ and $(\tilde{x}, \tilde{y}) \not\succeq^i (x, y)$.

preted as the personalized price vector for the public goods for agent i. In many places we will be interested in the summation of a prices or quantity vector over agents. In order to save space, we will indicate this with bold face type. Thus,

$$\mathbf{q} \equiv \sum_i q^i .$$

Note that we do not assume that prices are positive, but we exclude the zero vector from the price space. Define the *marginal rate of substitution correspondence for consumer* i, $\text{MRS}^i : C^i \to\to \Pi$, by:[3]

$$\text{MRS}^i(x^i, y^i) \equiv$$
$$\left\{(p,q) \in \Pi \mid (p,q^i) \cdot (x^i,y^i) < (p,q^i) \cdot (\tilde{x}^i,\tilde{y}^i) \;\forall\, (\tilde{x}^i,\tilde{y}^i) \in C^i \text{ s.t. } (\tilde{x}^i,\tilde{y}^i) \succ^i (x^i,y^i)\right\} .$$

Define also the *weak marginal rate of substitution correspondence for consumer* i, $\text{WMRS}^i : C^i \to\to \Pi$, by:

$$\text{WMRS}^i(x^i, y^i) \equiv$$
$$\left\{(p,q) \in \Pi \mid (p,q^i) \cdot (x^i,y^i) \leq (p,q^i) \cdot (\tilde{x}^i,\tilde{y}^i) \;\forall\, (\tilde{x}^i,\tilde{y}^i) \in C^i \text{ s.t. } (\tilde{x}^i,\tilde{y}^i) \succ^i (x^i,y^i)\right\} .$$

The MRS correspondence is simply the set of hyperplanes supporting, but not intersecting, the strictly preferred set. This means that all consumption bundles on a supporting hyperplane in $\text{MRS}^i(x^i, y^i)$ are not strongly preferred to (x^i, y^i) and all bundles below the hyperplane are inferior. The WMRS correspondence, on the other hand, is the set of hyperplanes supporting the weakly preferred set but possibly intersecting the strictly preferred set. (If it were not for the restriction on the price space, the WMRS would equal minus the normal cone to the weakly preferred set; see Rockafellar 1970.) This means that all consumption bundles below the hyperplane are inferior, but bundles on the hyperplane may be strongly preferred to (x^i, y^i). Obviously, the marginal

[3] It may seem odd at first that we define the supporting prices for an agent to elements of Π, which has dimension $L + IM$. We do this mainly to simplify notation, but it is easy to give an economic interpretation. Following Foley, we can think of public goods as jointly produced private goods. In this case, agents maximize their preferences over the entire $L+IM$ dimensional joint commodity space and therefore require this many prices to define their budget constraint. Of course, agents are satiated in all but $L + M$ of the commodities and so do not care about the other $(I-1)M$. Thus, their optimization is equivalent to optimization in the $L + M$ subspace given in the definition of MRS.

rate of substitution set $\mathrm{MRS}^i(x^i, y^i)$ is always a subset of the weak marginal rate of substitution set $\mathrm{WMRS}^i(x^i, y^i)$. The reason we need the WMRS correspondence is that the weakly preferred set may not be closed at the boundary, and so the MRS correspondence may be empty. The WMRS correspondence is never empty-valued, since it is the supporting hyperplane set at a boundary point of a closed, convex set. Also note that if $(p, q) \in \mathrm{MRS}^i(x^i, y^i)$ and the agent has income $(p, q^i) \cdot (x^i, y^i)$, then (x^i, y^i) is a preference maximizing choice over the budget set. On the other hand if $(p, q) \in \mathrm{WMRS}^i(x^i, y^i)$ then we are only guaranteed that (x, y) minimizes expenditure over the set of consumption bundles which are not inferior to (x, y). If the consumption bundle is interior, and the utility function differentiable, then these correspondences are nonempty, single valued, and of course, identical.

[**Figure 1 here**]

In the example depicted in Figure 1, the agent's indifference curves intersect the public commodity axis with a vertical slope, and terminate at their intersection with this axis. Otherwise, the preferences are standard, satisfying all of the assumptions A, as well as most other assumptions commonly made on preferences. At every point on the public commodity axis, the weak marginal rate of substitution correspondence has a singleton value of $(1, 0)$.[4] In other words, the vertical axis supports the preferred set. Since the WMRS correspondence contains the MRS correspondence, and the unique line of support intersects the preferred set, the marginal rate of substitution correspondence is empty-valued.

We represent each firm $f \in \mathcal{F}$ by a production set $P^f \subset \Re^L \times \Re^M_+$. A typical production plan will be written (z^f, g^f), where z is a net output vector of private commodities and g^f is the output vector of public commodities.

Define the *marginal rate of transformation correspondence for* P^f, $\mathrm{MRT}^f : P^f \to\to$

[4] Properly speaking, we should have indicated all the elements of the supporting vector here, in accordance with the definition of MRS and WMRS. However, in all discussions of examples we only indicate the components relating to the commodities consumed by the agent in question, to enhance clarity.

Π, as follows:

$$\text{MRT}^f(z^f, g^f) \equiv$$
$$\left\{(p,q) \in \Pi \mid (p,\mathbf{q}) \cdot (z^f, g^f) \geq (p,\mathbf{q}) \cdot (\tilde{z}^f, \tilde{g}^f) \; \forall \, (\tilde{z}^f, \tilde{g}^f) \in P^f \right\}.$$

If not for the restriction on the price space, MRT would be the normal cone to the production set (Rockafellar 1970, page 15).

The *comprehensive hull* of a set in $\Re^L \times \Re^M_+$ is defined as follows:

$$\text{comp}(Z) \equiv \left\{(z,g) \in \Re^L \times \Re^M_+ \mid \exists (\tilde{z}, \tilde{g}) \in Z \text{ s.t. } (z,y) \leq (\tilde{z}, \tilde{g})\right\}.$$

For all $f \in \mathcal{F}$ we assume:

B1) P^f is a nonempty, closed set.

B2) P^f is a convex set.

B3) $P^f = \text{comp}(P^f)$ (Free disposal).

We define the *global production set* in the usual way:

$$\mathbf{P} \equiv \left\{ (\mathbf{z}, \mathbf{g}) \in \Re^L \times \Re^M_+ \,\middle|\, (\mathbf{z}, \mathbf{g}) \equiv \sum_f (z^f, g^f) \text{ and } (z^f, g^f) \in P^f \; \forall \, f \in \mathcal{F} \right\}.$$

Note that we follow the convention established for summations over agents and so

$$\mathbf{z} \equiv \sum_f z^f \quad \mathbf{g} \equiv \sum_f g^f \text{ and } \mathbf{P} \equiv \sum_f P^f.$$

Now define the *aggregate marginal rate of transformation correspondence* $\text{MRT} : \mathbf{P} \rightarrow\!\!\rightarrow \Pi$ by

$$\text{MRT}(\mathbf{z}, \mathbf{g}) \equiv \{(p,q) \in \Pi \mid (p, \mathbf{q}) \cdot (\mathbf{z}, \mathbf{g}) \geq (p, \mathbf{q}) \cdot (\tilde{\mathbf{z}}, \tilde{\mathbf{g}}) \; \forall \, (\tilde{\mathbf{z}}, \tilde{\mathbf{g}}) \in \mathbf{P}\}.$$

We make the additional assumption:

B4) $\mathbf{P}$ is closed.

Notice that $\mathbf{P}$ inherits convexity and comprehensiveness from the individual P^f sets.

An allocation is a list $a = ((x^1, y^1), \ldots, (x^I, y^I), (z^1, g^1) \ldots (z^F, g^F)) \in C^1 \times \cdots \times C^I \times P^1 \times \cdots \times P^F$. Let A denote the set of feasible allocations:

$$A \equiv \left\{ a \in C^1 \times \cdots \times C^I \times P^1 \times \cdots \times P^F \,\middle|\right.$$

$$\left.\sum_f z^f = \sum_i (x^i - \omega^i) \text{ and } \sum_f g^f \equiv \mathbf{g} = y^i \;\forall\, i \in \mathcal{I} \right\}.$$

Although we retain the superscript i for the consumption of public commodities by agent i, the definition of a feasible allocation requires that all agents consume exactly the same vector of public commodities. Thus, agents cannot freely dispose of an undesirable public good, for example.

The set of *Pareto efficient allocations* is defined as

$$PE \equiv$$

$$\left\{ a \in A \mid \nexists\, \hat{a} \in A \text{ s.t. } \forall i \in \mathcal{I}, (\hat{x}^i, \hat{y}^i) \succeq^i (x^i, y^i) \text{ and } \exists\, j \in \mathcal{I} \text{ s.t. } (\hat{x}^j, \hat{y}^j) \succ^j (x^j, y^j) \right\}.$$

Let Δ^{I-1} denote the $I-1$ dimensional simplex:

$$\Delta^{I-1} \equiv \left\{ \theta \in \Re^I \mid \sum_i \theta^i = 1, \text{ and } \theta^i \geq 0 \;\forall\, i \in \mathcal{I} \right\}.$$

We denote a profit share system for a private ownership economy by $\theta = (\theta^1, \ldots \theta^f, \ldots \theta^F) \in \Delta^{I-1} \times \ldots \times \Delta^{I-1} \equiv \Theta$ where $\theta^{i,f}$ is interpreted as consumer i's share of the profits of firm f.

An allocation and price vector $(a, p, q) \in A \times \Pi$ is said to be a *Lindahl equilibrium relative to the endowment* $\omega \in \Re^{I \times L}$ *and profit shares* $\theta \in \Theta$ if and only if:

a. for all $f \in \mathcal{F}$, $(p, q) \in \mathrm{MRT}^f(z^f, g^f)$.

b. for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y^i)$ and $(p, q^i)\cdot(x^i, y^i) = p\cdot\omega^i + \sum_f \theta^{i,f}(p, \mathbf{q})\cdot(z^f, g^f)$.

Note that given the definitions of MRS^i and MRT^f, and the fact that local non-satiation implies that each agent will exhaust his income, these are equivalent to profit and preference maximization, respectively. Feasibility is already required by the definition of an allocation. Define the Lindahl equilibrium allocation correspondence $LE : \Re^{I \times L} \times \Theta \to\to A$ as follows:

$$LE(\omega, \theta) \equiv$$

$$\{a \in A \mid \text{ for some } (p, q) \in \Pi, (a, p, q) \text{ is a Lindahl equilibrium for } \omega \text{ and } \theta\}.$$

3. Results

In this section, we give a formal presentation of our results. A verbal summary our main point (the generalized Samuelson conditions) can be found in the concluding section. We begin by showing that the private commodity prices must be nonnegative.

Lemma 1. *For all* $(\mathbf{z}, \mathbf{g}) \in \mathbf{P}$ *and all* p *such that there exists* q *with* $(p, q) \in \mathrm{MRT}(\mathbf{z}, \mathbf{g})$, *it is the case that* $p \geq 0$.

Proof/

Suppose not; then for some $(\mathbf{z}, \mathbf{g}) \in \mathbf{P}$ and $(p, q) \in \Pi$ such that $(p, q) \in \mathrm{MRT}(\mathbf{z}, \mathbf{g})$, there is a private commodity $\ell \in \mathcal{L}$ such that $p_\ell < 0$. By free disposal, for all $\delta > 0$ $(\mathbf{z}_1, \ldots, \mathbf{z}_\ell - \delta, \ldots, \mathbf{z}_L, \mathbf{g}) \in \mathbf{P}$. But $(p, \mathbf{q}) \cdot (\mathbf{z}_1, \ldots, \mathbf{z}_\ell - \delta, \ldots, \mathbf{z}_L, \mathbf{g}) > (p, \mathbf{q}) \cdot (\mathbf{z}, \mathbf{g})$, contradicting the definition of $\mathrm{MRT}(\mathbf{z}, \mathbf{g})$.

■

The following standard lemma states that, given an allocation $a \in A$ and prices $(p, q) \in \Pi$, $(\mathbf{z}, \mathbf{g})$ maximizes profits over the global production set $\mathbf{P}$ at prices $(p, \mathbf{q})$ if and only if (z^f, g^f) maximizes the profits of each firm $f \in \mathcal{F}$ at these prices. This allows us to state the subsequent theorems in terms of maximizing profits over the global production set instead of going to the extra step of considering each firm. Since this lemma is standard, we omit the proof to save space.

Lemma 2. *Given* $(\mathbf{z}, \mathbf{g}) \in \mathbf{P}$ *and* $(p, q) \in \Pi$, $(p, \mathbf{q}) \cdot (\mathbf{z}, \mathbf{g}) \geq (p, \mathbf{q}) \cdot (\bar{\mathbf{z}}, \bar{\mathbf{g}})$ *for all* $(\bar{\mathbf{z}}, \bar{\mathbf{g}}) \in \mathbf{P}$ *if and only if for all* $f \in \mathcal{F}$ *there exists* (z^f, g^f) *such that* $(p, \mathbf{q}) \cdot (z^f, g^f) \geq (p, \mathbf{q}) \cdot (\bar{z}^f, \bar{g}^f)$ *for all* $(\bar{z}^f, \bar{g}^f) \in P^f$ *and* $\sum_f (z^f, g^f) = (\mathbf{z}, \mathbf{g})$.

We now give the first necessity theorem.

Theorem 1. *For all* $a \in PE$, *there exists a price vector* $(p, q) \in \Pi$ *such that (a)* $(p, q) \in \mathrm{MRT}(\mathbf{z}, \mathbf{g})$ *and, (b) for all* $i \in \mathcal{I}$, $(p, q) \in \mathrm{WMRS}^i(x^i, y^i)$.

Proof/

Following Foley, we define an artificial production set in which public commodities are treated as strictly jointly produced private commodities:

$$AP \equiv \left\{ (\tilde{z}, \tilde{g}^1, \ldots, \tilde{g}^I) \in \Re^L \times \Re^{IM} \mid \tilde{g}^1 = \cdots = \tilde{g}^I = \tilde{g} \text{ and } (\tilde{z}, \tilde{g}) \in \mathbf{P} \right\}.$$

AP is closed, convex, and nonempty as a consequence of $\mathbf{P}$ possessing these properties. Next we define the socially preferred set of the allocation a:

$$SP(a) \equiv \left\{ (\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \in \Re^L \times \Re^{IM} \;\middle|\; \forall\, i \in \mathcal{I},\ \exists\, \tilde{x}^i \text{ with } (\tilde{x}^i, \tilde{y}^i) \in C^i \text{ s.t.} \right.$$

$$\left. \tilde{z} = \sum_{i=1}^{I} (\tilde{x}^i - \omega^i),\ \forall\, i \in \mathcal{I}, \left(\tilde{x}^i, \tilde{y}^i\right) \succeq^i (x^i, y^i) \text{ and } \exists\, j \in \mathcal{I} \text{ s.t. } \left(\tilde{x}^j, \tilde{y}^j\right) \succ^j (x^j, y^j) \right\}.$$

The socially preferred set inherits convexity, and by continuity and nonsatiation it has a nonempty interior.

a. Since $a = ((x^1, y^1), \ldots, (x^I, y^I), (z^1, g^1) \ldots (z^F, g^F)) \in PE$ by assumption, $SP(a) \cap AP = \emptyset$. Then by the Minkowski Separation Theorem (Takayama 1985, p. 44), there exists a price vector $(p, q^1, \ldots, q^I) \neq 0$ with $\| p \| < \infty$, and a scalar r, such that:

(i) For all $(\tilde{z}, \tilde{g}^1, \ldots, \tilde{g}^I) \in AP$, $p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{g}^i \leq r$.

(ii) For all $(\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \in \text{closure}(SP(a))$, $p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{y}^i \geq r$.

By continuity and nonsatiation, $(z, y^1, \ldots, y^I) \in \text{closure}(SP(a))$. By hypothesis, $(z, y^1, \ldots, y^I) \in AP$. It follows from (i) and (ii) that $p \cdot z + \sum_i q^i \cdot y^i = r$. Therefore, for all $(\tilde{z}, \tilde{g}^1, \ldots, \tilde{g}^I) \in AP$:

$$p \cdot z + \sum_i q^i \cdot g^i = r \geq p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{g}^i.$$

Since $(p, q) \neq 0$, this establishes part (a) of the theorem.

b. Now suppose that part (b) is false. Then there exists $j \in \mathcal{I}$ such that for some $(\bar{x}^j, \bar{y}^j) \in C^j$ it is the case that $(\bar{x}^j, \bar{y}^j) \succ^j (x^j, y^j)$ and $(p, q^j) \cdot (\bar{x}^j, \bar{y}^j) < (p, q^j) \cdot (x^j, y^j)$. Hence,

$$\left(\sum_{i \neq j} (x^i - \omega^i) + (\bar{x}^j - \omega^j), y^1, \ldots, \bar{y}^j, \ldots, y^I \right) \in SP(a)$$

and

$$p \cdot \sum_{i \neq j}(x^i - \omega^i) + p \cdot (\bar{x}^j - \omega^j) + \sum_{i \neq j} q^i \cdot y^i + q^j \cdot \bar{y}^j$$
$$< \left(p, q^1, \ldots, q^I\right) \cdot \left(\sum_i (x^i - \omega^i), y^1, \ldots, y^I\right)$$

a contradiction to (ii) above.

■

We note that, as in Milleron (1972, page 431), we do not actually need the closedness of the production set, but without it Lindahl equilibria may fail to exist, in which case the whole line of inquiry along second welfare theorem lines rests on a shaky foundation. Optimality conditions very similar to this were used in Conley (1994) for a one private, M public good economy. No proof of their validity was offered, however.

As a corollary to this theorem we state a version of the second welfare theorem. In particular, we show that we can decentralize any Pareto efficient allocation through prices for some set of endowments and profit shares such that the production of each firm is profit maximizing under the prices, and each agent's consumption bundle minimizes expenditure over the weakly preferred set. This is not quite the same thing as decentralizing the allocation as a Lindahl equilibrium since agents are not necessarily maximizing preferences over the budget set. To get this stronger result, slightly stronger conditions are needed. We show this below. See Debreu (1959) for details.

Corollary 1.1 *(weak second welfare theorem) For all $a \in PE$, there exists a price vector $(p, q) \in \Pi$ an endowment vector $\hat{\omega}$ such that $\sum_i \omega^i = \sum_i \hat{\omega}^i$, and a profit share system $\hat{\theta} \in \Theta$ such that (a) $(p, q) \in \mathrm{MRT}(\mathbf{z}, \mathbf{g})$, (b) for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{WMRS}^i(x^i, y^i)$, and (c) $(p, q^i) \cdot (x^i, y^i) = p \cdot \hat{\omega}^i + \sum_f \hat{\theta}^{i,f}(p, \mathbf{q}) \cdot (z^f, g^f)$.*

Proof/

We know by Theorem 1, there exist prices $(p, q) \in \Pi$ such that (a)$(p, q) \in \mathrm{MRT}(\mathbf{z}, \mathbf{g})$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{WMRS}^i(x^i, y^i)$. For all $i \in \mathcal{I}$, let

$$\hat{\omega}^i \equiv \frac{(p, q^i) \cdot (x^i, y^i)}{\sum_j (p, q^j) \cdot (x^j, y^j)} \sum_j \omega^j,$$

and for all $i \in \mathcal{I}$, and all $f \in \mathcal{F}$, let

$$\hat{\theta}^{i,f} \equiv \frac{(p, q^i) \cdot (x^i, y^i)}{\sum_j (p, q^j) \cdot (x^j, y^j)}.$$

Note that by construction for all $f \in \mathcal{F}$, $\sum_i \theta^{i,f} = 1$, and $\sum_i \omega^i = \sum_i \hat{\omega}^i$. It only remains to show that $(p, q^i) \cdot (x^i, y^i) = p \cdot \hat{\omega}^i + \sum_f \hat{\theta}^{i,f}(p, \mathbf{q}) \cdot (z^f, g^f)$. But

$$p \cdot \hat{\omega}^i + \sum_f \hat{\theta}^{i,f}(p, \mathbf{q}) \cdot (z^f, g^f) =$$

$$p \cdot \frac{(p, q^i) \cdot (x^i, y^i)}{\sum_j (p, q^j) \cdot (x^j, y^j)} \sum_j \omega^j + \sum_f \frac{(p, q^i) \cdot (x^i, y^i)}{\sum_j (p, q^j) \cdot (x^j, y^j)} (p, \mathbf{q}) \cdot (z^f, g^f) =$$

$$(p, q^i) \cdot (x^i, y^i) \frac{p \cdot \sum_j \omega^j + \sum_f (p, \mathbf{q}) \cdot (z^f, g^f)}{\sum_j (p, q^j) \cdot (x^j, y^j)} =$$

$$(p, q^i) \cdot (x^i, y^i) \frac{p \cdot \sum_j \omega^j + \sum_f (p, \mathbf{q}) \cdot (z^f, g^f)}{p \cdot (\sum_j \omega^j + \sum_f z^f) + \mathbf{q} \cdot \mathbf{g}} = (p, q^i) \cdot (x^i, y^i)$$

■

Khan and Vohra (1987) consider a similar model of a public goods economy. They do not assume convexity in preferences and production, and so are more general in this respect. On the other hand, they assume monotonicity of preferences in public commodities instead of nonsatiation. This excludes the examples like the garbage incinerator mentioned in the introduction from the domain of problems they are able to treat. They prove a version of the second welfare theorem employing a notion of supporting vector set equivalent, under convexity, to our WMRS(x^i, y) (Khan and Vohra 1987, page 236).

Next, we give a second necessity theorem. We strengthen the hypothesis to require that all agents have a cheaper point in the consumption set. This allows us to conclude that there will exist supporting prices in the MRS correspondence of each agent, instead of just the WMRS. This means that the prices are fully decentralizing.

Theorem 2. *If $a \in A$ is a Pareto efficient allocation, then for every $i \in \mathcal{I}$ such that (a) $p \cdot x^i > 0$, or (b) there exists m such that $q^i_m < 0$, or (c) there exists m such that $q^i_m > 0$ and $y_m > 0$, where (p, q^i) are the prices established by Theorem 1, we have that $(p, q) \in \mathrm{MRS}^i(x^i, y^i)$.*

Proof/

(a) Suppose that for some $i \in \mathcal{I}$, $p \cdot x^i > 0$ and $(p, q) \notin \mathrm{MRS}^i(x^i, y^i)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y^i)$ and $(p, q^i) \cdot (x^i, y^i) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. Since $p \geq 0$ by Lemma 1, and $x^i \geq 0$ because $(x^i, y) \in C^i$, $p \cdot x^i > 0$ implies that there exists $\ell \in \mathcal{L}$ such that $p_\ell > 0$ and $x^i_\ell > 0$.

Denote the open line segment between two points by $L((x^i, y^i), (\bar{x}^i, \bar{y}^i))$. By the convexity of preferences and the linearity of the budget constraint, for all $(\tilde{x}^i, \tilde{y}^i) \in L((x^i, y^i), (\bar{x}^i, \bar{y}^i))$, we have $(\tilde{x}^i, \tilde{y}^i) \succ^i (x^i, y^i)$ and $(p, q^i) \cdot (x^i, y^i) \geq (p, q^i) \cdot (\tilde{x}^i, \tilde{y}^i)$. For $(\tilde{x}^i, \tilde{y}^i)$ close enough to (x^i, y^i), $\tilde{x}^i_\ell > 0$. By the continuity of preferences, there exists $\epsilon > 0$ such that $(\tilde{x}^i_1, \ldots, \tilde{x}^i_\ell - \epsilon, \ldots, \tilde{x}^i_L, \tilde{y}^i) \succ^i (x^i, y^i)$. Since $p_\ell > 0$, there follows $(p, q^i) \cdot (\tilde{x}^i_1, \ldots, \tilde{x}^i_\ell - \epsilon, \ldots, \tilde{x}^i_L, \tilde{y}^i) < (p, q^i) \cdot (\tilde{x}^i, \tilde{y}^i) \leq (p, q^i) \cdot (x^i, y^i)$, leading to a contradiction to (ii) in Theorem 1 (b).

(b) Suppose now that for some $i \in \mathcal{I}$, $\exists m$ s.t. $q^i_m < 0$ and $(p, q) \notin \mathrm{MRS}^i(x^i, y^i)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y^i)$ and $(p, q^i) \cdot (x^i, y^i) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. By the continuity of preferences, there exists $\epsilon > 0$ such that $(\bar{x}^i, \bar{y}^i_1, \ldots, \bar{y}^i_m + \epsilon, \ldots, \bar{y}^i_M) \succ^i (x^i, y^i)$. Since $q^i_m < 0$, this leads to the same contradiction as in part (a).

(c) Finally, suppose that for some $i \in \mathcal{I}$, $\exists m$ s.t. $q^i_m > 0$ and $y_m > 0$ and $(p, q) \notin \mathrm{MRS}^i(x^i, y^i)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y^i)$ and $(p, q^i) \cdot (x^i, y^i) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. We can now mimic the proof of (a) above, with y^i_m in the place of x^i_ℓ and q^i_m in the place of p_ℓ.

■

The reason that an extra assumption is required to obtain the full support is illustrated in the following example. Consider an economy with two agents, one private and one public commodity, one firm with one-to-one linear technology, and endowment of one unit of the private commodity for each agent. Agent 1 has preferences exactly as in Figure 1, and agent 2 has translated Cobb-Douglas preferences such that the slope of agent 2's indifference curve at $(x^2, y^2) = (3/2, 1/2)$ is -1. Then the allocation $(x^1, y^1, x^2, y^2, z^1, g^1) = (0, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ is Pareto efficient, but $\mathrm{WMRS}^1(0, \frac{1}{2})$ contains only the vector $(1, 0)$, which intersects the strictly preferred set of agent 1.[5] Therefore, the Samuelson prices arising from Theorem 1 are not separating prices, and this failure occurs for agent 1 who violates all three of the conditions of Theorem 2. This allows us to state a stronger second welfare theorem.

Corollary 2.1 *(strong second welfare theorem) If $a \in A$ is a Pareto efficient allocation such that for all agents $i \in \mathcal{I}$, (x^i, y^i) is in the interior of C^i, then there exists a price vector $(p, q) \in \Pi$ an endowment vector $\hat{\omega}$, and a profit share system θ such that $a \in LE(\hat{\omega}, \theta)$ and $\sum_i \omega^i = \sum_i \hat{\omega}^i$.*

Proof/

Since for all agents $i \in \mathcal{I}$, (x^i, y^i) is in the interior of C^i, the hypothesis of Theorem 2 is satisfied. Therefore, there exist prices $(p, q) \in \Pi$ such that (a)$(p, q) \in \mathrm{MRT}(\sum_{i=1}^{I}(x^i - \omega^i), \mathbf{g})$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y^i)$. But (a) means all firms maximize profits under the prices, and (b) means each consumer i chooses (x^i, y^i) when he maximizes his preferences while having income $(p, q^i) \cdot (x^i, y^i)$. But we know from the argument given in the proof of Corollary 1.1 that it is possible to divide endowments and profits so that each agent has exactly this income, and the social endowment is exactly exhausted.

■

We now give our sufficiency theorem.

Theorem 3. *If $a \in A$ is a feasible allocation and there exists a price vector (p, q) such that (a) $(p, q) \in \mathrm{MRT}(\sum_{i=1}^{I}(x^i - \omega^i), \mathbf{g})$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y^i)$,*

[5] This implies that $\mathrm{MRS}^1(0, \frac{1}{2}) = \emptyset$.

then a is Pareto efficient.

Proof/

Suppose that the hypotheses of the Theorem are met but a is not Pareto efficient. Then there exists a feasible allocation $\tilde{a} \in A$ such that for all $i \in \mathcal{I}$ $(\tilde{x}^i, \tilde{y}^i) \succeq^i (x^i, y^i)$, and for some $j \in \mathcal{I}$, we have $(\tilde{x}^j, \tilde{y}^j) \succ^j (x^j, y^j)$.

First observe that if $(p, q) \in \text{MRS}^i(x^i, y^i)$, and $(\tilde{x}^i, \tilde{y}^i) \succeq^i (x^i, y^i)$, it is the case that $(p, q^i) \cdot (\tilde{x}^i, \tilde{y}^i) \geq (p, q^i) \cdot (x^i, y^i)$. Suppose not, then by local nonsatiation, for all $\epsilon > 0$ there exists $(\hat{x}, \hat{y}) \in C^i$ such that $\| (x^i, y^i) - (\hat{x}, \hat{y}) \| < \epsilon$ and $(\hat{x}, \hat{y}) \succ^i (x^i, y^i)$. But for small enough ϵ, $(p, q^i) \cdot (\hat{x}^i, \hat{y}^i) < (p, q^i) \cdot (x^i, y^i)$, contradicting the definition of MRS. Also, by definition of MRS, since $(\tilde{x}^j, \tilde{y}^j) \succ^j (x^j, y^j)$, we have $(p, q^i) \cdot (\tilde{x}^i, \tilde{y}^i) > (p, q^i) \cdot (x^i, y^i)$. Summing up over all agents gives,

$$\sum_i p \cdot \tilde{x}^i + \mathbf{q} \cdot \tilde{\mathbf{g}} > \sum_i p \cdot x^i + \mathbf{q} \cdot \mathbf{g}. \tag{i}$$

But by (a),

$$\sum_i p \cdot \tilde{x}^i + \sum_i q^i \cdot \tilde{y}^i - \sum_i p \cdot \omega^i = (p, \mathbf{q}) \cdot (\sum_i (\tilde{x}^i - \omega^i), \tilde{\mathbf{g}}) \leq$$

$$(p, \mathbf{q}) \cdot (\sum_i (x^i - \omega^i), \mathbf{g}) = \sum_i p \cdot x^i + \sum_i q^i \cdot y^i - \sum_i p \cdot \omega^i. \tag{ii}$$

Now (ii) yields a contradiction of (i), which proves the theorem.

■

Finally, we get the first welfare theorem as an immediate consequence of this.

Corollary 3.1 *If $a \in LE(\omega, \theta)$, a is Pareto efficient.*

Proof/

By the definition of Lindahl equilibrium, there exists a price vector (p, q) such that (a) $(p, q) \in \text{MRT}(\sum_{i=1}^{I} (x^i - \omega^i), \mathbf{g})$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \text{MRS}^i(x^i, y^i)$. But then by Theorem 3, a is Pareto efficient.

■

4. Some comments on the literature

Milleron (1972) contains a version of our Theorem 1 (page 430, Theorem 2.1). However, he is mostly interested in the welfare theorems as they relate to public competitive equilibrium, the existence of Lindahl equilibrium, and the non-convergence of the core to the set of Lindahl equilibria, along with some incentive issues. He does not characterize Pareto efficient allocations in terms of Samuelson prices, which is the main focus of the present paper. We note also that the second welfare theorem as applied to public competitive equilibrium is much weaker than the one for Lindahl equilibrium, since the set of public competitive equilibria is generally much larger than the set of Lindahl equilibria of an economy.

The relationship of the necessary condition for efficiency derived here and the one derived in Campbell and Truchon (1988) is worth explaining at some length.[6] They assumed that there is only one private good and a finite number of public goods, K in their notation. Production is carried out by one firm, which faces cost $\gamma(y)$ in terms of the private good in order to produce a vector y of public goods, where γ is a differentiable convex increasing function. The production set of the unique firm is $P = \{(z, y) \in \Re^{1+K} \mid z + \gamma(y) \leq 0\}$. Preferences are represented by quasi-concave differentiable utility functions, where the marginal utility of the private good is positive everywhere for every consumer. Each consumer i faces a lower bound b_i for his consumption of the private good. The marginal rate of substitution for public good k relative to the unique private good for consumer i is denoted by π^i_k, and the vector of these for consumer i is denoted π^i. The partial derivative of γ with respect to the kth public good is denoted by γ_k.[7]

The Samuelson-type condition in Campbell and Truchon is condition GOC, which can be expressed as follows: for all k, there exist $v_k \leq \gamma_k$, and, for all i, there exists a

[6] We thank the referee for proposing that the relationship with Campbell and Truchon's results be presented in this way.

[7] At the boundary, all of these derivatives are one-sided derivatives taken from the interior.

vector ψ^i in the box $[0,\pi^i] \subset \Re^K$ such that:

$$\sum_i \psi_k^i = v_k, \quad (\gamma_k - v_k)y_k = 0, \quad k = 1,\ldots,K, \quad \text{and} \ (\pi^i - \psi^i)(x_i - b_i) = 0, \ \text{for all} \ i.$$

In writing the MRT and WMRS correspondences, given the existence of only one private good desired by all agents, it is convenient to normalize its price to be equal to one. Then the definitions of these correspondences are, under the Campbell and Truchon assumptions:

$$\text{WMRS}^i(x,y) = \{(1,q^1,\ldots,q^I) \in \Re^{1+IK} \mid q^i = \pi^i \text{ if } x_i > b_i \text{ and } q^i \in [0,\pi^i] \text{ if } x_i = b_i\},$$

$$\text{MRT}(z,y) = \{(1,q^1,\ldots,q^I) \in \Re^{1+IK} \mid \sum_i q_k^i = \gamma_k \text{ if } y_k > 0 \text{ and } \sum_i q_k^i \leq \gamma_k \text{ if } y_k = 0\}.$$

These sets are well-defined under the Campbell and Truchon assumptions. Then the relationship of the two sets of results under these assumptions is as follows: there exists a vector $(1,\psi^1,\ldots,\psi^I)$ that satisfies condition GOC if and only if $(1,\psi^1,\ldots,\psi^I) \in \text{MRT}(-\gamma(y),y)$ and $(1,\psi^1,\ldots,\psi^I) \in \text{WMRS}^i(x_i,y)$ for all i.

Finally, some notes on papers that are tangentially relevant. Saijo (1990) addresses a quite different point arising from Campbell and Truchon than we do; namely, he shows that the robustness of boundary Pareto efficient allocations observed by Campbell and Truchon is not a phenomenon specific to public good economies, since it also happens in exchange economies. Manning (1993, chapter 3) contains an extension of Foley's (1970) results to economies with local public goods, using assumptions based on Foley's, such as constant returns to scale and ruling out the private goods boundary. Manning (1994) develops the analysis of local public goods further by utilizing techniques based on the present paper.

5. Conclusion

In conclusion, our main purpose in this paper is to provide Samuelson conditions for economies with many public and private goods without assuming monotonicity or differentiability. The conditions we provide treat both interior and boundary optima in a unified way. Intuitively, our results are easy to understand. Consider a one private one public good economy where the private good price is normalized to one. At boundary allocations, their are many price lines which support the weakly preferred set (the same is true at kinks in the interior of the consumption set.) The set of slopes of these lines of support can be interpreted as the set of marginal willingnesses to pay (MWP) for public good. Our conditions say that if an allocation is Pareto optimal, it is possible to find a selection from each agent's MWP correspondence such that their sum is an element of each firm's marginal rate of transformation correspondence (which may also be set-valued at kinks or at boundaries). In the body of the paper, we generalize this idea to many goods, and use the use same basic approach to provide sufficient conditions for Pareto optimality, and prove first and second welfare theorems that include boundary allocations.

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