Coasian Equilibrium†

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and

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Abstract

We consider a general equilibrium economy with public goods and externalities. Following Boyd and Conley (1997), we treat externality markets directly instead of indirectly through Arrovian commodities. Because such direct externality markets are not subject to the nonconvexities that Starrett (1972) shows are fundamental to Arrow's externality markets, this new approach admits the use of largely standard methods to prove welfare and existence theorems in an economy with externalities. We extend the Boyd and Conley model to allow firms to benefit from public goods and be damaged by externalities, and to allow consumers to produce externalities. We state a first welfare theorem and prove the existence of a competitive equilibrium. Taken together, this can be viewed as a type of general equilibrium Coase Theorem. Considered as a special case, these theorems also represent a significant generalization of existing results for pure public goods economies.

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1. Introduction

Coase's (1960) implied assertion that, if transaction costs are zero, any assignment of property rights leads to an efficient outcome, has strong intuitive appeal. But it is well-known that externalities can cause ordinary competitive markets to fail. Arrow (1970) recognized that this failure arises from the incompleteness of markets caused by the presence of externalities. To complete these markets, Arrow extended the standard commodity space to include artificial commodities which serve as proxies for externalities. The resulting economy can be transformed into a special case of the Arrow-Debreu-McKenzie economy for which standard results apply. Starrett (1972), however, demonstrated that Arrow's method necessarily introduces nonconvexities into the production sets. As a result, the existence and second welfare theorems to which Arrow appeals depend on an assumption which is logically inconsistent with Arrovian commodities. Unfortunately, game theoretic and other non-price based studies of economies with externalities have also concluded that equilibria are not necessarily efficient.[1]

Can the intuitive appeal of the Coase hypothesis and the apparently failed attempts to provide a theoretical foundation for it be reconciled? Hurwicz (1995) suggests that the truth of Coase's hypothesis depends critically on the institutions under which property rights are traded and on the type of equilibrium concept applied. In that spirit, Boyd and Conley (1997) suggest an alternative price-based approach to economies with externalities. They proposed that markets be completed directly by distributing property rights for externality production itself, rather than indirectly through artificial Arrovian commodities.[2] The main advantage of this new approach is that properly bounding the endowment of externality property rights frees the direct externality

[1] Papers in this spirit include Aivazian *et al.* (1987), Harrison and McKee (1980), and Hoffman *et. al.* (1994), Hurwicz (1999) and Chari and Jones (2000). For a further discussion of these results, see Boyd and Conley (1997).

[2] This is similar to an example provided by Laffont (1988). He seemed to believe, however, that such direct externality markets were still subject to Starrett's fundamental nonconvexities. So it is not completely clear what Laffont envisioned.

markets from the fundamental nonconvexities identified by Starrett. Therefore, this new approach allows the use of largely standard methods to prove welfare theorems for economies with externalities. A second major advantage to the direct market approach is that, unlike Arrovian markets, it provides proper incentives for the use of abatement and other externality avoidance technologies.

The first economic contribution of this paper is to provide a significant generalization of Boyd and Conley (1997) in order to allow: 1) public goods to enter into firms' production sets;[3] 2) firms also to be affected by externalities (instead of just consumers); and 3) consumers to produce externalities. While these seem like natural features for an economy with externalities to possess, we are aware of no general equilibrium treatment of an economy with these properties elsewhere in the literature.

The second economic contribution of this paper is to prove for the first time the existence of the competitive equilibrium introduced in Boyd and Conley (1997). Combined with the first welfare theorem, these two theorems can be taken as a general equilibrium Coase theorem.[4] Of course, we are not the first to explore the existence of equilibrium in public goods economies. The earliest work of which we are aware is Foley (1970) who showed the existence of a generalized Lindahl equilibrium by enlarging the commodity spaces and appealing to an existence result from a private goods economy. Milleron (1972) considered a more general pure public goods economy, and formalized Foley's proof. Roberts (1973) and Khan and Vohra (1985) extended Foley's result to economies with a measure space of agents. Khan and Vohra (1987) and Bonnisseau (1991) prove the existence of a Lindahl-Hotelling equilibrium (MC or AC producer prices with Lindahl consumer prices) for cases of increasing-returns-to-scale technology or general nonconvexities. However, all of these existence results are for economies in which consumers do not contribute to public good (or bad) levels, firms do not use

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[3] Diamantaras and Wilkie (1994) also allow public goods to be inputs in production and show the existence of ratio equilibrium for such an economy.

[4] We do not address the stronger claim which has also come to be consider a Coase Theorem, namely that the outcome is independent of the initial endowment of rights.

public goods (or bads) as inputs, or initial endowments of public goods are not considered. In this respect, the proof provided here is the most general one of which we are aware.

The third contribution of this paper is technical. The generalizations descibed above create several technical difficulties in proving the standard theorems. These difficulties all root in the fact that firms both produce and consume public goods, and that both consumers and firms benefit from abatement while they, themselves, produce externalities. Our proof technique is inspired by Foley (1970) and if, like Foley and others, we make a separation between the production and consumption of public commodities, then Foley's proofs could be applied almost directly. But without that separation, the global production set is not the direct sum of firms' production sets and care must taken to define the equilibrium concept in such a way that the agents do not inefficiently internalize their public commodity production choices. (For example, firms must believe that no connection exists between the public goods they produce and the levels of public inputs they use). A greater difficulty arises in defining artificial production, consumption, and socially prefered sets so that the separating prices in the artificial economy support equilibrium and efficient allocations in the real economy.

Unfortunately, these factors make the construction we propose somewhat complex and we have, therefore, largely relegated it to the appendix. A more elegant construction would certainly be desirable, but we have no promising leads on a simpler approach. It would need to address the fact that the set of choice variables for firms is larger than for consumers because they include both production and consumption of public goods while individuals are only consumers of public goods. As a result, the public goods variable cannot be made to do double duty as in Foley, and a different set of artificial commodities must be introduced which are of no utility value to consumers. Interested readers who read the appendix will also notice several other factors that prevent the direct application of Foley's technique.

The plan of this paper is as follows. In section two, we describe our model and state a first welfare theorem. In section three, we prove the existence of a competitive equilibrium by defining an associated economy which has a quasi-equilibrium for which Debreu proved existence and then show this quasi-equilibrium corresponds to an equilibrium of our original economy.

2. An Externality Rights Model

We consider a model with I individual consumers and F firms. We use the convention $\mathcal{I} \equiv \{1, \ldots, I\}$ for consumers, and $\mathcal{F} \equiv \{I+1, \ldots, I+F\}$ for firms. Subscripts are used to represent firms and consumers and superscripts to represent commodities. There are N^c *private commodities* which include private goods and bads.[5] In addition there are N^g *public goods*, and N^r *public externality rights*. For example, for externalities such as smoke, these externality rights are interpreted as permission to generate a specified level of smoke.[6]

A typical consumption bundle will be denoted $(x_i^c, x_i^G, x_i^r, x_i^R)$ where $x_i^c \in \Re^{N^c}$ is a bundle of private commodities, $x_i^G \in \Re^{N^g}$ is a bundle of public goods, $x_i^r \in \Re^{N^r}$ is a bundle of privately used externality rights, and $x_i^R \in \Re^{N^r}$ is a bundle of publicly held externality rights (for example, abatement of public bads).[7] Notice that the last two components of a consumption bundle represent two different uses of the same commodity: externality rights. It is key to this analysis to distinguish between private uses and public uses of externality rights. We imagine an economy in which each consumer derives benefits from his own private use of externality rights, for example

[5] The directed externalities discussed in Boyd and Conley (1997) are a special case of a private bad.

[6] These externality rights also admit interpretation as positive externalities. For example, for the externality generated by planting trees, these public externality rights might be interpreted as compensation vouchers "earned" by planting a specified number of trees. To be concrete in our discussions, however, we will refer mainly to negative externalities such as smoke.

[7] We use uppercase superscripts to denote public aspects of goods or prices in this model and lowercase superscripts to denote private aspects. Furthermore, i and j will be used to denote consumers and f and k will be used to denote firms.

burning leaves in the fall, but is also harmed by the collective private use of rights on the part of all consumers and firms. We capture this in the consumption bundle by letting x_i^r denote the private use of rights, and x_i^R denote net level of publicly held rights which are not used and may therefore be thought of as the total level of abatement. Thus, x_i^R is a kind of public good.[8] Each agent $i \in \mathcal{I}$ is characterized by an endowment of private goods and externality rights, $\omega_i = (w_i^c, 0, w_i^r, 0)$, and a preference relation $\succeq_i$ over the consumption set $X_i \subset \Re^{N^c+N^g+2N^r}$. The aggregate endowment is $\omega = \sum_i \omega_i = (\sum_i \omega_i^c, 0, \sum_i \omega_i^r, 0)$. It is important to emphasize, however, that this externality rights endowment does not predetermine the level of smoke in the economy. The fraction of rights that are left unused and therefore become a public good called "abatement" is determined endogenously through the market.

As noted by Boyd and Conley, it is both the structure of externality rights and the boundedness of the externality rights endowment that differentiate this model from Arrow's model and allows us to escape the fundamental nonconvexities described by Starrett (1972).[9] It does not matter whether this bound is imposed by some human agency in the spirit of a pollution "benchmark," or is the result of physical natural limit on the possibilities to pollute.[10] Cres (1996) also notices that some sort of bounding of property rights is necessary for externality markets to work. He formalizes a notion that appears in Starrett's paper by proposing that agents be given endowments of rights by setting a kind of status quo as a benchmark. Thus, for example, agents might be entitled to the expectation that they work in a smoke free environment. Any deviation from this standard would require a market exchange of rights. *In contrast to the current paper, however, these benchmarks and tradable rights are set in terms*

[8] It would also have been possible to put the total use of externality rights by the society into the consumption bundle instead of it's opposite, the total unused rights. We choose this approach simply for notational convenience.

[9] It is worth repeating a point from Boyd and Conley that merely bounding the Arrovian commodities is not sufficient for avoiding fundamental nonconvexities. Furthermore, bounding the rights endowment clearly does not gaurantee convexity, it only avoids Starrett's fundamental nonconvexities.

[10] Osana (1962) discusses economically reasonable sufficient conditions for attaining the boundedness of attainable sets in economies with externalities.

of Arrovian commodities. Not surprisingly, his results are largely negative and echo the major message of our work. In particular, he shows that equilibrium exists only under very restrictive conditions and concludes that the way that agents are endowed and the institutional framework in which commodities are traded have a large impact of the possibly that markets can work efficiently in the presence of externalities. It is especially worth noting that Cres shows that simply bounding endowments in an Arrovian model is not enough.

We make the following assumptions on X_i and $\succeq_i$ for all $i \in \mathcal{I}$:

A1) $\succeq_i$ is complete and transitive

A2) $\succeq_i$ is continuous (the upper and lower contour sets are closed relative to X_i)

A3) if $x_i \succeq_i \tilde{x}_i$, then for all $\lambda \in [0,1]$, $\lambda x_i + (1-\lambda)\tilde{x}_i \succeq_i \tilde{x}_i$ (weak convexity)

A4) for all $x_i \in X_i$, and for all $\epsilon \geq 0$ there exists $\tilde{x}_i \in X_i$ such that $\parallel x_i - \tilde{x}_i \parallel \leq \epsilon$ and $\tilde{x}_i \succ_i x_i$ (local nonsatiation).

Each firm $f \in \mathcal{F}$ is represented by a production set $Y_f \subset \Re^{N^c+2N^g+2N^r}$. A typical production plan will be written $(y_f^c, y_f^g, y_f^G, y_f^r, y_f^R)$ where $y_f^c \in \Re^{N^c}$ is the net output bundle of private commodities, $y_f^g \in \Re^{N^g}$ is the gross output bundle of privately produced public goods, $y_f^G \in \Re^{N^g}$ is the input bundle of public goods, $y_f^r \in \Re^{N^r}$ is the input bundle of privately used externality rights, and $y_f^R \in \Re^{N^r}$ is an input bundle which equals the total level of externality rights consumed across all the agents in the economy.[11] As with consumers, firms both benefit from using up externality rights (y_f^r) and are harmed by the total level of rights collectively used up in the economy (y_f^R).

It is important to note that for consumers, x_i^R is the level of abatement whereas for firms, y_f^R is the level of smoke. This has a significant technical implication. *It means that preferences of agents are defined taking the total social endowment of rights as fixed.* Increasing the total social endowment of smoke rights, for example, would mean that a given level of abatement would leave more smoke in the air, agents worse

[11] Although y_f^r is described as a firm's private *use* of externality rights, it can also be the firm's production of externality rights. In this case we might imagine a firm that specializes in cleaning up pollution, for example. This interpretation does not affect the model in any way.

off, and marginal rates of substitution between commodities different in general. Thus, while the rights endowment can be set at any level, the preferences considered in this model must take this choice as parametric in subsequent analysis. Also note that firms' production sets include production of public goods by other firms. In both cases, this was the most notationally convenient way to deal with highly general externalities considered in this paper. Neither of these conventions restricts the model in any way.

We assume for all $f \in \mathcal{F}$:

B1) Y_f is a nonempty, closed set

B2) for all $y_f,\ \tilde{y}_f \in Y_f$ and all $\lambda \in [0,1]$, $\lambda y_f + (1-\lambda)\tilde{y}_f \in Y_f$ (convexity).

See Boyd and Conley (1997) for a demonstration that this convexity assumption is not inconsistent with externality markets modeled in this way. It must be admitted, however, that nonfundamental nonconvexities as described by Baumol (1972) are still possible and therefore, convexity is a much stronger assumption here than in an economy without externalities.

Two complications arise in describing the global production opportunities. First, firms use public commodities as inputs and hence the global set is not generally the sum of all the firms' sets. Second, consumers also contribute to externality production, and so the global production opportunities must be indexed by the consumers' aggregate private use of rights. We define the *global production set relative to x^r*, an arbitrary level of consumers' private use of externality property rights, as follows:

$$\mathbf{Y}(x^r) \equiv \Bigg\{ \mathbf{y} \equiv (y^c, y^g, y^G, y^r, y^R) \in \Re^{N^c+2N^g+2N^r} \Bigg|$$

$$(y^c, y^g, y^G, y^r, y^R) = (\sum_f y^c_f, \sum_f y^g_f, -\sum_f y^g_f, \sum_f y^r_f, \sum_f y^r_f - x^r)$$

$$\text{and for all } f \in \mathcal{F},\ (y^c_f, y^g_f, y^G, y^r_f, y^R) \in Y_f \Bigg\}.$$

In words, a global production bundle (relative to a given level of externality rights used by consumers) consists of an aggregate level of net private commodities production, an aggregate level of public goods production, a level of public goods input which equals the level produced, an aggregate input of externality rights, and the total level of used rights that must equal the sum of firms' and consumers' uses. By convention, y^g is generally positive because it is an output vector, and y^G, y^r, and y^R are generally negative because they are input vectors.

We make the following additional assumption:

B3) $\mathbf{Y}(x^r)$ is upper semi-continuous

An allocation is a list $a = (x_1, \ldots, x_I, y_{I+1}, \ldots, y_{I+F})$. The set of feasible allocations A consists of all allocations a such that

1. for all $i \in \mathcal{I}$, $x_i \in X_i$
2. for all $f \in \mathcal{F}$, $y_f \in Y_f$
3. $\sum_i x_i^c = \sum_i \omega_i^c + \sum_f y_f^c$
4. for all $i \in \mathcal{I}$ and all $f \in \mathcal{F}$, $\quad x_i^G = -y_f^G = \sum_k y_k^g$
5. for all $i \in \mathcal{I}$ and all $f \in \mathcal{F}$, $x_i^R = \sum_j \omega_j^r + y_f^R = \sum_j \omega_j^r - \sum_j x_j^r + \sum_k y_k^r$.

Conditions one and two require that the allocation be feasible for each consumer and producer. Condition three requires that the net production of private goods equals the consumption. Condition four requires that the production of public goods equals the amount consumed by each consumer, and that each firm experiences the total public goods production of all firms. Finally, condition five requires that the total level of externality rights is divided between abatement and externality uses, and that each firm experiences the total level of externalities generated by all the consumers and firms while each consumer experiences the resulting level of abatement.[12]

The set of *Pareto efficient allocations* is defined as

$$PE \equiv \big\{a \in A \mid \text{there is no } \hat{a} \in A \text{ s.t. } \hat{x}_i \succeq_i x_i \text{ for all } i \in \mathcal{I}, \\ \text{and } \hat{x}_j \succ_j x_j \text{ for some } j \in \mathcal{I}\big\}.$$

The price space is denoted by

$$\Pi \equiv$$

[12] In interpreting condition five, recall that by convention x_i^R and x_i^R are positive because they are consumption vectors, but y_f^R and y_f^r are generally negative because they are inputs for firms.

$$\left\{(p^c, p_1^G, \ldots, p_I^G, p_{I+1}^G, \ldots, p_{I+F}^G, p_1^R, \ldots, p_I^R, p_{I+1}^R, \ldots, p_{I+F}^R) \in \Re^{N^c+(I+F)N^g+(I+F)N^r} \backslash 0\right\}.$$

Note that each private commodity has a common price for all agents whereas public commodities have personalized prices. Furthermore, prices for the private aspects of public goods and externality rights have not been explicitly defined because those prices are derived from the personalized prices of public goods and abatement. That is, given a price vector $\mathbf{p} \in \Pi$, the individualized price faced by consumer i is

$$p_i = \Big(p^c, p_i^G, \sum_{j\in\mathcal{I}} p_j^R + \sum_{k\in\mathcal{F}} p_k^R, p_i^R\Big),$$

where $\sum_j p_j^R + \sum_k p_k^R$ is the price of externality rights and p_i^R is agent i's personalized price on abatement. Note that generally, $p_j^R \geq 0$, and $p_k^R \geq 0$, since p_j^R is the price a consumer pays for the public good "abatement" while $-p_k^R$ is compensation a firm receives for the public bad "smoke".[13] Similarly, the individualized price faced by firm f is

$$p_f = \Big(p^c, \sum_{j\in\mathcal{I}} p_j^G + \sum_{k\in\mathcal{F}} p_k^G,\ p_f^G\ , \sum_{j\in\mathcal{I}} p_j^R + \sum_{k\in\mathcal{F}} p_k^R, p_f^R\Big).$$

It will be helpful to denote the common prices on the private aspects of the public commodities as follows:

$$p^g \equiv \sum_{j\in\mathcal{I}} p_j^G + \sum_{k\in\mathcal{F}} p_k^G \quad \text{and} \quad p^r \equiv \sum_{j\in\mathcal{I}} p_j^R + \sum_{k\in\mathcal{F}} p_k^R.$$

Recall that production possibilities are determined by the aggregate levels of *used* externality rights. However, consumers' utilities depend on the level of *unused* rights. To coordinate these two sides of the rights markets, we require firms and consumers alike to pay according to the aggregate abatement level that is equal to the total available rights minus the aggregate level of smoke.[14] Therefore, firm f producing y_f

[13] If this were not the case, abatement would be a bad and the externality would be positive. While this would not create difficulties in model, it may not be very interesting to study a positive externality that can be produced only if rights are purchased at a negative price.

[14] Alternatively, we could have made firms' production possibilities depend on the abatement level. Formally, however, this would require that the global production set be indexed by the externality rights endowment, which is less conventional than indexing it by the amount of smoke generated by con-

and facing prices p_f and aggregate social rights endowment $\sum_i \omega_i^r$ makes profits of

$$\pi_f\Big(y_f, p_f, \sum_i \omega_i^r\Big) = p^c y_f^c + p^g y_f^g + p_f^G y_f^G + p^r y_f^r - p_f^R\Big(\sum_i \omega_i^r + y_f^R\Big).$$

Let Δ^{I-1} denote the $I-1$ dimensional simplex:

$$\Delta^{I-1} \equiv \Big\{\theta \in \Re^I \Big| \sum_i \theta_i = 1, \text{ and } \theta_i \geq 0 \,\forall\, i \in \mathcal{I}\Big\}.$$

We denote a profit share system for a private ownership economy by $\theta = (\theta_{I+1}, \ldots, \theta_{I+F}) \in \Delta^{I-1} \times \cdots \times \Delta^{I-1} \equiv \Theta$ where θ_{if} is interpreted as consumer i's share of the profits of firm f.

The budget set of agent i depends on endowments, firm shares, profits, and prices. Omitting the arguments of the profit functions, this is given by:

$$B_i(\omega, \theta_i, \pi, p_i) \equiv \Big\{(x_i^c, x_i^G, x_i^r, x_i^R) \in X_i \;\Big|\; p_i x_i \leq p_i \omega_i + \sum_f \theta_{if}\pi_f\Big\}.$$

An allocation and price vector $(a, \mathbf{p}) \in A \times \Pi$ is said to be a *competitive equilibrium relative to endowments* ω and *profit shares* $\theta \in \Theta$ if and only if:

(a) for all $i \in \mathcal{I}$, $x_i \in B_i(\omega_i, \theta_i, \pi(a, \mathbf{p}, \sum_i \omega_i^r), p_i)$ and $x_i \succeq_i \hat{x}_i$ for all $\hat{x}_i \in B_i(\omega_i, \theta_i, \pi(a, \mathbf{p}, \sum_i \omega_i^r), p_i)$;

(b) for all $f \in \mathcal{F}$, $\pi_f(y_f, p_f, \sum_i \omega_i^r) \geq \pi_f(\hat{y}_f, p_f, \sum_i \omega_i^r)$ for all $\hat{y}_f \in Y_f$.

We begin by stating a first welfare theorem for this competitive equilibrium. A second welfare theorem also holds for this model. The proofs for both welfare theorems are notationally dense. And the method of proof for the second welfare theorem is very similar to the one used for the existence theorem below. Thus, in order to not tax the reader unduly, we have omitted these proofs from the current paper. Complete proofs are available from the authors upon request. We state the first welfare theorem anyway due to its importance in interpreting our results in terms of the Coase hypothesis.

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sumers, as we have done. Furthermore, note from the definition of profits that, in general, firms will not desire to shut down and ask that an infinite amount of smoke be produced because y_f^R is bounded below by ω^r. This implies that there is a bound on the compensation that firms have available to them in exchange for allowing smoke to be produced. This is how we avoid Starrett's fundamental nonconvexities in our model.

Theorem 1. *If* $(a, \mathbf{p})$ *is a competitive equilibrium then* $a \in PE$.

In words, Theorem 1 says that, if transaction costs are zero, market exchanges of property rights lead to a Pareto efficient outcome. Note that the distribution of externality property rights is arbitrary and so, although a different efficient allocation is likely to be reached if the distribution of endowments changes, the fact that an efficient outcome is reached does not depend on a particular allocation.[15] Provided that an equilibrium exists, the first welfare theorem is essentially a Coase Theorem.

3. Existence of a Competitive Equilibrium

In this section we turn the question of existence of competitive equilibrium. Our approach is to construct an associated economy for which we can find a quasi-equilibrium using Debreu (1962). This method was introduced by Foley (1970) and developed in more detail by Milleron (1972). We then show that the quasi-equilibrium of the associated economy corresponds to a competitive equilibrium in our original economy. It is important to note, however, that the use of Foley's technique in this model is not a simple adaptation. We discuss the differences in more detail below.

We add the following assumptions in order to prove the existence of a competitive equilibrium. For all $i \in \mathcal{I}$:

A5) X_i has a lower bound

A6) X_i is closed and convex

A7) $(\omega_i^c, 0, \omega_i^r, 0) \in X_i$

and for all $f, k \in \mathcal{F}$:

B5) $(0, 0, y_f^G, 0, y_f^R) \in Y_f$ whenever there exist $y_k \in Y_k$ for all $k \neq f$ and $x_i \in X_i$ for all $i \in \mathcal{I}$ such that $y_f^G = \sum_k y_k^g$ and $y_f^R = \sum_k y_k^r - \sum_i x_i^r$.

[15] This theorem is easily extended to include the case in which firms also have endowments of property rights. See Boyd and Conley (1997).

B6) for all $x^r \in \Re^{N^r}$, $AC(\mathbf{Y}(x^r)) \cap \Re_+^{N^c+2N^g+2N^r} = \{0\}$ where AC denotes the Asymptotic Cone defined as

$$AC(\mathbf{Y}(x^r)) \equiv \left\{ b \in \Re^{N^c+2N^g+2N^r} \;\middle|\; \mathbf{Y}(x^r) + \lambda b \in \mathbf{Y}(x^r) \;\text{ for all }\; \lambda \geq 0 \right\}$$

Assumption B5 is interpreted as a shut-down possibility. That is, regardless of the level of ambient smoke or public goods produced by any other firm or consumer, each firm can always shut down and produce zero output of private commodities and public goods. Assumption B6 is interpreted as the asymptotic impossibility of free production for any given level of public inputs.

For each consumer and firm we introduce one artificial commodity for each consumer's net consumption of public commodities and externalities and one artificial commodity for each firm's net production of public commodities and externalities.

For each $i \in \mathcal{I}$ construct an associated consumption set

$$X_i^* \equiv \Big\{ (\hat{x}^c, \hat{x}_1^G, .., \hat{x}_I^G, \hat{y}_{I+1}^G, .., \hat{y}_{I+F}^G, \hat{x}_1^R, .., \hat{x}_I^R, \hat{y}_{I+1}^R, .., \hat{y}_{I+F}^R) \in \Re^{N^c+(I+F)N^g+(I+F)N^r} \Big|$$

$$\text{there is an } x_i \in X_i \;\text{ such that}$$

$$\hat{x}^c = x_i^c, \;\; \hat{x}_i^G = x_i^G, \;\; \hat{x}_{j\neq i}^G = 0, \;\;\; \hat{y}_f^G = 0 \;\text{ for all }\; f \in \mathcal{F},$$

$$\hat{x}_i^R = x_i^r + x_i^R, \;\; \hat{x}_{j\neq i}^R = x_i^r, \;\text{ and }\;\; \hat{y}_f^R = x_i^r \;\text{ for all }\; f \in \mathcal{F} \Big\}.$$

Each $\hat{x}_j^G$ and $\hat{y}_f^G$ are zero because agent i does not contribute to the production of public goods. In contrast, agent i's use of externality rights contributes to the overall level of the public input of abatement and so the definition of each $\hat{x}_j^R$ and $\hat{y}_f^R$ reflect agents i's private consumption of rights and corresponding contribution to abatement.

We now define a function which maps elements of X_i to elements of X_i^*. Let $\Gamma_i : X_i \to X_i^*$ be defined by

$$\Gamma_i(x_i^c, x_i^G, x_i^r, x_i^R) = (x_i^c, \overbrace{0, .., x_i^G, ., 0}^{I}, \overbrace{0, ..0}^{F}, \overbrace{x_i^r, .., x_i^r + x_i^R, .., x_i^r}^{I}, \overbrace{x_i^r, .., x_i^r}^{F}).$$

It is easily verified that Γ_i is linear in the sense that $\Gamma_i(x_i + \hat{x}_i) = \Gamma_i(x_i) + \Gamma_i(\hat{x}_i)$ and $\Gamma_i(\lambda x_i) = \lambda \Gamma_i(x_i)$ for all $\lambda \in \Re$. The following lemma shows that an element of X_i corresponds to one and only one element of X_i^*.

Lemma 1. *For all $i \in \mathcal{I}$, $\Gamma_i : X_i \to X_i^*$ is bijective.*

Proof/

See the Appendix.

The bijection $\Gamma_i : X_i \to X_i^*$ induces a well-defined preference relation $\succeq_i^*$ on X_i^* such that $x_i \succeq_i \tilde{x}_i$ if and only if $\mathbf{x}_i \succeq_i^* \tilde{\mathbf{x}}_i$ where $\Gamma_i(x_i) = \mathbf{x}_i$ and $\Gamma_i(\tilde{x}_i) = \tilde{\mathbf{x}}_i$. The associated endowment of agent i is $\eta_i = \Gamma_i(\omega_i) = (\omega_i^c, 0, .., 0, \omega_i^r, .., \omega_i^r)$. It will be demonstrated below that, given the way each X_i^* is defined, the global associated consumption set is $X^* = \sum_i X_i^*$. The aggregate associated endowment is $\eta = \sum_i \eta_i$.

For each $f \in \mathcal{F}$ construct an associated production set

$$Y_f^* \equiv \Big\{ (\hat{y}^c, \hat{x}_1^G, .., \hat{x}_I^G, \hat{y}_{I+1}^G, .., \hat{y}_{I+F}^G, \hat{x}_1^R, .., \hat{x}_I^R, \hat{y}_{I+1}^R, .., \hat{y}_{I+F}^R) \in \Re^{N^c + (I+F)N^g + (I+F)N^r} \Big|$$

$$\text{there is an } y_f \in Y_f \text{ such that}$$

$$\hat{y}^c = y_f^c, \quad \hat{x}_i^G = y_f^g \text{ for all } i \in \mathcal{I}, \quad \hat{y}_{k \neq f}^G = y_f^g, \quad \hat{y}_f^G = y_f^g + y_f^G,$$

$$\hat{x}_i^R = y_f^r \text{ for all } i \in \mathcal{I}, \quad \hat{y}_{k \neq f}^R = y_f^r, \quad \text{and} \quad \hat{y}_f^R = y_f^r - \sum_i \omega_i^r - y_f^R \Big\}.$$

Note that the definition of each $\hat{x}_i^G$ and $\hat{y}_f^G$ reflects the fact that firm f's production of public goods simultaneously affects all consumers and firms who consume that good. Similarly, firm f's private use of externality rights simultaneously affects the public level of abatement through each $\hat{x}_i^R$ and the public level of smoke through each $\hat{y}_k^R$. We demonstrate below that by defining production sets in this way the associated global production set, $\mathbf{Y}^* = \sum_f Y_f^*$, corresponds directly with feasibility in our original economy. Finally, note that for any price $\mathbf{p}$, firm f's profits are simply $\pi_f = \mathbf{p} \cdot \mathbf{y}_f^*$ for $\mathbf{y}_f^* \in Y_f^*$.

We now define a function which maps elements of Y_f to elements of Y_f^*. Let $\Upsilon_f : Y_f \to Y_f^*$ be defined by

$$\Upsilon_f(y_f^c, y_f^g, y_f^G, y_f^r, y_f^R) = (y_f^c, \overbrace{y_f^g, .., y_f^g}^{I}, \overbrace{y_f^g, .., y_f^g + y_f^G, .., y_f^g}^{F},$$

$$\overbrace{y_f^r, .., y_f^r}^{I}, \overbrace{y_f^r, .., y_f^r - \sum_i \omega_i^r - y_f^R, .., y_f^r}^{F}).$$

As with Γ_i, it is easily verified that Υ_f is affine. The following lemma shows that an element of Y_f corresponds to one and only one element of Y_f^*.

Lemma 2. *For all $f \in \mathcal{F}$, $\Upsilon_f : Y_f \to Y_f^*$ is bijective.*

Proof/

See the Appendix.

We have thus defined the associated private ownership economy

$$\mathcal{E}^* = \Big\{ \{X_i^*, \succeq_i^*, \eta_i\}_{i \in \mathcal{I}}, \{Y_f^*\}_{f \in \mathcal{F}}, \theta \Big\}$$

from our original economy

$$\mathcal{E} = \Big\{ \{X_i, \succeq_i, \omega_i\}_{i \in \mathcal{I}}, \{Y_f\}_{f \in \mathcal{F}}, \theta \Big\}.$$

An allocation for $\mathcal{E}^*$ is a list $a^* = (\mathbf{x_1}, .., \mathbf{x_I}, \mathbf{y_{I+1}}, .., \mathbf{y_{I+F}})$. Here we state the definition of a feasible allocation for A^* and in the next lemma we show that this definition correctly corresponds with the definition of feasibility in our original economy. The set of feasible allocations A^* for $\mathcal{E}^*$ consists of all allocations a^* such that

1. for all $i \in \mathcal{I}$, $\mathbf{x}_i \in X_i^*$
2. for all $f \in \mathcal{F}$, $\mathbf{y}_f \in Y_f^*$
3. $\sum_i \mathbf{x}_i = \sum_f \mathbf{y}_f + \sum_i \eta_i$.

The following lemma shows that any feasible bundle for $\mathcal{E}^*$ corresponds to one and only one feasible bundle for $\mathcal{E}$.

Lemma 3. *There is a bijection between A and A^*.*

Proof/

See the Appendix.

An allocation and price vector $(\mathbf{a}^*, \mathbf{p}) \in A^* \times \Pi$ is said to be a *quasi-equilibrium relative to endowments* η and *profit shares* $\theta \in \Theta$ if and only if:

(α) for all $i \in \mathcal{I}$, $\mathbf{p} \cdot \mathbf{x_i^*} \leq \mathbf{p} \cdot \eta_i + \sum_f \theta_{if} \mathbf{p} \cdot \mathbf{y_f^*}$ and either $\mathbf{x}_i^* \succeq_i^* \mathbf{x}_i$ for all

$$\mathbf{x}_i \in \left\{ \hat{\mathbf{x}}_i \in X_i^* \middle| \mathbf{p} \cdot \hat{\mathbf{x}}_i \leq \mathbf{p} \cdot \eta_i + \sum_f \theta_i \mathbf{p} \cdot \mathbf{y}_f^* \right\}$$

or

$$\mathbf{p} \cdot \mathbf{x_i^*} = \mathbf{p} \cdot \eta_i + \sum_f \theta_{if} \mathbf{p} \cdot \mathbf{y}_f^* = \min \{\mathbf{p} \cdot \mathbf{x_i} \mid \mathbf{x_i} \in X_i^*\}.$$

(β) for all $f \in \mathcal{F}$, $\mathbf{p} \cdot \mathbf{y}_f^* \geq \mathbf{p} \cdot \mathbf{y}_f$ for all $\mathbf{y}_f \in Y_f^*$

We are now prepared to prove that there exists a quasi-equilibrium of the associated economy. We prove this theorem by demonstrating that our assumptions on the original economy imply that Debreu's (1962) conditions for existence of a quasi–equilibrium are satisfied for $\mathcal{E}^*$.

Theorem 2. *There exists a quasi-equilibrium of $\mathcal{E}^*$.*

Proof/

See the Appendix.

Note that we could also define a quasi-equilibrium of $\mathcal{E}$ in the appropriate way and then prove that the quasi-equilibrium of $\mathcal{E}^*$ corresponds directly to a quasi-equilibrium of $\mathcal{E}$. As we are not particularly interested in the quasi-equilibrium, we state but do not prove the following corollary but we prove a parallel result later to show that the equilibrium of $\mathcal{E}^*$ corresponds to a competitive equilibrium of $\mathcal{E}$.

We introduce the following additional assumptions to show that the quasi-equilibrium of $\mathcal{E}^*$ is also an equilibrium of $\mathcal{E}^*$:

A8) for all $i \in \mathcal{I}$ and for all $\mathbf{p} \in \Pi$, there exists $x_i \in X_i$ such that $p_i x_i < p_i \omega_i$ (there are cheaper points)

Theorem 3. *There exists an equilibrium of $\mathcal{E}^*$.*

Proof/

See the Appendix.

It only remains to be shown that the equilibrium of $\mathcal{E}^*$ corresponds to a competitive equilibrium of our original economy, $\mathcal{E}$.

Theorem 4. *There exists a competitive equilibrium of $\mathcal{E}$.*

Proof/

See the Appendix.

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Appendix

Lemma 1. *For all* $i \in \mathcal{I}$, $\Gamma_i : X_i \to X_i^*$ *is bijective.*

Proof/

a. First note that X_i^* is the image of X_i under Γ_i. This immediately implies that Γ_i is surjective.
b. Next we show that Γ_i is injective. Choose $x_i, \bar{x}_i \in X_i$ such that $\Gamma_i(x_i) = \Gamma_i(\bar{x}_i)$. Expanding this we get

$$\Gamma_i(x_i) = (x_i^c, 0, .., x_i^G, .., 0, x_i^r, .., x_i^r + x_i^R, .., x_i^r) =$$
$$\Gamma_i(\bar{x}_i) = (\bar{x}_i^c, 0, .., \bar{x}_i^G, .., 0, \bar{x}_i^r, .., \bar{x}_i^r + \bar{x}_i^R, .., \bar{x}_i^r)$$

which holds if and only if term by term equality holds. Therefore, $x_i^c = \bar{x}_i^c$, $x_i^G = \bar{x}_i^G$, $x_i^r = \bar{x}_i^r$, $x_i^R = \bar{x}_i^R$ and so we finally have that $x_i = \bar{x}_i$. This shows that Γ_i is injective. □

Lemma 2. *For all $f \in \mathcal{F}$, $\Upsilon_f : Y_f \to Y_f^*$ is bijective.*

Proof/

a. First note that Y_f^* is the image of Y_f under Υ_f. This immediately implies that Υ_f is surjective.
b. Next we show that Υ_f is injective. Choose $y_f, \bar{y}_f \in Y_f$ such that $\Upsilon_f(y_f) = \Upsilon_f(\bar{y}_f)$. Expanding this we get

$$(y_f^c, y_f^g, .., y_f^g + y_f^G, .., y_f^g, y_f^r, .., y_f^r - \sum_i \omega_i^r - y_f^R, .., y_f^r) =$$
$$(\bar{y}_f^c, \bar{y}_f^g, .., \bar{y}_f^g + \bar{y}_f^G, .., \bar{y}_f^g, \bar{y}_f^r, .., \bar{y}_f^r - \sum_i \omega_i^r - y_f^R, .., y_f^r)$$

which holds if and only if term by term equality holds. Therefore, $y_f^c = \bar{y}_f^c$, $y_f^g = \bar{y}_f^g$, $y_f^G = \bar{y}_f^G$, $y_f^r = \bar{y}_f^r$, $y_f^R = \bar{y}_f^R$ and so we finally have that $y_f = \bar{y}_f$. This shows that Υ_f is injective. □

Lemma 3. *There is a bijection between A and A^*.*

Proof/

We need to show that $a = (x_1, .., x_I, y_{I+1}, .., y_{I+F}) \in A$ if and only if

$$a^* = \Big(\Gamma_1(x_1), .., \Gamma_I(x_I), \Upsilon_{I+1}(y_{I+1}), .., \Upsilon_{I+F}\Big) \in A^*.$$

By Lemmas 1 and 2, it is clear that a satisfies conditions one and two of the definition of A if and only if a^* satisfies conditions one and two of the definition of A^*. It only remains to be shown that conditions three, four, and five of the definition of A hold if and only if

$$\sum_i \Gamma_i(x_i) = \sum_f \Upsilon_f(y_f) + \sum_i \Gamma_i(\omega_i).$$

Subtracting the endowment from both sides and expanding this gives

$$(\sum_i x_i^c - \sum_i \omega_i^c, x_1^G, .., x_I^G, 0, .., 0, \sum_i x_i^r + x_i^R - \sum_i \omega_i^r, .., \sum_i x^r + x_I^R - \sum_i \omega_i^r,$$
$$\sum_i x_i^r - \sum_i \omega_i^r, .., \sum_i x_i^r - \sum_i \omega_i^r) =$$

$$(\sum_f y_f^c, \sum_f y_f^g, .., \sum_f y_f^g, \sum_f y_f^g + y_{I+1}^G, .., \sum_f y_f^g + y_{I+F}^G, \sum_f y_f^r, .., \sum_f y_f^r,$$
$$\sum_f y_f^r - \sum_i \omega_i^r - y_{I+1}^R, .., \sum_f y_f^r - \sum_i \omega_i^r - y_{I+F}^R).$$

This is true if and only if term by term equality holds, that is,

$$\sum_i x_i^c - \sum_i \omega_i^c = \sum_f y_f^c,$$

$$x_i^G = -y_f^G = \sum_k y_k^g \text{ for all } i \in \mathcal{I} \text{ and for all } f \in \mathcal{F},$$

$$\sum_j x_j^r - x_i^R - \sum_j \omega_j^r = \sum_k y_k^r \text{ for all } i \in \mathcal{I} \quad \text{and,}$$

$$\sum_j x_j^r - \sum_j \omega_j^r = \sum_k y_k^r - \sum_j \omega_j^r - y_f^R \text{ for all } f \in \mathcal{F}.$$

These are precisely conditions three, four, and five of the definition of A. □

Theorem 2. *There exists a quasi-equilibrium of $\mathcal{E}^*$.*

Proof/
To prove this result, we carefully state each assumption of Debreu's (1962) theorem in the context of our model. We then prove that each of these assumptions hold for $\mathcal{E}^*$.

(a.1) $AC(X^*) \cap -AC(X^*) = \{0\}$.
This result follows from $AC(X^*) \subset \Re_+^{N^c+N^g+2N^r}$ which follows trivially from the fact that, for any nonempty convex set $C \in \Re^N$ which is bounded below, $AC(C) \subset \Re_+^N$.

(a.2) *For every $i \in \mathcal{I}$, X_i^* is closed and convex.*
First write $X_i = X_i^c \times X_i^G \times X_i^r \times X_i^R \subset \Re^{N^c+N^g+2N^r}$. Note that since X_i is closed that X_i^c, X_i^G, X_i^r, and X_i^R are closed. Then since $AC(X_i) \subset \Re_+^{N^c+N^g+2N^r}$ by (b) above, by Debreu (1959) Proposition 1.9(7) we know that $AC(X_i^c) \subset \Re_+^{N^c}$, $AC(X_i^G) \subset \Re_+^{N^G}$, $AC(X_i^r), AC(X_i^R) \subset \Re_+^{N^r}$. It follows immediately that $AC(X_i^r) \cap -AC(X_i^R) = 0$ which implies that $X_i^r + X_i^R$ is closed by Debreu (1959) Proposition 1.9(9). Finally, X_i^* is closed because it is the product of closed sets. Next, recall that $\Gamma_i : X_i \to X_i^*$ is an affine bijection. Thus it is routine to verify that each X_i^* is convex given that each X_i is.

Let agent i's feasible consumption set of the associated economy be defined as

$$\bar{X}_i^*(A^*) \equiv \left\{ x_i^* \in X_i^* \mid \text{ there exists } a^* \in A^* \text{ s.t. } a^* = (x_1^*, .., x_i^*, .., x_I^*, y_{I+1}^*, .., y_{I+F}^*) \right\}$$

and similarly define $\bar{X}_i(\mathcal{A})$.

(b.1) *For every* $\bar{\mathbf{x}}_i \in \bar{X}_i^*$*, there is a* $\mathbf{x}_i \in X_i^*$ *such that* $\mathbf{x}_i \succ_i^* \bar{\mathbf{x}}_i$.
Local nonsatiation implies that for all $\bar{x}_i \in \bar{X}_i$ there exists a $x_i \in X_i$ such that $x_i \succ_i \bar{x}_i$. But $\Gamma_i(x_i) \in X_i^*$ and $\Gamma_i(\bar{x}_i) \in \bar{X}_i^*$ by Debreu's Lemma 6. Finally, $\Gamma_i(x_i) \succ_i^* \Gamma_i(\bar{x}_i)$ because Γ_i preserves order.

(b.2) *For every* $\mathbf{x'}_i \in X_i^*$*, the sets* $\{\mathbf{x}_i \in X_i^* | \mathbf{x}_i \succeq_i \mathbf{x'}_i\}$ *and* $\{\mathbf{x}_i \in X_i^* | \mathbf{x'}_i \succeq_i \mathbf{x}_i\}$ *are closed in* X_i^**.*
This follows directly from Lemma 3 and assumption A1).

(b.3) *For every* $\mathbf{x'}_i \in X_i^*$*, the set* $\{\mathbf{x}_i \in X_i^* | \mathbf{x}_i \succeq_i \mathbf{x'}_i\}$ *is convex.*
This follows directly from Lemma 3 and assumption A3).

(c.1) $(\eta + Y^*) \cap X^* \neq \emptyset$.
Since $0 \in Y^*$ we have that $\eta \in (\eta + Y^*)$. And $\eta \in X^*$ by the fact that $(w_i^c, 0, w_i^r, 0) \in X_i$ for all $i \in \mathcal{I}$. Therefore, $\eta \in (\eta + Y^*) \cap X^*$.

(c.2) *If* D *is the smallest cone containing points* $\sum_i (\mathbf{x}_i - \eta_i)$ *where* $\mathbf{x}_i$ *is strictly maximal on* $\bar{X}_i^*$ *for all* $i \in \mathcal{I}$ *then* $(\eta_i + closure(Y^*) - D) \cap X_i^* \neq \emptyset$.
It is easily verified that A^* is compact and, therefore, $\bar{X}_i^*$ is compact. So if A^* is nonempty and since $\succeq_i^*$ is continuous, $\bar{X}_i^*$ has a maximal element, $\mathbf{x_i}$. Then since $0 \in D$ we only need to show that $\eta_i \in (\eta_i + closure(Y^*) - D) \cap X_i^*$. Furthermore, $closure(Y^*) + \eta = closure(Y^* + \eta)$ and since X^* is closed by (a.2) above, we know that $closure(Y^* + \eta) \cap X^* = closure[(Y^* + \eta) \cap X^*]$. Therefore, the claim reduces to showing that $(Y^* + \eta) \cap X^*$ is closed.
Now consider the set $\ddot{Y}(x^r)$ defined as follows:

$$\ddot{Y}(x^r) \equiv \Big\{ (z^c, z^g, \ldots, z^g, 0, \ldots, 0, z^r, \ldots, z^r, z^R, \ldots, z^R) \in \Re^{N^c + (I+F)N^g + (I+F)N^r} \Big| $$

$$(z^c - \sum_i \omega_i^c, z^g, -z^g, z^r - \sum_i \omega_i^r, z^R - \sum_i \omega_i^r) \in \mathbf{Y}(x^r) \Big\}.$$

Note that since $\mathbf{Y}(x^r)$ is closed for all x^r and is upper semi-continuous, then $\ddot{Y}(x^r)$ is also closed and upper semi-continuous. Now consider the coordinate projection $\Psi : X^* \to \Re^{N^c + (I+F)N^g + (I+F)N^r}$ defined as follows:

$$\Psi(\hat{x}^c, \hat{x}_1^G, \ldots, \hat{x}_I^G, \hat{y}_{I+1}^G, \ldots, \hat{y}_{I+F}^G, \hat{x}_1^R, \ldots, \hat{x}_I^R, \hat{y}_{I+1}^R, \ldots, \hat{y}_{I+F}^R) =$$

$$(0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, \hat{y}_F^R)$$

and note that Ψ is continuous. Then by Debreu (1959) Proposition 1.8(2), $\ddot{Y} \circ \Psi$ is also upper semi-continuous. Then by construction

$$\forall x \in X^* \cap (Y^* + \eta), \quad x \in \ddot{Y}(\Psi(x)) \tag{$*$}$$

$$\text{and } \forall x \in X^*, \quad \ddot{Y}(\Psi(x)) \subset Y^* + \eta. \tag{$**$}$$

By (d.2), we know that $AC(X^*) \cap AC(Y^*) = \{0\}$. By Debreu (1959) Proposition 1.9(8) we therefore know that $X^* \cap (Y^* + \eta)$ bounded. Then the Bolzano-Weierstrass theorem tells us that every sequence in $X^* \cap (Y^* + \eta)$ has a convergent subsequence. Let $\{z_k\}$ be a convergent sequence in $X^* \cap (Y^* + \eta)$ and suppose that it converges to z_0. Since X^* is closed, it must be the case that $z_0 \in X^*$. We need to show that $z_0 \in Y^* + \eta$ in order to prove the claim. By $(*)$ above, we know that each $z_k \in \ddot{Y}(\Psi(z_k))$ and by upper semi-continuity of $\ddot{Y} \circ \Psi$, $z_0 \in \ddot{Y}(\Psi(z_0))$. By $(**)$, $z_0 \in X^*$ implies that $z_0 \in Y^* + \eta$.

(d.1) *for all* $f \in \mathcal{F}$, $0 \in Y_f^*$
This follows directly from $(0, 0, 0, 0, -\sum_i \omega_i^r) \in Y_f$ for all $f \in \mathcal{F}$.

(d.2) $AC(X^*) \cap \; AC(Y^*) = \{0\}$.
By assumption (B6) we know $AC(Y(x^r)) \cap \Re_+^{N^c+2N^g+2N^r} = \{0\}$. Therefore $AC(Y^*) \cap \Re_+^{N^c+(I+F)N^g+(I+F)N^r} = \{0\}$. By (a.1) above we know that $AC(X^*) \subset \Re_+^{N^c+(I+F)N^g+(I+F)N^r}$. Therefore $AC(X^*) \cap \; AC(Y^*) = \{0\}$.

Under the above conditions Debreu's (1962) conditions for a quasi-equilibrium of $\mathcal{E}^*$ hold, and therefore the theorem is proven. □

Theorem 3. *There exists an equilibrium of* $\mathcal{E}^*$.

Proof/
By the previous theorem there exists a quasi-equilibrium of $\mathcal{E}^*$. Then by assumption (B8), all consumers have positive wealth. Therefore, the quasi-equilibrium is also an equilibrium. □

Theorem 4. *There exists a competitive equilibrium of* $\mathcal{E}$.

Proof/
Let $(a^*, \mathbf{p})$ where $a^* = (\mathbf{x}_1, .., \mathbf{x}_I, \mathbf{y}_{I+1}, .., \mathbf{y}_{I+F})$ be an equilibrium for $\mathcal{E}^*$. Now let

$$a = (\Gamma_1^{-1}(\mathbf{x}_1), .., \Gamma_I^{-1}(\mathbf{x}_I), \Upsilon_{I+1}^{-1}(\mathbf{y}_{I+1}), .., \Upsilon_{I+F}^{-1}(\mathbf{y}_{I+F})) = (x_1, .., x_I, y_{I+1}, .., y_{I+F}).$$

Notice that $\mathbf{p} \cdot \mathbf{x_i} = p_i \cdot x_i = p^c x_i^c + p_i^G + p^r x_i^r + p_i^R x_i^R$ so (α) implies that $\Gamma_i^{-1}(\mathbf{x}_i)$ for each $i \in \mathcal{I}$ satisfies part (a) of the definition of a competitive equilibrium. Similarly, since $\pi_f = \mathbf{p} \cdot \mathbf{y_f} = p_f \cdot y_f$, (β) implies that $\Upsilon_f^{-1}(\mathbf{y}_f)$ for each $f \in \mathcal{F}$ satisfies part (b). And finally, by Lemma (6), feasibility of a^* implies feasibility of a. Therefore, $(a, \mathbf{p})$ is a competitive equilibrium of $\mathcal{E}$. □