Axiomatic Foundations for Compromise Theory†

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Abstract

We provide a characterization of the symmetric and weighted versions of the Euclidean compromise solution for multiobjective optimization problems first proposed by P.L.Yu. These solutions minimize a measure of the distance between a feasible set and the utopia point of that set, and are closely related to solutions that have been proposed in the literature that studies bargaining with claims. We provide new axiomatizations of the symmetric and weighted Nash Bargaining Solutions and our characterizations of the compromise solutions employ axioms that are, in a certain sense, dual to those used in these new axiomatizations of the Nash solutions. We also provide characterizations for solutions that are dual to the Egalitarian and Kalai-Smorodinsky solutions.

1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where $S \subset \Re^n$ is a set of feasible payoffs, and $d \in S$ (the *disagreement point*) is interpreted as the payoff that agents receive if they fail to reach an agreement. In *bargaining theory* one is typically interested in finding a solution that shares the gains among the agents as measured from the disagreement point. Bargaining theory is only one of many possible approaches to solving social choice problems. Sharing the surplus according to a notion of fairness captured by a set of axioms has a great deal of appeal. However, the application of this method depends critically on the existence of a disagreement point from which to measure these gains.

When such a disagreement point is not part of the data, then an alternative approach is available in which an endogenously defined reference point replaces the disagreement point in the construction of the solution. See Thomson (1981b) or more recently, Conley, McLean, and Wilkie (1997), for further discussion. Another possibility can be found in the bargaining with claims literature initiated by Chun and Thomson (1992). In this approach, each agent is assumed to have an exogenously given claim on the social surplus. If it is not possible to simultaneously satisfy all of these competing claims, then agents must reach a compromise and settle on some feasible payoff profile. Also see Bossert (1992, 1993) and Herrero (1994) for interesting papers in this spirit. We discuss the exact relationship between the current paper and the bargaining with claims literature in more detail below.

In this paper, we study a third alternative that is similar in spirit to the theory of bargaining with claims. In this approach, agents must reach a compromise based on an endogenously determined but generally infeasible *utopia point* whose coordinates correspond to the maximum feasible payoffs attainable by the players. We will provide axiomatizations of several solutions based on this idea and, following Yu (1973) and Freimer and Yu (1976), we will refer to these as *compromise solutions.*

A multiobjective programming problem is defined by a feasible set of actions, X, and n objective functions. These objectives generally rank the alternatives in X differently. A typical example of a multiobjective programming problem is the following. Suppose that there were several alternative methods of manufacturing a new computer chip. Each method is associated with different levels of cost, time to completion, reliability, environmental damage, need for newly trained personnel, and other variables that the firm would like to maximize or minimize (these are the multiple objectives). These functions may be used to map each process into a point in this objective space. Since it is possible to manufacture chips using several processes at once, the problem amounts to finding the best linear combination of these points. Thus, multiobjective programming problems reduce to choosing an optimal point from an n-dimensional, convex feasible set. The obvious approach of defining a utility function over the objective space is rejected as nonoperational, either because the decisions are made by committee and hence we run into Arrow's impossibility theorem, or because of the difficulty of articulating or eliciting preferences.

Such problems are generally solved in the multiobjective programming literature by sharing losses over the different objectives as measured from an infeasible "aspiration point" according to some geometric rule. See Arrow and Raynaud (1986), Lee (1972), and Yu (1985) for further discussion. For an extensive bibliography of the applications of multiobjective programming and compromise theory, see White (1990).

The primary contribution of this paper is an axiomatic characterization for both the symmetric and the weighted Euclidean compromise solutions in the general framework of bargaining problems with nonempty, compact, convex sets of attainable payoffs. In addition to this we explore the relationship between the Euclidean Compromise Solution and a variant of the Nash Bargaining Solution in which a (generally infeasible) *nadir point* replaces the disagreement point of the classical theory (see Conley, McLean and Wilkiel, 1997). The complementary nature of the two solutions is apparent from the definitions: players share losses with respect to a utopia point in the compromise approach while they share gains with respect to a nadir point in the usual Nash approach. We provide new axiomatizations of the symmetric and weighted Nash Bargaining Solutions in the absence of a disagreement point and then give characterizations of the compromise solutions that employ axioms that are, in a certain sense, dual to those used in our new axiomatizations of the Nash solutions. In a related paper, Rubinstein and Zhou (1999) axiomatize a solution that minimizes the Euclidean distance between a feasible set and a fixed point. In contrast, we adapt Yu's widely used approach of using an endogenously determined aspiration point. Voorneveld, van den Nouweland and McLean (2008) also provide an alternative characterization using a consistency based approach.

We also provide a characterization of a solution that is closely related to Chun's equal loss solution (see Chun, 1988) utilizing the utopia point in a way that complements the use of the disagreement point in the standard axiomatization of the Egalitarian Solution provided in Kalai (1977). Finally, we show that that Raiffa Solution axiomatized in Kalai and Smorodinsky (1975) can be characterized in two ways using different versions of scale covariance formulated in terms of the nadir point or the utopia point.

2. Preliminaries

In this section we present some of the basic definitions and notation that will be used throughout the rest of the paper. Let $N \equiv \{1, \ldots, n\}$ denote the index set. Given a point $z \in \Re^n$, a set $S \subset \Re^n$ is said to be *z-comprehensive* if $z \leq x \leq y$ and $y \in S$ imply $x \in S$.

The *comprehensive hull of a set* $S \subset \Re^n$*, with respect to a point* $z \in \Re^n$, is the smallest z-comprehensive set containing S:[1]

$$comp(S; z) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists\, y \in S \text{ such that } z \leq x \leq y\}.$$

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[1] We use the following conventions to indicate vector inequalities in $\Re^n$: $x \geq y$ means $x_i \geq y_i$ for all i; $x > y$ means $x \geq y$ and in addition $x \neq y$; $x \gg y$ means $x_i > y_i$ for all $i \in N$, where subscripts indicate the components of a vector.

The convex hull of a set $S \subseteq R^n$ will be denoted

$$con(S).$$

Let

$$\Sigma \equiv \{S \subseteq \Re^n \mid S \text{ is nonempty and compact } \}.$$

The *utopia point* of a feasible set $S \in \Sigma$ is defined as $u(S) = (u_1(S), .., u_n(S)) \in \Re^n$ where

$$u_i(S) = \max_{x \in S} x_i$$

and the *nadir point* is defined as $m(S) = (m_1(S), .., m_n(S)) \in \Re^n$ where[2]

$$m_i(S) = \min_{x \in S} x_i.$$

We now define several classes of feasible sets that will serve as the domains for the various solutions that we present in this paper. Let

$$\Sigma^{con} \equiv \{S \in \Sigma \mid S \text{ is convex } \}$$

$$\Sigma_0^{con} \equiv \{S \in \Sigma^{con} | x \gg m(S) \text{ for some } x \in S\}$$

$$\Sigma^{comp} \equiv \{S \in \Sigma | comp(S; m(S)) = S \text{ and S has nonempty interior } \}.$$

Note that we do not require convexity of the feasible set in Σ^{comp},

Define the *strong Pareto set of* S as:

$$P(S) \equiv \{x \in S \mid y > x \text{ implies } y \notin S\},$$

and the *weak Pareto set of* S as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}.$$

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[2] We use m rather than n to avoid confusion with the n used to indicate the diminsion for the Euclidiean space in which the solutions are defined.

If $b, x \in \Re^n$, then $b{*}x \in \Re^n$ will denote the vector with i^{th} component $(b{*}x)_i = b_i x_i$ and

$$b * S \equiv \{b * x | x \in S\}.$$

If $b \in \Re^n_{++}$, then $b^{-1} = (\frac{1}{b_1}, .., \frac{1}{b_n})$. Finally, let e denote the vector $(1, .., 1)$ and for each i, let e_i denote the standard basis vector in $\Re^n$.

A *permutation operator*, π, is a bijection from N to N. Π^n is the class of all such operators. Let $\pi(x) = (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \ldots, x_{\pi^{-1}(n)})$ and $\pi(S) = \{y \in \Re^n \mid y = \pi(x) \text{ and } x \in S\}$.

3. The Compromise and Nash Bargaining Solutions

Recall that a bargaining problem is a pair (S, d), where $d \in S$, and there exists some $x \in S$ such that $x \gg d$. If $\mathcal{B}$ denotes the set of all bargaining problems, then a solution is a map that associates with each $(S, d) \in \mathcal{B}$ a point $\Phi(S, d) \in S$.

The Nash bargaining solution, introduced in Nash (1950), is defined as follows:

$$Nash(S, d) \equiv \left\{ \underset{\substack{x \in S \\ x \geq d}}{\text{argmax}} \prod_{i \in N} (x_i - d_i) \right\}.$$

A *multi-objective choice problem* consists simply of a feasible set S in the n-dimensional objective space. A solution is a map that associates with each problem S in some domain of problems, $\mathcal{M}$, a unique feasible point, $F(S) \in S$.

Note that in this domain, there is no disagreement point. The natural generalization of the Nash solution is therefore one in which the nadir point replaces the disagreement point as the agents' minimal expectation of payoffs and define the solution accordingly. This follows the reference function approach presented in Conley, McLean and Wilkie (1997). Thus, we define the *Nash Bargaining Solution* Σ_0^{con} as follows:

$$NB(S) \equiv \underset{\substack{x \in S \\ x \geq m(S)}}{\operatorname{argmax}} \prod_{i \in N} (x_i - m_i(S)).$$

A nonsymmetric version can be defined as follows: for each $w \in \Re^n_{++}$, the *weighted Nash Bargaining Solution* on $NB^w : \Sigma_0^{con} \to \Re^n$ is defined as follows: for each $S \in \Sigma_0^{con}$,

$$NB^w(S) = \underset{\substack{x \in S \\ x \geq m(S)}}{\operatorname{argmax}} \prod_{i \in N} (x_i - m_i(S))^{w_i}.$$

The logic of the Nash Bargaining Solution for choice problems is the same as that for bargaining problems with an exogenous disagreement point: given S, agents bargain over gains relative to the (possibly infeasible) profile of individually rational payoffs $m(S)$. In this paper, we propose a "dual" approach in which agents bargain over losses relative to a (possibly infeasible) profile of utopia payoffs.

The specific solution that we study here is one of a class introduced in Yu (1973) and extensively studied in Yu(1985). Formally, for all $S \in \Sigma_0^{con}$ and for each $1 < p < \infty$, let $Y_p(S)$ be defined by:

$$Y_p(S) \equiv \underset{x \in S}{\operatorname{argmin}} \left(\sum_{i \in N} (u_i(S) - x_i)^p \right)^{\frac{1}{p}}.$$

When $p = 2$, this solution minimizes the Euclidean distance between the utopia point and the feasible set. "Since this is the case with which we deal in this paper, we will simplify the notaion and write $Y(S)$ for $Y_2(S)$ and we will refer to the function $Y : \Sigma^{con} \to R^n$ as the *Euclidean Compromise solution*.[3]

Formally, for all $S \in \Sigma^{con}$ let $Y(S)$ be defined by:

$$Y(S) \equiv \underset{x \in S}{\operatorname{argmin}} \left(\sum_{i \in N} (u_i(S) - x_i)^2 \right)^{\frac{1}{2}}.$$

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[3] Note that $Y_p(S)$ can be defined for $p = 1$ or ∞, although the solution will not be point valued in these cases. The case of $p = 1$ is important because it defines the utilitarian solution. See Thomson (1981a) for an interesting discussion and characterization of this solution. The case of $p = \infty$ corresponds to minimizing the Chebychev distance from the utopia point to the feasible set. We will study this solution in more detail in the next section.

The class of solutions proposed by Yu can be generalized to a nonsymmetric version as follows: for each $1 < p < 1$, for each $w \in \Re^n_{++}$, and each $S \in \Sigma^{con}$, let $Y^w_p(S)$ be defined as:

$$Y^w_p(S) \equiv \arg\min_{x \in S} \left(\sum_{i \in N} w_i [u_i(S) - x_i]^p \right)^{\frac{1}{p}} .$$

The resulting solution $Y^w_p : \Sigma^{con} \to \Re^n$ minimizes the distance between the utopia point and the feasible set where distance is defined in terms of a weighted metric on $\Re^n$: Clearly, $Y^w_p = Y_p$ if w is a positive scalar multiple of e. We will again simplify the notation and write $Y^w(S)$ for $Y^w_2(S)$: In particular, the solution $Y^w : \Sigma^{con} \to \Re^n$ is defined for each $S \in \Sigma^{con}$ as

$$Y^w(S) \equiv \arg\min_{x \in S} \left(\sum_{i \in N} w_i [u_i(S) - x_i]^2 \right)^{\frac{1}{2}} .$$

We now present standard axioms that will be used in the characterizations that follow:

Weak Pareto Optimality (WPO): $F(S) \in WP(S)$.

Pareto Optimality (PO): $F(S) \in P(S)$.

Symmetry (SYM): If $\pi(S) = S \ \ \forall \pi \in \Pi^n$, then $F_i(S) = F_j(S), \ \forall i, j$.

Translation Invariance (TINV): For all $x \in \Re^n$, $F(S + x) = F(S) + x$.

Scale Covariance (SCOV): $F(\lambda * S) = \lambda * F(S)$ for all $\lambda \in \Re^n_{++}$

n-Independence of Irrelevant Alternatives (n-IIA) Suppose that $S, T \in \Sigma$ with $S \subseteq T$ and $n(S) = n(T)$. Then $F(T) \in S$ implies $F(S) = F(T)$.

u-Independence of Irrelevant Alternatives (u-IIA): Suppose that $S, T \in \Sigma$ with $S \subseteq T$ and $u(S) = u(T)$. Then $F(T) \in S$ implies $F(S) = F(T)$.

The axioms PO, TINV, SCOV are obvious translations of Nash's axioms from bargaining theory to the choice problem framework. Furthermore, n-IIA and u-IIA are adaptations of Nash's original IIA axiom for bargaining problems to the choice framework with the disagreement point d replaced by $n(S)$ or $u(S)$. Given these remarks, it is not surprising that one can characterize the NB solution on Σ_0^{con} using PO, TINV, SYM, SCOV and n-IIA. The next result is a consequence of Theorem 4 in Conley, McLean, Wilkie (1997).

Theorem 1. *A solution $F : \Sigma_0^{con} \rightarrow \Re^n$ satisfies PO, SYM, TINV, SCOV and n-IIA if and only if $F = NB$.*

Proof/

See Theorem 4 in Conley, McLean, Wilkie, 1997.

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Next, we introduce two axioms that will be used in the characterizations of the classes of weighted compromise and Nash bargaining solutions. Note that these are very weak axioms since we will require that they apply only to an extremely limited domain of problems. Define

$$\Delta = con\{e^1, .., e^n\}$$

and

$$\Gamma = con\{e - e^1, .., e - e^n\}.$$

Individual Rationality[4] (IR): $F_i(\Delta) > 0$ for all $i \in N$

Individual Fairness (IF): $F_i(\Gamma) < 1$ for all $i \in N$

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[4] Here we use the term individual rationality in a manner analagous to that of Roth (1977). A more appropriate term would be strict individual rationality.

One can think of these as weak forms of symmetry and the motivation is straightforward. Imagine a group of n social choice theorists sitting down at a bar. At first the waitress brings a single beer. They don't want it to go to waste, but it only seems fair if anyone drinks, no one should be left out. Thus, they impose IR and require that everyone has at be allowed least a sip. Next, the waitress brings them *n-1* beers. Each can reasonably claim they are entitled to an entire beer, but of course it is not feasible to satisfy all these claims. Thus, they agree to impose IF which says that if it is infeasible to satisfy all claims, that no one should have their entire claim satisfied while others have to make do with only a partial settlement. Note that the domain of these two axioms is limited to very special subsets of symmetric problems where this motivation is especially compelling.

The technical roles played by IR (in the axiomatization of the class of weighted Nash solutions) and by IF (in the axiomatization of the class of weighted compromise solutions) are similar. In each case, the axiom is used to identify a positive weight vector that defines a specific member of the class of weighted solutions. Axioms with a similar role have been used to axiomatize a variety of weighted generalizations of symmetric solutions for TU and NTU games. These include the positivity axiom used by Kalai and Samet (1987) in their axiomatization of the class of weighted Shapley values and the feasibility axiom used by Kalai (1977) in his axiomatization of the class of weighted Nash bargaining solutions. Indeed, we could use IR in place of SYM in Theorem 1 and characterize the class of weighted Nash bargaining solutions on Σ_0^{con} using a proof identical to that in Kalai (1977).[5]

As a first approach to finding a desirable compromise solution on Σ_0^{con} in which losses are measured from the utopia point, it seems natural to propose the following analogue of the Nash bargaining solution.

$$G(S) \equiv \arg\min_{x \in S} \prod_{i \in N} [u_i(S) - x_i]$$

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[5] It important to point out that every weighted Nash solution satisfies IR but may not satisfy IF. Similarly, every weighted compromise solution satisfies IF but may not satisfy IR.

Since $G(S) \neq \emptyset$ for each $S \in \mathcal{C}$, G is a well defined set-valued mapping on Σ_0^{con}. Unfortunately, this correspondence need not be single valued and is therefore not a solution according to our definition. Furthermore, there is no selection from this correspondence that satisfies the symmetry or u-IIA axioms. These problems arise because

$$f(x_1, .., x_n) = \ln \prod_{i \in N} [u_i(S) - x_i] = \sum_{i \in N} \ln[u_i(S) - x_i]$$

defines a strictly concave function on S and its minimum is therefore attained at an extreme point (or possibly several extreme points) of S. We now ask the following question: is there any solution on Σ_0^{con} that satisfies Nash's axioms of PO, SYM, TINV and SCOV, and u-IIA (instead of n-IIA)? Lemma 1 shows that there is no such solution.

Lemma 1. *On the domain Σ_0^{con}, there is no solution that satisfies PO, SYM, u-IIA and SCOV.*

Proof/

Let $n = 2$ and consider the following problem:

$$S = con\{(-8, 0), (-8, -8), (0, -8), (-4, 0), (0, -4)\}$$

$$T = con\{(-4, 0), (-4, -12), (0, -12), (-2, 0), (0, -6)\}$$

By PO and SYM, it follows that $F(S) = (-2, -2)$. Furthermore, $T = b * S$ where $b = (\frac{1}{2}, \frac{3}{2})$ so SCOV implies that $F(T) = (-1, -3)$. Now let

$$C = con\{(-4, 0), (-1, -3), (0, -6), (-4, -6)\}.$$

Note that $C \subseteq S, u(C) = u(S) = (0, 0)$ and $F(S) \in C$. Hence, we can apply u-IIA and deduce that $F(C) = (-2, -2)$. However, it is also true that $C \subseteq T, u(C) = u(T) = (0, 0)$ and $F(T) \in C$. Applying u-IIA again, it follows that $F(C) = (-1, -3)$, a contradiction.

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This result shows that there is no scale covariant choice solution that is “dual” to the NB solution. However, the feasible sets associated with multiobjective programs are not necessarily generated by vonNeumann-Morgenstern utility functions and the SCOV axiom loses some of its appeal as an axiom on this domain. Scale covariance may be interpreted as a cardinal proportional division of the surplus axiom. Viewed as a proportionality requirement, scale covariance becomes less compelling, especially in light of the impossibility result. Several alternatives are possible. We will modify scale covariance and weaken the domain of application of the axiom. In particular, the proportionality axiom that replaces scale covariance will be restricted to certain “transferable utility problems” in which the trade off between any two objectives is constant.

If $p \in \Re^n_{++}$ and a is a number, let $H(p,a) = \{x \in \Re^n \mid p \cdot x = a\}$ denote the associated hyperplane. We now define two special classes of choice problems and two additional axioms defined for these special classes.

Let

$$\mathcal{L}^+ = \{S \in \Sigma_0^{con} \mid P(S) = H(p,a) \cap [n(S) + \Re^n_+] \text{ for some } p \in \Re^n_{++}\}$$

and let

$$\mathcal{L}^- = \{S \in \Sigma_0^{con} \mid P(S) = H(p,a) \cap [u(S) - \Re^n_+] \text{ for some } p \in \Re^n_{++}\}.$$

In words, $S \in \mathcal{L}^+$ if the Pareto set of the problem is equal to a hyperplane that extends exactly to edges of the positive orthant defined using the nadir point as the origin. Similarly, $S \in \mathcal{L}^-$ if the Pareto set of the problem is equal to a hyperplane that extends exactly to the edges of the negative orthant defined using the utopia point as the origin. Given this, we define the following two proportionality axioms:

Proportional Gains: (PGAIN): Suppose that $S \in \mathcal{L}^+$ and $\lambda \in \Re^n_{++}$. If $S' = \lambda * S$, then

$$\lambda_j [F_i(S') - n_i(S')][F_j(S) - n_j(S)] = \lambda_i [F_j(S') - n_j(S')][F_i(S) - n_i(S)]$$

for all i, j.

Proportional Losses: (PLOSS): Suppose that $S \in \mathcal{L}^-$ and $\lambda \in \Re^n_{++}$. If $S' = \lambda * S$, then

$$\lambda_i[u_i(S') - F_i(S')][u_j(S) - F_j(S)] = \lambda_j[u_j(S') - F_j(S')][u_i(S) - F_i(S)]$$

for all i, j.

Since $n(\lambda * S) = \lambda * n(S)$ whenever $\lambda \in \Re^n_{++}$ and $S \in \Sigma^{con}$, it follows that PGAIN is weaker than SCOV. Note that the "locations" of λ_i and λ_j have been switched in the statements of the two axioms. To interpret these axioms, suppose $S \in \mathcal{L}^+$ and that $\lambda \in \Re^n_{++}$ with $\lambda_i/\lambda_j = 2$. In the rescaled problem $\lambda * S = S'$, player/objective i has twice the relative weight (or "importance") of player/objective j as in the original problem S. According to PGAIN,

$$\frac{F_i(S') - n_i(S')}{F_j(S') - n_j(S')} = 2\left[\frac{F_i(S) - n_i(S)}{F_j(S) - n_j(S)}\right]$$

That is, the utility **gain** to player i relative to that of player j in the problem $\lambda * S = S'$ should be twice the utility **gain** to player i relative to that of player j in the original problem S.

Now suppose $S \in \mathcal{L}^-$ and again, suppose that $\lambda \in \Re^n_{++}$ with $\lambda_i/\lambda_j = 2$. According to PLOSS,

$$\frac{u_i(S') - F_i(S')}{u_j(S') - F_j(S')} = \tfrac{1}{2}\left[\frac{u_i(S) - F_i(S)}{u_j(S) - F_j(S)}\right]$$

Hence, the utility **loss** to player i relative to that of player j in the problem $\lambda * S = S'$ should be half the utility **loss** to player i relative to that of player j in the original problem S. Therefore, the player/objective who has become twice as important enjoys a doubling of his relative gain in PGAIN but only suffers a halving of his relative loss in PLOSS.

We now provide an alternative characterization of the weighted Nash Bargaining solution on Σ^{con}.

Theorem 2. *A solution $F : \Sigma_0^{con} \rightarrow \Re^n$ satisfies PO, IR, TINV, n-IIA and PGAIN if and only if there exists $w \in \Re^n_{++}$ such that $F = NB^w$.*

Proof/

See appendix.

As an immediate corollary of Theorem 2, we obtain the following characterization of the symmetric Nash Bargaining Solution. Corollary 1 differs from Theorem 1 in that PGAIN has replaced the stronger SCOV.

Corollary 1. *A solution $F : \Sigma_0^{con} \rightarrow \Re^n$ satisfies PO, IR, TINV, n-IIA PGAIN and SYM if and only if there $F = NB$.*

Proof/

Left to reader.

We now turn to the weighted and symmetric Euclidean compromise solutions. In order to characterize the function Y on Σ^{con}, we need the following continuity axiom:

Continuity (CONT): $F : \Sigma^{con} \rightarrow \Re^n$ is continuous with respect to the Hausdorff metric topology on Σ^{con}.

Theorem 3. *A solution $F : \Sigma^{con} \rightarrow \Re^n$ satisfies PO, TINV, u-IIA, IF, CONT and PLOSS if and only if there exists $w \in \Re^n_{++}$ such that $F = Y^w$.*

Proof/

See appendix.

Corollary 2. *A solution $F : \Sigma^{con} \rightarrow \Re^n$ satisfies PO, TINV, u-IIA, SYM, CONT and PLOSS if and only if $F = Y$.*

Proof/

See appendix.

Note that the continuity axiom cannot be dropped from Theorem 3. To see this, let $w = (1, 2, 3)$ and define a solution $F : \Sigma^{con} \rightarrow \Re^n$ as follows:

$$F(S) = \begin{cases} Y(S) & \text{if } Y(S) < u(S) \\ Y^w(S) & \text{otherwise.} \end{cases}$$

It is easy to verify that F satisfies PO, TINV, u-IIA, and PLOSS. However, F is not continuous. For each $\varepsilon \geq 0$, define

$$S_\varepsilon = \{(1,0,0),(0,1,0),(0,0,\varepsilon)\}.$$

so that

$$u(S_\varepsilon) = (1,1,\varepsilon).$$

If $\varepsilon < 1/2$, then $Y(S_\varepsilon) = (1/2,1/2,0)$ so $F(S_\varepsilon) = (1/2,1/2,0)$ whenever $0 \leq \varepsilon < 1/2$. Note that $S_\varepsilon \to S_0$ in the Hausdorff metric topology as $\varepsilon \to 0$. Since $F(S_0) = (1/3,2/3,0)$ but $F(S_\varepsilon) \to (1/2,1/2,0)$, it follows that F is not continuous.

Theorem 3 and Corollary 2 are the main results of this paper and provide the first axiomatic characterization of the symmetric and weighted Euclidean Compromise solutions of which we are aware. In addition, a comparison of these results demonstrates a clear duality between the Nash Bargaining Solution and the Euclidean Compromise Solution. We continue to explore related duality ideas in the next section.

4. Duality and the Egalitarian and Kalai-Smorodinsky Solutions.

The Egalitarian Bargaining solution was characterized on the domain of comprehensive problems by Kalai (1977). In our domain of choice problems, the nadir point again replaces the disagreement point and the *Egalitarian Bargaining Solution* is defined as:

$$EB(S) \equiv m(S) + t^* e, \text{ where } t^* = max\{t \in \Re_+ \mid m(S) + te \in S\}.$$

As we mentioned in the previous section, the solution Y^p for $p = \infty$ corresponds to minimizing the Chebyshev distance from the utopia point to the feasible set. However, strictly speaking, Y^∞ does not define a *solution* when $n > 2$ since it is not necessarily single valued. Consider the following example in $\Re^3$. Let

$$S = conv((0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1),(0,1,1)).$$

Observe that $u(S) = (1, 1, 1)$, and for every point $x \in conv((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 1))$, we have that $||u(S) - x||_\infty = \frac{1}{2}$. Thus, there is a continuum of points that minimizes distance with respect to the *sup norm.* The failure to be decisive in recommending a unique point disqualifies the sup norm as a basis for a compromise solution. The *egalitarian compromise* we propose in this paper is actually a selection from this set of minimizers:

$$EC(S) \equiv u(S) - (t^*)e, \text{ where } t^* = \min\{t \in \Re_+ \mid u(S) - te \in S\}.$$

This selection is justified for several reasons. First, for those choice sets with $P(S) = WP(S)$, it holds that $Y^\infty(S)$ is a singleton and that $Y^\infty(S) = EC(S)$. Second, the selection $EC(\cdot)$ is continuous. Third, we provide an axiomatic justification for the assertion that EC is dual to the EB solution of Kalai, (1977). Unfortunately, there is one problem with this selection: when $n > 2$, it does not necessarily exist. Consider the set:

$$S = con((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, \tfrac{1}{4}), (1, 0, \tfrac{1}{4}), (0, 1, \tfrac{1}{4})).$$

Observe that $u(S) = (1, 1, \frac{1}{4})$. However, the set of points of equal loss from $u(S)$ does not intersect S. One possibility solution to this problem is to change the domain. The sub-domain of Σ^{comp} on which EC exists is the following:

$$\Sigma^{EC} \equiv \{S \in \Sigma^{comp} | \text{ for some } y \in \Re, u(S) - (y, y, \ldots, y) \in S\}.$$

Note that this solution is also well defined if the feasible set satisfies free disposal. Such problems are not compact, however, and therefore are not a subset of Σ^{comp}.

The following monotonicity axiom is complementary to the axiom used by Kalai (1977) to characterize the egalitarian bargaining solution.

> *Compromise Strong Monotonicity* (CSM) Let $S, S' \in \Sigma^{EC}$ be such that $u(S) = u(S')$ and $S \subset S'$, then $F(S) \leq F(S')$.

Theorem 4. *A solution $F : \Sigma^{EC} \to \Re^n$ satisfies SYM, TINV, WPO, and CSM if and only if $F \equiv EC$.*

Proof/

See appendix.

This result is a minor adaptation of the characterization of the *equal loss bargaining rule* in Chun (1988). Formally, the difference stems from the fact that Chun defines his rule for the class of *bargaining problems* that are convex and comprehensive. His solution shares losses between agents from the *ideal point,* which is the utopia point of the set that results when attention is restricted to allocations that dominate the disagreement point. The disagreement point is used in an essential way by Chun but the differences in technical detail are minor and we include Theorem 4 for the sake of completeness.[6]

We now turn our attention to the Raiffa solution that was proposed by Raiffa and axiomatized in Kalai and Smorodinsky (1975). For each $S \in \Sigma^{comp}$, define:

$$RC(S) \equiv t^*m(S) + (1-t^*)u(S), \text{ where } t^* = min\{t \in \Re_+ \mid tm(S) + (1-t)u(S) \in S\}.$$

To axiomatize this solution, we use a natural generalization of the restricted monotonicity axiom used by Roth (1980) in his characterization of the Kalai-Smorodinsky solution.

Compromise Restricted Monotonicity (CRM) Let $S, S' \in \Sigma^{comp}$ be such that $u(S) = u(S')$, $m(S) = m(S')$, and $S \subset S'$. Then $F(S) \leq F(S')$.

We also need to modify the Scale Covariance axiom as we cannot refer to the disagreement point. A natural adaptation of the axiom is replace the disagreement point with the nadir point.

Nadir Point Scale Covariance (NSC): For all $S \in \Sigma^{comp}$, and for all $b \in \Re^n{}_{++}$, if $S' = b * S$ then (a) for all j, such that $F(S)_j = m(S)_j$ it holds that $F(S')_j = m(S')_j$, and (b) for all j such that $F(S)_j > m(S)_j$, it holds for all i that $\frac{b_j(F(S')_i - m(S')_i)}{b_i(F(S')_j - m(S')_j i} = \frac{F(S)_i - m(S)_i}{F(S)_j - m(S)_j}$.

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[6] Also see Bossert (1992, 1993) for alternative formulations and characterizations of similar solution concepts.

The next result is a straightforward extension of Roth's characterization of the Kalai-Smorodinsky solution to the domain Σ^{comp} and we omit the proof.

Theorem 5. *A solution* $F : \Sigma^{comp} \to R^n$ *satisfies WPO, SYM, NSC, TINV and CRM if and only if* $F = RC$.

Alternatively, one can axiomatize the Raiffa compromise solution using a utopia point version of scale covariance.

> *Utopia Point Scale Covariance* (USC): For all S, and $b \in \Re^n{}_{++}$, if $S' = b * S$, then (a) for all j, such that $F(S)_j = u(S)_j$ it holds that $F(S')_j = u(S')_j$, and (b) for all j such that $F(S)_j < u(S)_j$, it holds for all i that $\frac{b_j(u(S')_i - F(S')_i)}{b_i(u(S')_j - F(S')_j)} = \frac{u(S)_i - F(S)_i}{u(S)_j - F(S)_j}$.

Theorem 6. *A solution* $F : \Sigma^{comp} \to \Re^n$ *satisfies WPO, SYM, USC, TINV, and CRM, if and only if* $F = RC$

Proof/

See appendix

There are several related papers in the literature. Chun and Thomson (1992) define and characterize the *Proportional solution.* This differs from the solution above in that the line of compromise is drawn between the given claims and disagreement points instead of the utopia and nadir points. Somewhat closer in spirit is Herrero (1994). Here the line of compromise is drawn between some given claims point and endogenous reference point called the "natural reference point." In contrast, we endogenize both ends of the line of compromise.

5. Conclusion

In this paper, we have provided an axiomatic foundation for the compromise approach to multiobjective choice problems and we have shown that this approach is a natural complement to that of traditional bargaining theory. If we take the traditional position that we should choose from among competing solution concepts based on the axioms needed for their characterizations, then we need only decide whether the axioms should apply to gains or losses. If agents make optimistic but infeasible demands, then the compromise approach in which agents begin at a utopia point and make concessions until a feasible payoff profile is attained would be appropriate. If on the other hand, agents insist on some minimum payoff as a starting point in negotiations and bargain for gains until a Pareto optimal payoff profile is attained, then the Nash approach is appropriate. We conclude by pointing out that it is possible to generalize these results using Roth-Thomson idea of reference functions. Equivalent results could be shown for a broad class of solutions in which compromises are made from any infeasible reference point which satisfies certain regularity conditions. This and other related issues are taken up in Conley, McLean, and Wilkie (1997).

Appendix

Theorem 2. *A solution $F : \Sigma_0^{con} \rightarrow \Re^n$ satisfies PO, IR, TINV, n-IIA and PGAIN if and only if there exists $w \in \Re^n_{++}$ such that $F = NB^w$.*

Proof/

Part 1: For each $w \in R^n_{++}$, the function $NB^w : \Sigma_0^{con} \rightarrow R^n$ satisfies PO, TINV, n-IIA. IR and PGAIN. The proofs for PO, TINV and n-IIA are straightforward and PGAIN follows from the fact that NB^w satisfies S.COV. Since each $w_i > 0$, it follows that

$$NB^{w_i}(\Delta) = \frac{w_i}{(\sum_{j \in N} w_j)} < 1$$

so IR is satisfied.

Part 2: Suppose that $F : \Sigma_0^{con} \rightarrow R^n$ satisfies PO, TINV, n-IIA. IR and PGAIN and define $w = F(\Delta)$. The IR axiom implies that $w \in R^n_{++}$ we will now show that $F(S) = NB^w(S)$ for all $S \in \Sigma_0^{con}$.

Step 1: Applying TINV, we will assume that $m(S) = 0$. Let $x = NB^w(S)$ and note that $x - m(S) \in \Re^n_{++}$. From the definition of $NB^w(S)$, it follows that S is supported at x by the hyperplane $H(p, a)$ where $p = w * (x - m(S))^{-1} = w * x^{-1}$ and $a = w \cdot e$.

Let $S' = p * S$. Note that $w \in S'$ and that the hyperplane $H(e, a)$ supports S' at the point w. Now define

$$T' = \{z \in \Re^n_+ | \ e \cdot z \leq a\}$$

and note that

$$P(T') = conv\{ae_1, .., ae_n\} = (ae) * \Delta.$$

Next, we show that

$$F(T') = NB^w(T').$$

Let $A = P(T') = (ae) * \Delta$. Since $\Delta \in \mathcal{L}^+$, we can apply PGAIN and conclude that

$$a[F_i(A) - m_i(A)][F_j(\Delta) - m_j(\Delta)] = a[F_j(A) - m_j(A)][F_i(\Delta) - m_i(\Delta)]$$

Hence,

$$F_i(A)w_j = F_j(A)w_i.$$

Since PO implies that $\sum_{i \in N} F_i(A) = a$, we conclude that

$$F_i(A) = a\frac{w_i}{\sum_{j \in N} w_j} = NB^w_i(A).$$

Since $A \subseteq T', m(A) = 0 = m(T')$ and $F(T') \in P(T') = A$, we can apply n-IIA and conclude that

$$F_i(T') = F_i(A) = a\frac{w_i}{\sum_{j \in N} w_j} = NB^w_i(T').$$

Step 2: Let $T = p^{-1} * T$ and define $y = F(T)$. Since $T' \in \mathcal{L}^+$, we can again apply PGAIN and conclude that

$$p_i(y_i - m_i(T))[F_j(T') - m_j(T')] = p_j(y_j - m_j(T))[F_i(T') - m_i(T')].$$

Since $m(T) = m(T') = 0$ and $F_i(T') = a\frac{w_i}{\sum_{j \in N} w_j}$, we obtain

$$p_i y_i w_j = p_j y_j w_i.$$

Furthermore, $p = w * x^{-1}$ so

$$p_j x_j y_i = p_j y_j x_i.$$

Since $y = F(T) \in P(T) \subseteq H(p, a)$ and $x \in H(p, a)$, it follows that $x = y$ and, therefore, $F(T) = NB^w(S)$.

Step 3: To complete the proof, note that $S \subseteq T$ (since $S' \subseteq T'$), $m(S) = 0 = m(T)$ and $F(T) = x \in S$. Applying n-IIA, we conclude that $F(S) = F(T) = NB^w(S)$.

■

Theorem 3. *A solution* $F : \Sigma^{con} \to \Re^n$ *satisfies PO, TINV, u-IIA, IF, CONT and PLOSS if and only if there exists* $w \in \Re^n_{++}$ *such that* $F = Y^w$.

Proof/

Part 1: For each $w \in R^n_{++}$, the function $Y^w : \Sigma^{con} \to R^n$ satisfies PO, TINV, u-IIA, CONT, IF and PLOSS. The proofs for PO, TINV and u-IIA are again straightforward and CONT is an immediate application of Berge's Theorem. Since each $w_i > 0$, it follows that

$$Y^w(\Gamma) = 1 - \frac{1}{w_i(\sum_{j \in N} \frac{1}{w_j})} < 1$$

so Y^w satisfies IF. Finally, we must show that PLOSS is satisfied. Choose $w \in R^n_{++}$. Suppose that $S \in \mathcal{L}^-, \lambda \in R^n_{++}$ and $S' = \lambda * S$. Since $S \in \mathcal{L}^-$, there exists a number a and $p \in R^n_{++}$ such that $P(S) = H(p,a) \cap [u(S) - \Re^n_+]$. Direct computation shows that the weighted projection problem

$$\min_{z \in H(p,a)} \sum_{i \in N} w_i [u_i(S) - z_i]^2$$

has a unique solution x given by

$$x = u(S) - \left[\frac{(p \cdot u(S) - a)}{\sum_{j \in N} (\frac{p_j^2}{w_j})} \right] (w^{-1}) * p.$$

Next, we show that $x \in P(S)$. To see this, note that $p \cdot u(S) - a \geq 0$. It follows that, for each i,

$$u(S) - \left[\frac{(p \cdot u(S) - a)}{p_i} \right] e_i \in H(p,a) \cap [u(S) - \Re^n_+].$$

Since

$$x = \sum_{i \in N} \left(u(S) - \left[\frac{(p \cdot u(S) - a)}{p_i} \right] e_i \right) \left(\frac{(\frac{p_i^2}{w_i})}{\sum_{j \in N} (\frac{p_j^2}{w_j})} \right)$$

and $H(p,a) \cap [u(S) - \Re^n_+]$ is convex, we conclude that $x \in H(p,a) \cap [u(S) - \Re^n_+] = P(S)$ from which it follows that $Y^w(S) = x$.

Next, note that $P(S') = H(p', a) \cap [u(S') - \Re^n_+]$ where $p' = (\lambda^{-1}) * p$. Using the argument given above, it follows that

$$Y^w(S') = u(S') - \left[\frac{(p' \cdot u(S') - a)}{\sum_{j \in N} (\frac{p_j'^2}{w_j})} \right] (w^{-1}) * p'.$$

Therefore,

$$\lambda_i [u_i(S') - Y_i^w(S')][u_j(S) - Y_j^w(S)] = \lambda_j [u_j(S') - Y_j^w(S')][u_i(S) - Y_i^w(S)]$$

and the result follows.

Part 2: Suppose that $F : \Sigma^{con} \to R^n$ satisfies PO, TINV, u-IIA. IF, CONT and PLOSS. For each i, define

$$w_i = \frac{1}{1 - F_i(\Gamma)}$$

and note that $w \in R^n_{++}$ as a consequence of IF. We will now show that $F(S) = Y^w(S)$ for all $S \in \Sigma^{con}$ and the argument will be partitioned into three cases.

Case 1: Suppose that $u(S) - Y^w(S) \in \Re^n_{++}$.

Step 1.1: Applying TINV, we will assume that $u(S) = 0$. Let $x = Y^w(S)$. From the definition of $Y^w(S)$, it follows that S is supported at x by the hyperplane $H(p, a)$ where $p = w * (u(S) - x) = -w * x$ and $p \cdot x$. Note that $a < 0$.

Let $S' = p * S$ and note that the hyperplane $H(e, a)$ supports S' at the point $p * x$. Let β be a point of equal coordinates such that $z > \beta$ for all $z \in S'$ and define

$$T' \equiv \{z \in \Re^n | \beta \leq z \leq 0 \text{ and } e \cdot z \leq a\}.$$

Geometrically, the set T' is the hypercube between β and the origin with a symmetric slice removed from the upper corner. In addition, $P(T') = conv\{ae_1, .., ae_n\}$. Next, we show that $F(T') = Y^w(T')$.

To see this, let $T'' = T' - ae$ and note that $u(T'') = -ae$. Furthermore,

$$P(T'') = conv\{ae_1, .., ae_n\} - ae = conv\{(-a)(e - e_1), .., (-a)(e - e_n)\} = (-ae) * \Gamma.$$

Let $B = P(T'') = (-ae) * \Gamma$. Since $\Gamma \in \mathcal{L}^-$, we can apply PLOSS and conclude that

$$[u_i(B) - F_i(B)][u_j(\Gamma) - F_j(\Gamma)] = [u_j(B) - F_j(B)][u_i(\Gamma) - F_i(\Gamma)].$$

Hence,

$$[a + F_i(B)](1 - F_j(\Gamma)) = [a + F_j(B)](1 - F_i(\Gamma)).$$

Since PO implies that $\sum_{i \in N} F_i(B) = -(n-1)a$, we conclude that

$$F_i(B) = -a\left[1 - \frac{1 - F_i(\Gamma)}{\sum_{j \in N}[1 - F_j(\Gamma)]}\right] = -a\left[1 - \frac{1}{w_i(\sum_{j \in N} \frac{1}{w_j})}\right].$$

Since $B \subseteq T'', u(B) = ae = u(T'')$ and $F(T'') \in P(T'') = B$, we can apply u-IIA and conclude that

$$F_i(T'') = F_i(B) = -a\left[1 - \frac{1}{w_i(\sum_{j \in N} \frac{1}{w_j})}\right]$$

Applying TINV, we conclude that

$$F_i(T') = F_i(T'') + a = a\left[\frac{1}{w_i(\sum_{j \in N} \frac{1}{w_j})}\right] = Y^w_i(T').$$

for each i.

Step 1.2: Let $T = p^{-1} * T'$ and define $y = F(T)$. Since $T' \in \mathcal{L}^-$, we can again apply PLOSS and conclude that

$$\frac{1}{p_i}(u_i(T) - y_i)[u_j(T') - F_j(T')] = \frac{1}{p_j}(u_j(T) - y_j)[u_i(T') - F_i(T')].$$

Since $u(T) = u(T') = 0$ and $F_i(T') = a\left[\frac{1}{w_i(\sum_{j\in N}\frac{1}{w_j})}\right]$, we obtain

$$p_j y_i w_i = p_i y_j w_j$$

Furthermore, $p = -w * x$ so

$$y_i p_j x_j = x_i p_j y_j.$$

Since $y = F(T) \in P(T) \subseteq H(p, a)$ and $x \in H(p, a)$, it follows that $x = y$ and, therefore, $F(T) = Y^w(S)$.

Step 1.3: To complete the proof, note that $S \subseteq T$ (since $S' \subseteq T'$), $u(S) = 0 = u(T)$ and $F(T) = x \in S$. Applying u-IIA, we conclude that $F(S) = F(T) = Y^w(S)$.

Case 2: Suppose that $u(S) - Y^w(S) \notin \Re^n_{++}$ and $u(S) \neq Y^w(S)$.
Step 2.1: Let $x = Y^w(S)$ and $p = w * [u(S) - Y^w(S)] = w * [u(S) - x]$. Next define

$$J = \{i \in N | u_i(S) - Y_i^w(S) = 0\}$$

and note that $J \neq N$. Finally, choose $\varepsilon > 0$ and define

$$p_\varepsilon = p + \varepsilon \sum_{i\in J} w_i e_i,$$

$$u_\varepsilon = u(S) - \left(\frac{p_\varepsilon \cdot (w^{-1} * p)}{(w^{-1} * p_\varepsilon) \cdot p_\varepsilon}\right)\left(w^{-1} * p_\varepsilon\right)$$

and

$$S_\varepsilon = conv(S \cup \{u_\varepsilon\}).$$

Note that $u(S_\varepsilon) = u(S)$ since $u_\varepsilon << u(S)$. We will prove that $Y^w(S_\varepsilon) = u_\varepsilon$. Obviously, $u_\varepsilon \in S_\varepsilon$ so it is enough to show that

$$(w * [u(S_\varepsilon) - u_\varepsilon]) \cdot z \leq (w * [u(S_\varepsilon) - u_\varepsilon]) \cdot u_\varepsilon \text{ whenever } z \in S_\varepsilon.$$

Step 2.2: Claim:

$$p_\varepsilon \cdot z \leq p_\varepsilon \cdot u_\varepsilon \text{ whenever } z \in S.$$

Proof: First, note that

$$p_\varepsilon \cdot u_\varepsilon = p_\varepsilon \cdot u(S) - p_\varepsilon \cdot (w^{-1} * p) = p_\varepsilon \cdot [x + (w^{-1} * p)] - p_\varepsilon \cdot (w^{-1} * p) = p_\varepsilon \cdot x.$$

Now choose $z \in S$. Since $x = Y^w(S)$, it follows that $p \cdot z \leq p \cdot x$ and, therefore,

$$\sum_{i \notin J} p_i z_i \leq \sum_{i \notin J} p_i x_i.$$

Since $z_i \leq u_i(S) = Y_i^w(S) = x_i$ if $i \in J$, we conclude that

$$\sum_{i \notin J} p_i z_i + \sum_{i \in J} w_i \varepsilon z_i \leq \sum_{i \notin J} p_i x_i + \sum_{i \in J} w_i \varepsilon x_i.$$

Therefore,

$$p_\varepsilon \cdot z \leq p_\varepsilon \cdot x = \ p_\varepsilon \cdot u_\varepsilon \text{ whenever } z \in S$$

and the proof of the claim is complete.

Step 2.3: Claim:

$$(w * [u(S_\varepsilon) - u_\varepsilon]) \cdot z \leq (w * [u(S_\varepsilon) - u_\varepsilon]) \cdot u_\varepsilon \text{ whenever } z \in S_\varepsilon.$$

Proof: If $z \in S_\varepsilon$, then there exist nonnegative numbers $\lambda_1, .., \lambda_m$ summing to one and points $y_1, .., y_m \in S \cup \{u_\varepsilon\}$ such that

$$z = \sum_{k \in N} \lambda_k y_k.$$

Applying Step 2.2, we deduce that

$$p_\varepsilon \cdot z = \sum_{k \in N} \lambda_k \left(p_\varepsilon \cdot y_k\right) \leq \sum_{k \in N} \lambda_k \left(p_\varepsilon \cdot u_\varepsilon\right) = p_\varepsilon \cdot u_\varepsilon.$$

Since $u(S_\varepsilon) = u(S)$, it follows that

$$\begin{aligned} &(w * [u(S_\varepsilon) - u_\varepsilon]) \cdot z \\ &= \left(\frac{p_\varepsilon \cdot (w^{-1} * p)}{(w^{-1} * p_\varepsilon) \cdot p_\varepsilon} \right) (p_\varepsilon \cdot z) \\ &\leq \left(\frac{p_\varepsilon \cdot (w^{-1} * p)}{(w^{-1} * p_\varepsilon) \cdot p_\varepsilon} \right) (p_\varepsilon \cdot u_\varepsilon) \\ &= (w * [u(S_\varepsilon) - u_\varepsilon]) \cdot u_\varepsilon \end{aligned}$$

and the proof of the claim is complete.

Step 2.4: As a consequence of Steps 2.1. 2.2 and 2.3, we conclude that

$$Y^w(S_\varepsilon) = u_\varepsilon = u(S) - \left(\frac{p_\varepsilon \cdot (w^{-1} * p)}{(w^{-1} * p_\varepsilon) \cdot p_\varepsilon}\right)(w^{-1} * p_\varepsilon).$$

Since $Y^w(S_\varepsilon) = u_\varepsilon << u(S) = u(S_\varepsilon)$, we can apply the conclusion of Case 1 of the proof to the choice problem S_ε and conclude that $F(S_\varepsilon) = Y^w(S_\varepsilon)$. As $\varepsilon \to 0$, S_ε converges to S in the Hausdorff metric topology so $Y^w(S_\varepsilon) \to Y^w(S)$. Applying CONT, it follows that $F(S) = Y^w(S)$ and the proof of Theorem 3 is complete.

Case 3: Suppose that $u(S) = Y^w(S)$. In this case, $u(S) \in S$ so PO implies that $F(S) = u(S)$. Therefore, $F(S) = Y^w(S)$ and the proof is complete.

■

Corollary 2. *A solution $F : \Sigma^{con} \to \Re^n$ satisfies PO, TINV, u-IIA, SYM, CONT and PLOSS if and only if $F = Y$.*

Proof/

It is easy to show that Y satisfies PO, TINV, u-IIA, SYM, CONT and Part 1 of the proof of Theorem 3 establishes that Y^w satisfies PLOSS for all $w \in R^n_{++}$, hence for $w = e$. Now suppose that $F : \Sigma^{con} \to R^n$ satisfies PO, TINV, u-IIA, SYM, CONT and PLOSS. Applying PO and SYM, it follows that

$$F(\Gamma) = \left(\frac{n-1}{n}\right)e.$$

Hence, $F_i(\Gamma) > 0$ for each i and we conclude that F satisfies IF. Applying Theorem 3, it follows that there exists $w \in \Re^n_{++}$ such that $F = Y^w$. Hence,

$$\frac{n-1}{n} = F_i(\Gamma) = Y^{w_i}(\Gamma) = 1 - \frac{1}{w_i(\sum_{j \in N} \frac{1}{w_j})}$$

and we conclude that

$$w_i = w_j$$

for all i and j. Therefore, $F = Y^w = Y^e = Y$.

■

Theorem 4. *A solution $F : \Sigma^{EC} \to \Re^n$ satisfies SYM, TINV, WPO, and CSM if and only if $F \equiv EC$.*

Proof/

The proof that EC satisfies the four axioms is elementary and is omitted. Conversely let F be a solution satisfying the four axioms. Given $S \in \Sigma^{EC}$, we can assume by TINV that the problem has been normalized such that $u(S) = 0$. Thus,

$EC(b * S) = \alpha e \equiv x$ for some $\alpha \leq 0$. By compactness there exists $\beta < 0$ such that $\beta e \ll m(S)$. Define the problem T as follows,

$$T \equiv comp(\{0\}; \beta e) \backslash \{x + \Re^n_{++}\}.$$

Note that $T \in \Sigma^{EC}, T$ is a symmetric problem and x is the only weakly Pareto optimal point of equal coordinates. Consequently, WPO and SYM imply that $F(T) = x$. By construction, $S \subseteq T$ and $u(S) = u(T) = 0$ so it follows from CSM that $F(S) \leq F(T) = x$.

If $x \in P(S)$, then WPO implies that $F(S) = F(T) = x$ and we are finished.

Suppose, on the other hand, that $x \in WP(S)$ but $x \notin P(S)$.

Suppose in this case that $F(S) = x' \in WP(S)$ and $x' < x$. Then consider the following two classes of problems:

$$S^\epsilon \equiv \{z \in S \mid z \not\gg (\alpha - \epsilon, \ldots, \alpha - \epsilon)\}$$

and

$$T^\epsilon \equiv \{z \in T \mid z \not\gg (\alpha - \epsilon, \ldots, \alpha - \epsilon)\}.$$

Note that for all $\epsilon > 0$ it holds that $S^\epsilon \subset T^\epsilon$ and for ϵ small enough, both S^ϵ and T^ϵ have non-empty interiors. Thus, as T^ϵ is symmetric and comprehensive, and $(\alpha - \epsilon, \ldots, \alpha - \epsilon)$ is the only symmetric, Pareto optimal payoff vector, it follows that $F(T^\epsilon) = (\alpha - \epsilon, \ldots, \alpha - \epsilon)$. As in the case above, $u(T^\epsilon) = u(S^\epsilon)$, and so by CSM, $F(S^\epsilon) \leq F(T^\epsilon)$. But by construction, $(\alpha - \epsilon, \ldots, \alpha - \epsilon)$ is the only element of $WP(S^\epsilon)$ that is less than or equal to $F(T^\epsilon)$. We conclude that $F(S^\epsilon) = (\alpha - \epsilon, \ldots, \alpha - \epsilon)$. However, comprehensiveness implies that $u(S) = u(S^\epsilon)$ for sufficiently small ϵ. Thus, by CSM, $F(S) \geq (\alpha - \epsilon, \ldots, \alpha - \epsilon)$. But the only point in $\Re^n$ which is less than or equal to $(\alpha, \ldots, \alpha)$ and greater than or equal to $(\alpha - \epsilon, \ldots, \alpha - \epsilon)$ for all ϵ is $(\alpha, \ldots, \alpha)$. We conclude that $F(S) = x$.

■

Theorem 6. *A solution $F : \Sigma^{comp} \rightarrow \Re^n$ satisfies WPO, SYM, USC, TINV and CRM if and only if $F = RC$*

Proof/

The proof that *RC* satisfies the axioms is elementary and is omitted. Conversely let F be a solution satisfying the five axioms. Given any $S \in \Sigma^{comp}$, without loss of generality we may assume by TINV the problem has been normalized such that $m(S) = 0$. Now let $b \in \Re^n_{++}$ be a linear transformation such that $b * u(S) = e \equiv y$. Then $RC(b * S) = \alpha e \equiv x$ for some $\alpha > 0$.

There are two cases to consider. First, Suppose that $\alpha = 1$. Then by SYM and WPO it holds that $F(b * S) = u(b * S) = RC(b * S)$. As the utopia point is scale invariant, $b * u(S) = u(b * S)$. Thus by USC, we have that $F(S) = u(S) = RC(S)$ as required.

Suppose instead that $\alpha < 1$. Let T be defined as:

$$T \equiv comp(\{y\};0)\backslash\{x + \Re^n_{++}\}.$$

Since T is symmetric and x is the only symmetric element of $WP(T)$, it follows from WPO and SYM that $F(T) = x$. Since $b * S \subseteq T, m(b * S) = m(T) = 0$ and $u(b * S) = u(T) = y$, it follows from CRM that $F(b * S) \leq F(T) = x$. Now let T' be defined as

$$T' \equiv comp(\{e_1, .., e_n, x\}; 0).$$

Since T' is symmetric and x is the only symmetric element of $WP(T')$, it follows from WPO and SYM that $F(T') = x$. Since $T' \subseteq S, m(b * S) = m(T') = 0$ and $u(b * S) = u(T') = y$, it follows from CRM that $F(b * S) \geq F(T') = x$. Therefore, $F(b * S) = x = RC(b * S)$.

Recall that from the definition of x, for each objective i we have that $F(b * S)_i < u(b * S)_i$. Thus, by USC, $F(S)$ must satisfy the following equation for all $i \in N$:

$$\frac{b_j(u(b * S)_i - F(b * S)_i)}{b_i(u(b * S)_j - F(b * S)_j)} = \frac{u(S)_i - F(S)_i}{u(S)_j - F(S)_j}.$$

We can rewrite this as

$$\frac{1 - \alpha}{1 - \alpha} = \frac{b_i(u(S)_i - F(S)_i)}{b_j(u(S)_j - F(S)_j)}.$$

By construction for all $i \in (1, \ldots, n)$, $\frac{1}{b_i} = u_i(S)$, and so

$$b_i F(S)_i = b_j F(S)_j.$$

We conclude that $F(S) = RC(S)$ since this is the only Pareto optimal and feasible point which satisfies this equation.

■

References

Arrow, K. and H. Raynaud (1986): *Social Choice and Multicriterion Decision Making.* MIT Press, Cambridge, Ma.

Bossert, W. (1992): "Monotonic Solutions for Bargaining Problems with Claims," *Economics Letters,* Vol. 39, pp 395-9.

Bossert, W. (1993): "Alternative Solutions to Bargaining Problems with Claims," *Mathematical Social Sciences,* Vol. 25, pp 205-20.

Chun, Y (1988): "The Equal-Loss Principle for Bargaining Problems.," *Economics Letters,* Vol. 26, pp. 103-106.

Chun, Y. and W. Thomson (1992): "Bargaining Problems with Claims," *Mathematical Social Sciences,* Vol. 24, pp. 19-33.

Conley, J. R. McLean, S. Wilkie (1997): "Reference Functions and Possibility Theorems for Cardinal Social Choice Problems.," *Social Choice and Welfare,* Vol. 14, pp. 65-78.

Conley, J. and S. Wilkie (1991): "The Bargaining Problem Without Convexity: Extending the Egalitarian and Kalai-Smorodinsky Solutions," *Economics Letters,* Vol. 36 pp. 365-369.

Freimer, M. and P.L. Yu (1976): "Some New Results on Compromise Solutions for Group Decision Problems," *Management Science,* Vol.22, pp 688-946.

Herrero, C (1994): "Endogenous Reference Points and the Adjusted Proportional Solution for Bargaining Problems with Claims," *Social Choice and Welfare,* Vol.15, pp. 113-119.

Kalai, E (1977): "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," *Econometrica,* Vol. 45, No.7, pp 1623-37.

Kalai, E. and D.Samet, (1987): "On Weighted Shapley Values," *International Journal of Game Theory,* Vol. 16 pp. 205-222..

Kalai, E. and M. Smorodinsky (1975): "Other Solutions to Nash's Bargaining Problem," *Econometrica,* Vol. 43, No.3, pp 513-8.

Lee, S.M (1972): *Goal Programming for Decision Analysis.* Auerbach.

Nash, J (1950): "The Bargaining Problem," *Econometrica,* Vol. 18, pp. 155-62.

Rubinstein, A and L. Zhou (1999): "Choice Problems with a "Reference Point"," *Math Social Sciences,* Vol. 37, pp. 205-209.

Roth, A (1977): "Individual Rationality and Nash Solution to the Bargaining Problem," *Mathematics of Operations Research,* Vol. 2, pp. 64-65..

Roth, A (1980): "An Impossibility Result Concerning n-Person Bargaining Games," *International Journal of Game Theory,* Vol. 8, pp. 129-132.

Thomson, W (1981a): "Nash's Bargaining Problem and Utilitarian Choice Rules," *Econometrica,* Vol. 49, No.2, pp. 535-38.

Thomson, W (1981b): "A Class of Solutions to Bargaining Problems," *Journal of Economic Theory,* Vol. 25, pp 431-441.

Voorneveld, M., A. van den Nouweland, R. McLean (2008): "An axiomatization of the Euclidean compromise solution," Manuscript.

White, D. J (1990): "A Bibliography on the Applications of Mathematical Programming Multiple-objective Methods," *Journal of the Operations Research society,* Vol. 41, No 8, pp 669-691.

Yu, P.L (1973): "A Class of Solutions For Group Decision Problems," *Management Science,* Vol. 19, No.8, pp. 936-946.

Yu, P.L (1985): *Multiple-Criteria Decision Making: Concepts, Techniques, and Extensions.* Plenum Press.