# HOW TO CONSTRUCT $\mathcal{H}_I$ ?

CLASSICAL LAGRANGIAN (USE OF GENERALIZED COORDINATES AND THEIR CONJUGATE MOMENTA)

CLASSICAL HAMILTONIAN

SUM OF UNCOUPLED HAMILTONIANS OF THE ATOM AND THE FIELDS

WITH THE REPLACEMENT OF ELECTRON MOMENTUM $\vec{p}_j \rightarrow \vec{p}_j + e\bar{A}(\bar{r}_j)$

VECTOR POTENTIAL

IN THE COULOMB GAUGE ($\bar{\nabla}\cdot\bar{A}=0$, CONVENIENT FOR THE QUANTIZATION OF THE FIELD)

COMPLETE HAMILTONIAN (THE SO-CALLED MINIMAL COUPLING)

$$\mathcal{H}' = \frac{1}{2m}\sum_j (\bar{p}_j + eA(\bar{r}_j))^2 + \frac{1}{2}\int \sigma(\bar{r})\phi(\bar{r})d\bar{r} + \frac{1}{2}\int(\varepsilon_0 E_T^2 + \mu_0^{-1}B^2)d\bar{r}$$

KINETIC ENERGIES OF ELECTRONS AND ENERGY OF INTERACTION WITH THE RADIATION FIELD

ELECTROSTATIC ENERGY OF ELECTRONS AND NUCLEI

$$\sigma(\bar{r}) = -\sum_j e\delta(\bar{r}-\bar{r}_j) + \mathcal{Z}e\delta(\bar{r})$$

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\left\{-\sum_j \frac{e}{|\bar{r}-r_j|} + \frac{\mathcal{Z}e}{r}\right\}$$

COULOMB INTERACTION BETWEEN VARIOUS PARTICLES

$(\bar{E}_L = -\bar{\nabla}\phi)$

ENERGY OF THE RADIATION FIELD

$\bar{H} \sim \bar{\nabla}\times\bar{A}$

$\bar{E}_T = -\dot{\bar{A}}$

WHERE IS $\mathcal{H}_I$ ?

$$\mathcal{H}_I' = \frac{e}{m}\sum_j \bar{A}(\bar{r}_j)\bar{p}_j + \frac{e^2}{2m}\sum_j \bar{A}(\bar{r}_j)^2$$

NOT CONVENIENT FOR CALCULATIONS

$\bar{A}(\bar{r}_j)$ DEPENDS ON THE CHOICE OF GAUGE

POSITIONS OF ALL $\bar{e}$ !

UNITARY TRANSFORMATION

$$\mathcal{H} = U^{-1}\mathcal{H}'U$$

$$\mathcal{H} = \mathcal{H}_E + \mathcal{H}_R + \mathcal{H}_I$$

(FIRST NON-VANISHING TERM OF H(0) AND TWO NON-VANISHING TERMS IN THE EXPANSION OF $E_T(0)$)

$$\mathcal{H}_E = \sum_j \frac{p_j^2}{2m} + \frac{1}{2}\int \sigma(\bar{r})\,\phi(\bar{r})\,d\bar{r}$$ ATOM

$$\mathcal{H}_R = \frac{1}{2}\int \left(\varepsilon_0 \bar{E}_T^2 + \mu_0^{-1}\bar{B}^2\right) d\bar{r}$$ RADIATION FIELD

$$\mathcal{H}_I = \mathcal{H}_{ED} + \mathcal{H}_{EQ} + \mathcal{H}_{MD} + \mathcal{H}_{NL}$$

ELECTRIC-DIPOLE INTERACTION

ELECTRIC QUADRUPOLE

$$\frac{1}{2}e\sum_j (\bar{r}_j\cdot\bar{\nabla})(\bar{r}_j\cdot\bar{E}_T(0))$$

MAGNETIC DIPOLE

$$\sim \bar{M}\cdot\bar{B}(0)$$

$$\bar{M} = \sum_j \bar{r}_j \times \bar{p}_j$$

ANGULAR MOM.

NEGLECTED

~~NON-LINEAR TERM~~

$$\sim \sum_j (\bar{r}_j \times \bar{B}(0))^2$$

(RELATIVELY) SMALLER THAN $\mathcal{H}_{ED}$ BY THE ORDER OF THE FINE STRUCTURE CONSTANT

ELECTRIC DIPOLE INTERACTION

$$\mathcal{H}_{ED} = e\sum_j \bar{r}_j\cdot\bar{E}_T(0) = e\bar{D}\cdot E_T(0)$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137}$$

$\Downarrow$

THE ELECTRIC DIPOLE APPROXIMATION

$$\mathcal{H}_I = \mathcal{H}_{ED}$$

## SECOND QUANTIZATION OF THE ATOMIC HAMILTONIAN

$$D = \sum_{ij} |i\rangle\langle i|D|j\rangle\langle j| = \sum_{ij} D_{ij}\,\underbrace{|i\rangle\langle j|}_{b_i^+ b_j}$$

$$D = \sum_{ij} D_{ij}\, b_i^+ b_j \quad , \quad D_{ij} = \langle i|D|j\rangle$$

## QUANTIZATION OF THE FIELD

# SECOND QUANTIZATION

INTRODUCES NO NEW PHYSICS!

IT IS JUST NEW AND VERY ELEGANT WAY OF TREATING MANY-PARTICLE SYSTEMS

AXIOM OF QUANTUM MECHANICS: ANTISYMMETRY PRINCIPLE

SATISFIED BY SLATER DETERMINANTS AND THEIR LINEAR COMBINATIONS

CAN WE SATISFY THE ANTISYMMETRY PRINCIPLE WITHOUT USING SLATER DETERMINANTS?

FORMALISM IN WHICH THE ANTISYMMETRY PROPERTY OF THE WAVE FUNCTION HAS BEEN TRANSFERRED ONTO THE ALGEBRAIC PROPERTIES OF CERTAIN OPERATORS

## WHY SECOND QUANTIZATION?

$$\mathcal{H}|i\rangle = \hbar\omega_i |i\rangle \qquad \mathcal{H} = \mathcal{H}^+$$

„FIRST QUANTIZATION"

$$\sum_i |i\rangle\langle i| = \mathbb{1}$$

$$\mathcal{H} = \mathbb{1}\cdot\mathcal{H}\cdot\mathbb{1} = \sum_i \sum_j |i\rangle\langle i|\mathcal{H}|j\rangle\langle j| \qquad , \langle i|j\rangle = \delta_{ij}$$

$$= \sum_i \hbar\omega_i |i\rangle\langle i|$$

$|i\rangle\langle j|$ ?

$$|i\rangle\langle j|l\rangle = |i\rangle\delta_{jl}$$

STATE $i$ IS CREATED

STATE $j$ IS ANNIHILATED

$$b_i^+ b_j |l\rangle = |i\rangle\delta_{jl}$$

$$\mathcal{H} = \sum_i \hbar\omega_i b_i^+ b_i$$

SECOND QUANTIZATION

$\left.\begin{array}{l}\text{CREATION}\\ \text{ANNIHILATION}\end{array}\right\}$ OF STATE!

$b_i^+ \; : \; \chi_i \;, \; i \equiv n l s m_l m_s$

SPIN-ORBITAL

CREATION OF ONE-PARTICLE STATE

ARBITRARY SLATER DETERMINANT: $|\chi_k, \dots \chi_l\rangle$

$$b_i^+ |\chi_k, \dots \chi_l\rangle = |\chi_i, \chi_k, \dots \chi_l\rangle$$

THE ORDER IN WHICH TWO CREATION OPERATORS ACT IS IMPORTANT!

$$b_i^+ b_j^+ |\chi_k, \dots \chi_l\rangle = b_i^+ |\chi_j, \chi_k, \dots \chi_l\rangle = |\chi_i, \chi_j, \chi_k, \dots \chi_l\rangle$$

$+$

$$b_j^+ b_i^+ |\chi_k, \dots \chi_l\rangle = |\chi_j, \chi_i, \chi_k, \dots \chi_l\rangle = -|\chi_i, \chi_j, \chi_k, \dots \chi_l\rangle$$

$$(b_i^+ b_j^+ + b_j^+ b_i^+) |\chi_k, \dots \chi_l\rangle = 0$$

OPERATOR RELATION:

$$b_i^+ b_j^+ + b_j^+ b_i^+ = 0 = \{b_i^+, b_j^+\}$$

ANTICOMMUTATOR

$$b_i^+ b_j^+ = - b_j^+ b_i^+$$

IF $i = j$ $\quad b_i^+ b_i^+ = - b_i^+ b_i^+ = 0$ PAULI EXCLUSION PRINCIPLE!

EXAMPLE:

$$b_1^+ b_1^+ |\chi_2 \chi_3\rangle = b_1^+ |\chi_1 \chi_2 \chi_3\rangle = |\underbrace{\chi_1 \chi_1}_{} \chi_2 \chi_3\rangle = 0$$

IN GENERAL

$$b_i^+ |\chi_k, \dots \chi_l\rangle = 0 \quad \text{IF} \quad i \in \{k, \dots l\}$$

$b_i$ ANNIHILATION, ADJOINT OF $b_i^+$ : $(b_i^+)^+ = b_i$

$$b_i |\chi_i \chi_k, \dots \chi_l\rangle = |\chi_k, \dots \chi_l\rangle$$

ACTS ONLY IF SPIN ORBITAL IS IMMEDIATELY TO THE LEFT

OTHERWISE:

$$\{b_i, b_j^+\} = \delta_{ij}$$

ANTICOMMUTATOR

$b_i b_j^+ = -b_j^+ b_i \quad i \neq j$

$b_i b_i^+ = 1 - b_i^+ b_i$ (FOR THE SAME SPIN ORBITAL)

**ALL THE PROPERTIES OF SLATER DETERMINANTS ARE REPRESENTED BY THE ANTICOMMUTATION RELATIONS OF CREATION AND ANNIHILATION OP.**

$$\boxed{\begin{aligned} \{b_i^+, b_j^+\} &= 0 \\ \{b_i, b_j\} &= 0 \\ \{b_i, b_j^+\} &= \delta_{ij} \end{aligned}}$$

**VACUUM STATE** = STATE OF A SYSTEM THAT CONTAINS NO PARTICLES

$| \rangle$

IT IS NORMALIZED $\quad \langle \; | \; \rangle = 1$

PROPERTIES:

$$b_i | \rangle = 0$$
$$\langle | b_i^+ = 0$$

APPLYING A SEQUENCE OF $b^+$ $\Rightarrow$ ANY STATE OF THE SYSTEM

$$|\chi_i\rangle = b_i^+ | \rangle$$

$$b_i^+ b_k^+ \dots b_l^+ | \rangle = |\chi_i \chi_k, \dots \chi_l\rangle$$

SECOND-QUANTIZED REPRESENTATION OF A SLATER DETERMINANT

FREE CLASICAL FIELD: $J_T = 0$

TRANSVERSE COMPONENT OF CURRENT DENSITY,
FOR INTERACTIONS WITH ATOMS $J_T$ IS DUE TO THE ATOMIC ELECTRONS

$$-\nabla^2 \bar{A} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0 \quad (*)$$

IN THIS REGION OF SPACE THE FIELD IS FREE

QUANTZATION OF THE ELECTROMAGNETIC FIELD MEANS: $\vec{A} \xrightarrow{\text{REPLACE}} \hat{\vec{A}}$

CUBIC CAVITY: BUT INSTEAD OF STANDING-WAVE SOLUTIONS, RUNNINGWAVES WITH PERIODIC BOUNDARY CONDITIONS

FOURIER SERIES FOR $\bar{A}$ IN THE CAVITY

$$\bar{A} = \sum_{\bar{k}} \left\{ \bar{A}_{\bar{k}}(t)\exp(i\bar{k}\bar{r}) + \bar{A}^*_{\bar{k}}(t)\exp(-i\bar{k}\bar{r}) \right\}$$

WAVE VECTOR $\vec{k}$: $k_x = \frac{2\pi n_x}{L}$, $k_y = \frac{2\pi n_y}{L}$, $k_z = \frac{2\pi n_z}{L}$

$$n_x, n_y, n_z = 0, \pm 1, \pm 2, \ldots$$

THE COULOMB GAUGE CONDITION:

$$\bar{k}\cdot\bar{A}_{\bar{k}}(t) = \bar{k}\cdot\bar{A}^*_{\bar{k}}(t) = 0$$

FIELD EQUATION (*)

$$k^2 \bar{A}_{\bar{k}}(t) + \frac{1}{c^2}\frac{\partial^2 \bar{A}_{\bar{k}}(t)}{\partial t^2} = 0$$

THE SAME FOR $\bar{A}^*_k$, SATISFIED INDEPENDENTLY

FOURIER COEFFICIENTS SATISFY THE SIMPLE HARMONIC EQUATION

$$(**) \quad \boxed{\frac{\partial^2 \bar{A}_{\bar{k}}(t)}{\partial t^2} + \omega_k^2 \bar{A}_k = 0}, \qquad \omega_k = ck$$

$$\Downarrow$$

**CONVERSION TO A QUANTUM MECHANICAL HARMONIC-OSCILLATOR EQUATION**

B14.4



To this end, let us evaluate the classical energy of the cavity normal mode specified by wavevector **k**. The solution of eqn (4.48) can be taken as

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$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{A}_{\mathbf{k}} \exp(-i\omega_{\mathbf{k}} t), \tag{4.50}$$

and the complete vector potential (eqn (4.43)) becomes

$$\mathbf{A} = \sum_{\mathbf{k}} \{\mathbf{A}_{\mathbf{k}} \exp(-i\omega_{\mathbf{k}} t + i\mathbf{k}.\mathbf{r}) + \mathbf{A}_{\mathbf{k}}^* \exp(i\omega_{\mathbf{k}} t - i\mathbf{k}.\mathbf{r})\}. \tag{4.51}$$

The cycle-averaged energy content of a single mode **k** is

$$\bar{\mathscr{E}}_{\mathbf{k}} = \frac{1}{2} \int_{\text{cavity}} (\epsilon_0 \overline{\mathbf{E}_{\mathbf{k}}^2} + \mu_0^{-1} \overline{\mathbf{B}_{\mathbf{k}}^2}) \, dV, \tag{4.52}$$

where the bars denote a cycle average, and $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$ are the electric and magnetic fields associated with the mode. From eqns (4.5), (4.8), and (4.51)

$$\mathbf{E}_{\mathbf{k}} = i\omega_{\mathbf{k}} \{\mathbf{A}_{\mathbf{k}} \exp(-i\omega_{\mathbf{k}} t + i\mathbf{k}.\mathbf{r}) - \mathbf{A}_{\mathbf{k}}^* \exp(i\omega_{\mathbf{k}} t - i\mathbf{k}.\mathbf{r})\}, \tag{4.53}$$

$$\mathbf{B}_{\mathbf{k}} = i\mathbf{k} \times \{\mathbf{A}_{\mathbf{k}} \exp(-i\omega_{\mathbf{k}} t + i\mathbf{k}.\mathbf{r}) - \mathbf{A}_{\mathbf{k}}^* \exp(i\omega_{\mathbf{k}} t - i\mathbf{k}.\mathbf{r})\}. \tag{4.54}$$

It is evident from eqns (4.46) and (4.49) that the magnitudes of $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$ are related by eqn (1.18) as expected for a free electromagnetic wave. Substitution into eqn (4.52) and evaluation of the time average gives

$$\bar{\mathscr{E}}_{\mathbf{k}} = 2\epsilon_0 V \omega_{\mathbf{k}}^2 \mathbf{A}_{\mathbf{k}}.\mathbf{A}_{\mathbf{k}}^*, \tag{4.55}$$

where $V = L.^3$

The mode variables $\mathbf{A}_{\mathbf{k}}$ and $\mathbf{A}_{\mathbf{k}}^*$ can be replaced by a generalized mode 'position' coordinate $Q_{\mathbf{k}}$ and a mode 'momentum' $P_{\mathbf{k}}$ in accordance with the transformations

$$\mathbf{A}_{\mathbf{k}} = (4\epsilon_0 V \omega_{\mathbf{k}}^2)^{-\frac{1}{2}} (\omega_{\mathbf{k}} Q_{\mathbf{k}} + iP_{\mathbf{k}}) \boldsymbol{\varepsilon}_{\mathbf{k}}. \tag{4.56}$$

and

$$\mathbf{A}_{\mathbf{k}}^* = (4\epsilon_0 V \omega_{\mathbf{k}}^2)^{-\frac{1}{2}} (\omega_{\mathbf{k}} Q_{\mathbf{k}} - iP_{\mathbf{k}}) \boldsymbol{\varepsilon}_{\mathbf{k}}. \tag{4.57}$$

The coordinates $Q_{\mathbf{k}}$ and $P_{\mathbf{k}}$ are scalar quantities, the directional properties of $\mathbf{A}_{\mathbf{k}}$ and $\mathbf{A}_{\mathbf{k}}^*$ having been separated by the introduction of a unit polarization vector $\boldsymbol{\varepsilon}_{\mathbf{k}}$ for the mode.

The single-mode energy (eqn (4.55)) is transformed by eqns (4.56) and (4.57) into

$$\bar{\mathscr{E}}_{\mathbf{k}} = \tfrac{1}{2}(P_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^2). \tag{4.58}$$

This is precisely the usual form of the energy of a classical harmonic oscillator. The problem of the vector potential associated with a cavity mode has thus been made formally equivalent to a classical harmonic-oscillator problem. The complete classical Hamiltonian for the cavity is formed by taking a sum over **k**, and the two independent directions of $\boldsymbol{\varepsilon}_{\mathbf{k}}$, of the single-mode expression (4.58).

B14.5



## 4.4. The quantum-mechanical harmonic oscillator

The electromagnetic field is now quantized by conversion of $Q_k$ and $P_k$ into quantum-mechanical position and momentum operators $\hat{q}_k$ and $\hat{p}_k$. As a preliminary to this conversion, it is convenient to develop the theory of the quantum-mechanical harmonic oscillator in the form most suitable for the field quantization.

The quantum-mechanical Hamiltonian for a one-dimensional harmonic oscillator of unit mass is

$$\hat{\mathscr{H}} = \tfrac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2), \tag{4.59}$$

where $\hat{p}$ and $\hat{q}$ obey the usual commutation relation

$$[\hat{q}, \hat{p}] = i\hbar. \tag{4.60}$$

Define a pair of operators $\hat{a}$ and $\hat{a}^\dagger$ to replace $\hat{q}$ and $\hat{p}$,

$$\hat{a} = (2\hbar\omega)^{-\frac{1}{2}}(\omega\hat{q} + i\hat{p}) \tag{4.61}$$

and

$$\hat{a}^\dagger = (2\hbar\omega)^{-\frac{1}{2}}(\omega\hat{q} - i\hat{p}) \tag{4.62}$$

or, conversely

$$\hat{q} = (\hbar/2\omega)^{\frac{1}{2}}(\hat{a} + \hat{a}^\dagger) \tag{4.63}$$

$$\hat{p} = -i(\hbar\omega/2)^{\frac{1}{2}}(\hat{a} - \hat{a}^\dagger). \tag{4.64}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are called, respectively, the destruction and creation operators for the harmonic oscillator. As will become clear, they are extremely useful, on account of their simple properties. They do not, however, represent observables of the harmonic oscillator, as is shown below.

From eqns (4.61) and (4.62) we have

$$\begin{aligned} \hat{a}^\dagger\hat{a} &= (2\hbar\omega)^{-1}(\hat{p}^2 + \omega^2\hat{q}^2 + i\omega\hat{q}\hat{p} - i\omega\hat{p}\hat{q}) \\ &= (\hbar\omega)^{-1}(\hat{\mathscr{H}} - \tfrac{1}{2}\hbar\omega), \end{aligned} \tag{4.65}$$

where eqns (4.59) and (4.60) have been used. Similarly

$$\hat{a}\hat{a}^\dagger = (\hbar\omega)^{-1}(\hat{\mathscr{H}} + \tfrac{1}{2}\hbar\omega). \tag{4.66}$$

The commutator of the new operators is easily found from these results,

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{4.67}$$

From eqn (4.65), the Hamiltonian can be written

$$\hat{\mathscr{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}). \tag{4.68}$$

The combination of operators $\hat{a}^\dagger\hat{a}$ occurs frequently; it is called the number operator of the oscillator, and we denote it

$$\hat{n} = \hat{a}^\dagger\hat{a}. \tag{4.69}$$

EIGENVALUE PROBLEM

$$\mathscr{H}|n\rangle = \hbar\omega(a^\dagger a + \tfrac{1}{2})|n\rangle$$

$$E_n = \hbar\omega(n + \tfrac{1}{2})$$

$$\begin{cases} a^+|n\rangle = |n+1\rangle : E_{n+1} \\ a|n\rangle = |n-1\rangle : E_{n-1} \end{cases}$$

$$a|0\rangle = 0$$

# ELECTROMAGNETIC FIELD IS QUATIZED!

ASSOCIATION OF A QUANTUM-MECHANICAL HARMONIC OSCILLATOR WITH EACH MODE $\vec{k}$ OF THE RADIATION FIELD

$\left.\begin{array}{l} a^+_{\vec{k}} \\ a_{\vec{k}} \end{array}\right\}$ OPERATORS THAT CREATE OR DESTROY A QUANTUM OF ENERGY $\hbar\omega_{\vec{k}}$ IN THE CAVITY ELECTROMAGNETIC-FIELD MODE OF WAVEVECTOR $\vec{k}$

$\Rightarrow$ CREATION OR DESTRUCTION OF PHOTONS WITH WAVEVECTORS $\vec{k}$

NUMBER OF PHOTONS $\vec{k}$

$$n_{\vec{k}} = 0, 1, 2, \ldots \qquad \mathcal{H}_R = \sum_{\vec{k}} \hbar\omega_{\vec{k}}\left(\hat{a}^+_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right)$$

THE EXCITATION LEVEL OF A CAVITY MODE $\vec{k}$ IS DETERMINED BY ITS EIGENSTATE $|n_{\vec{k}}\rangle$

PROPERTIES

$$a_{\vec{k}}|n_{\vec{k}}\rangle = \sqrt{n_{\vec{k}}}\,|n_{\vec{k}}-1\rangle \quad \text{(ABSORPTION)}$$

$$a^+_{\vec{k}}|n_{\vec{k}}\rangle = \sqrt{n+1}\,|n_{\vec{k}}+1\rangle \quad \text{(EMISSION + SPONTANEOUS EMISSON)}$$

A STATE OF TOTAL FIELD

$$|n_{k_1} n_{k_2} \ldots n_{k_l} \ldots\rangle = |n_{k_1}\rangle|n_{k_2}\rangle \ldots |n_{k_l}\rangle \ldots$$

DIFFERENT CAVITY MODES ARE INDEPENDENT

CLASSICAL VECTORS

$$A_{\vec{k}} = (4\varepsilon_0 V\omega^2_{\vec{k}})^{-\frac{1}{2}}(\omega_{\vec{k}} Q_k + i P_{\vec{k}})\varepsilon_{\vec{k}} \rightarrow$$

$$(\omega_{\vec{k}}\hat{q}_{\vec{k}} + i\hat{p}_{\vec{k}})\varepsilon_{\vec{k}} \rightarrow$$

$$\left(\frac{\hbar}{2\varepsilon_0} V\omega_{\vec{k}}\right)^{\frac{1}{2}} \hat{a}_{\vec{k}}\varepsilon_{\vec{k}}$$

$$A^*_{\vec{k}} \rightarrow \left(\frac{\hbar}{2\varepsilon_0} V\omega_{\vec{k}}\right)^{\frac{1}{2}} \hat{a}^+_{\vec{k}}\varepsilon_{\vec{k}}$$

QUANTUM MECHANICAL EXPRESSION FOR THE VECTOR POTENTIAL: (OPERATOR)

$$\hat{A} = \sum_{\vec{k}}\left(\frac{\hbar}{2\varepsilon_0} V\omega_{\vec{k}}\right)^{1/2}\varepsilon_{\vec{k}}\left\{\hat{a}_{\vec{k}}\exp(-i\omega_{\vec{k}}t + i\vec{k}\vec{r}) + \hat{a}^+_{\vec{k}}\exp(i\omega_{\vec{k}}t - i\vec{k}\vec{r})\right\}$$

$$\mathcal{H}_I \cong \mathcal{H}_{ED} = e\bar{D}\cdot E_T(0)$$

$$D = \sum_{i,j} D_{ij}\, b_i^+ b_j \quad , \; D_{ij} = \langle i|D|j\rangle$$

$$E_{\bar{k}} = \sim \varepsilon_{\bar{k}} \left\{ a_{\bar{k}} \exp(-i\omega_k t + i\bar{k}\bar{r}) - a_{\bar{k}}^+ \exp(i\omega_k t - i\bar{k}\bar{r}) \right\}$$

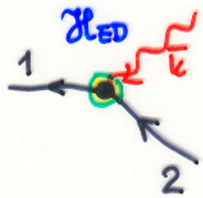
$$\mathcal{H}_{ED} = ie \sum_{\bar{k}} \sum_{ij} \left(\frac{\hbar\omega_{\bar{k}}}{2\varepsilon_0 V}\right)^{1/2} \varepsilon_{\bar{k}} D_{ij} \times$$

$$\left\{ \hat{a}_{\bar{k}} \exp(-i\omega_k t + i\bar{k}\bar{R}) - \hat{a}_{\bar{k}}^+ \exp(-i\omega_{\bar{k}} t - i\bar{k}\bar{R}) \right\} \hat{b}_i^+ \hat{b}_j$$

ABSORPTION (≡ DESTRUCTION) OF PHOTON $\bar{k}$

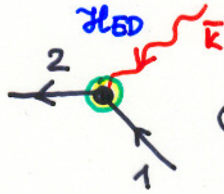
EMISSION (CREATION) OF PHOTON $\bar{k}$

TRANSITION FROM ATOMIC STATE $j$ TO ATOMIC STATE $i$



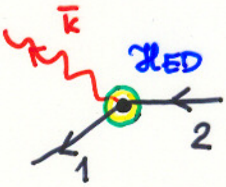

$a_{\bar{k}} b_1^+ b_2$

ABSORPTION



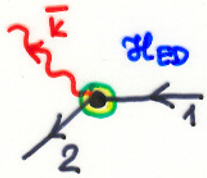

$a_{\bar{k}} b_2^+ b_1$

ABSORPTION




$a_{\bar{k}}^+ b_1^+ b_2$

EMISSION




$a_{\bar{k}}^+ b_2^+ b_1$

EMISSON

ONE PHOTON RADIATION PROCESSES

FIRST ORDER OF PERTURBATION EXPANSION

$$\langle n_k - 1, 2 | \mathcal{H}_{ED} | n_k, 1 \rangle = i\hbar g_{\bar{k}} \exp(-i\omega_{\bar{k}} t + i\bar{k}\bar{R}) \sqrt{n_k}$$

ABSORPTION

THIS MATRIX ELEMENT CONTRIBUTES TO THE TRANSITION RATE

$$\langle n_k + 1, 1 | \mathcal{H}_{ED} | n_k, 2 \rangle = -i\hbar g_k \exp(i\omega_{\bar{k}} t - i\bar{k}\bar{R}) \sqrt{n_{\bar{k}} + 1}$$

EMISSION

EVEN IF $n_k = 0$ IT IS POSSIBLE TO OBSERVE (AND DESCRIBE) SPONTANEOUS EMISSION!

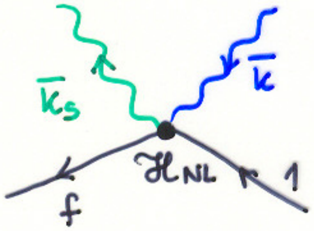
FIRST ORDER TERM = ONE PHOTON PROCESSES

# SECOND-ORDER PROCESSES: TWO PHOTONS INVOLVED

## KRAMERS-HEISENBERG FORMULA

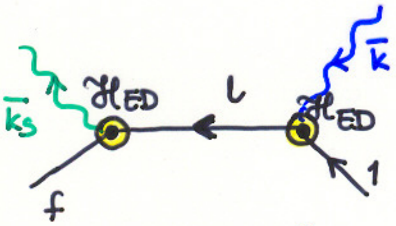
$$\frac{1}{\tau} = \frac{2\pi}{\hbar^2}\sum_f \sum_{\bar{k}_s} \Big| \overbrace{\langle n_{\bar{k}}-1, 1_{\bar{k}_s}, f | \mathcal{H}_{NL} | n_{\bar{k}}, 0_{\bar{k}_s}, 1\rangle}^{\text{FIRST ORDER}} +$$

$$\frac{1}{\hbar}\sum_l \langle n-1,1,f|\mathcal{H}_{ED}|l\rangle\langle l|\mathcal{H}_{ED}|n,0,1\rangle \Big|^2 \delta(\omega-\omega_s-\omega_f)$$

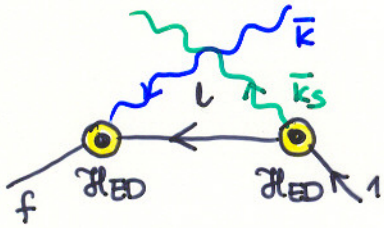



$a^{+}_{k_s} a_{\bar{k}} b^{+}_f b_1$ (RELATIVELY NEGLIGIBLE)

PHOTON SCATTERING

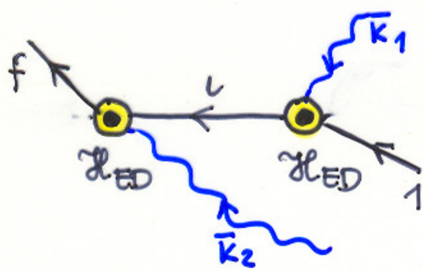



$a^{+}_{\bar{k}_s} a_{\bar{k}} b^{+}_f b_1$ (VIA INTERMEDIATE STATES)

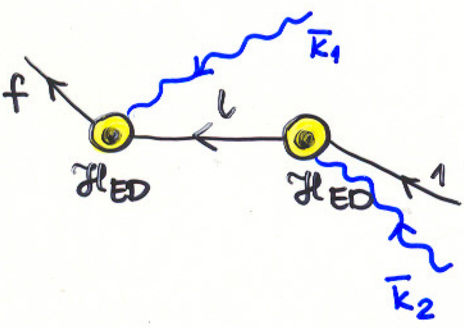



PHOTON SCATTERING

## TWO-PHOTON ABSORPTION




$a_{\bar{k}_2} a_{\bar{k}_1} b^{+}_f b_1$




$a_{\bar{k}_1} a_{\bar{k}_2} b^{+}_f b_1$

RODNEY LOUDON

# The quantum theory of light

SECOND EDITION

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