MIXED TENSOR OPERATORS

$$X_Q^{(K)} = \sum_{q_1 q_2} T_{q_1}^{(k_1)} U_{q_2}^{(k_2)} \langle k_1 q_1 k_2 q_2 | (k_1 k_2) K Q \rangle$$

$$\Downarrow$$

$$|KQ\rangle = \sum_{q_1 q_2} |k_1 q_1 k_2 q_2\rangle\langle k_1 q_1 k_2 q_2 | (k_1 k_2) K Q \rangle$$

SAME TRANSFORMATION PROPERTIES! (under operations of R_3)

ALSO TENSOR OPERATOR (COMMUTATORS ARE SATISFIED)

$T_{q_1}^{(k_1)}$, $U_{q_2}^{(k_2)}$ – COMPONENTS OF TWO TENSOR OPERATORS

MATRIX ELEMENTS

$T_{q_1}^{(k_1)}$ ACT ON PART 1

$U_{q_2}^{(k_2)}$ ACT ON PART 2

FOR EXAMPLE:

$T_{q_1}^{(k_1)}$ ACT ON ORBIT, $U_{q_2}^{(k_2)}$ ACT ON SPIN

OR

$T_{q_1}^{(k_1)}$ IS FUNCTION OF COORD. OF $1\bar{e}$, $U_{q_2}^{(k_2)}$ IS FUNCTION OF COORD. OF $2\bar{e}$

$$\langle \gamma j_1 j_2 J M_J | X_Q^{(K)} | \gamma' j_1' j_2' J' M_J' \rangle$$

ANGULAR – MOMENTUM QUANTUM NUMBERS OF THE TWO PARTS OF THE SYSTEM

1°. WIGNER – ECKART THEOREM:

$$= (-1)^{J-M_J} \begin{pmatrix} J & K & J' \\ -M_J & Q & M_J' \end{pmatrix} \langle \gamma j_1 j_2 J \| X^{(K)} \| \gamma' j_1' j_2' J' \rangle$$

REDUCED MATRIX ELEMENT

HOW TO EVALUATE IT ?

2° AT THE SAME TIME ...

$$\langle \gamma j_1 j_2 J M | X_Q^{(K)} | \gamma' j_1' j_2' J' M_J' \rangle =$$ W – E T. + TRANSFORMATION FROM COUPLED MOMENTA TO UNCOUPLED SCHEME:

$$|\gamma_1' j_1' j_2' J'\rangle = \sum_{m_1' m_2'} |j_1' m_1' j_2' m_2'\rangle\langle j_1' m_1' j_2' m_2' | j_1' j_2' J'\rangle$$

C-G ⇒ 3-j

$$\langle \gamma j_1 j_2 J | = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | \langle j_1 j_2 J | j_1 m_1 j_2 m_2 \rangle$$

C-G ⇒ 3-j

$$\cong \sum_{\substack{\cdots m_1 m_2 \\ m_1' m_2'}} \overset{\text{E-W TRANSFORM.}}{(3\text{-}j)(3\text{-}j)(3\text{-}j)} \langle \gamma j_1 m_1 j_2 m_2 | \underbrace{[T^{(k_1)} \times U^{(k_2)}]^{(k)}_Q}_{\text{STILL COUPLED!}} | \gamma' j_1' m_1' j_2' m_2' \rangle =$$

TENSORIAL PRODUCT

$$[T^{(k_1)} \times U^{(k_2)}]^{(k)}_Q = \sum_{q_1 q_2} (-1)^{-k_1+k_2+Q} [K]^{\frac{1}{2}} \begin{pmatrix} k_1 & k_2 & K \\ q_1 & q_2 & -Q \end{pmatrix} T^{(k_1)}_{q_1} U^{(k_2)}_{q_2}$$

UNCOUPLED

$$\cong \sum_{\cdots q_1 q_2} (3\text{-}j)(3\text{-}j)(3\text{-}j)(3\text{-}j) \langle \gamma j_1 m_1 j_2 m_2 | T^{(k_1)}_{q_1} U^{(k_2)}_{q_2} | \gamma' j_1' m_1' j_2' m_2' \rangle =$$

$$= \sum (3\text{-}j)(3\text{-}j)(3\text{-}j)(3\text{-}j) \langle j_1 m_1 | T^{(k_1)}_{q_1} | j_1' m_1' \rangle \langle j_2 m_2 | U^{(k_2)}_{q_2} | j_2' m_2' \rangle$$

„DEEPEST QUANTIZATION"

$$\cong \sum \underbrace{(3\text{-}j)(3\text{-}j)(3\text{-}j)(3\text{-}j)\overset{\text{W-E}}{(3\text{-}j)}\overset{\text{W-E}}{(3\text{-}j)}}_{9\text{-}j:\ \begin{Bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}} \langle j_1 \| T^{(k_1)} \| j_1' \rangle \langle j_2 \| U^{(k_2)} \| j_2' \rangle$$

REDUCED MATRIX ELEMENT FROM 1°

$$\boxed{\langle \gamma j_1 j_2 J \| X^{(K)} \| \gamma' j_1' j_2' J' \rangle = \sum_{\gamma''} \langle \gamma j_1 \| T^{(k_1)} \| \gamma'' j_1' \rangle \langle \gamma'' j_2 \| U^{(k_2)} \| \gamma' j_2' \rangle \times \{[J][K][J']\}^{1/2} \begin{Bmatrix} j_1 & j_1' & k_1 \\ j_2 & j_2' & k_2 \\ J & J' & K \end{Bmatrix}}$$

FOR SCALAR PRODUCT: $K=0 \Rightarrow$

$$\begin{Bmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{Bmatrix} = \frac{(-1)^{b+c+e+f}}{[e,f]^{1/2}} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$$

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = (-1)^{a+b+c+d+e+f+g+h+i} \begin{Bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{Bmatrix}$$

PARTICULAR CASES:

① $k_2=0$: $T^{(k)}$ ACTING ONLY ON PART 1

$$\boxed{\langle \gamma j_1 j_2 J \| T^{(k)} \| \gamma' j_1' j_2' J' \rangle = \delta(j_2, j_2')(-1)^{j_1+j_2+J'+k}[J,J']^{\frac{1}{2}} \begin{Bmatrix} J & k & J' \\ j_1' & j_2 & j_1 \end{Bmatrix} \langle \gamma j_1 \| T^{(k)} \| \gamma' j_1' \rangle}$$

② $k_1=0$: $U^{(k')}$ ACTING ONLY ON PART 2

$$\boxed{\langle \gamma j_1 j_2 J \| U^{(k')} \| \gamma' j_1' j_2' J' \rangle = \delta(j_1 j_1')(-1)^{j_1'+j_2'+J+k'}[J,J']^{\frac{1}{2}} \begin{Bmatrix} J & k' & J' \\ j_2' & j_1 & j_2 \end{Bmatrix} \langle \gamma j_2 \| U^{(k')} \| \gamma' j_2' \rangle}$$

IF IT IS IMPOSSIBLE TO SEPARATE T AND U:

$$\langle \gamma J M_J | X^{(K)}_Q | \gamma' J' M_J' \rangle = \sum_{q_1 q_2} (-1)^{k_1-k_2+Q}[K]^{\frac{1}{2}} \begin{pmatrix} k_1 & k_2 & K \\ q_1 & q_2 & -Q \end{pmatrix}$$

$$\langle \gamma J M_J | T^{(k_1)}_{q_1} U^{(k_2)}_{q_2} | \gamma' J' M_J' \rangle$$

$$\mathbb{1} = \sum_{\gamma'' J'' M_J''} |\gamma'' J'' M_J''\rangle\langle \gamma'' J'' M_J''|$$

+ WIGNER-ECKART THEOREM

$$\boxed{\langle \gamma J \| X^{(K)} \| \gamma' J' \rangle = [K]^{\frac{1}{2}}(-1)^{J+K-J'} \sum_{J'' \gamma''} \begin{Bmatrix} k_2 & K & k_1 \\ J & J'' & J' \end{Bmatrix} \times \langle \gamma J \| T^{(k_1)} \| \gamma'' J'' \rangle \langle \gamma'' J'' \| U^{(k_2)} \| \gamma' J' \rangle}$$

EXAMPLE: EVALUATE MATRIX ELEMENT (≡ ENERGY) OF COULOMB INTERACTION OPERATOR IN THE CASE OF f^2 CONFIGURATION

L-S COUPLING!

$$\frac{1}{r_{ij}} \Rightarrow \sum_k \frac{r_<^k}{r_>^{k+1}} \left(C_i^{(k)}\cdot C_j^{(k)}\right)$$

SCALAR PRODUCT

IN GENERAL:

$$\langle l^2 S M_S L M_L | (C_i^{(k)}\cdot C_j^{(k)}) | l^2 S' M_S' L' M_L'\rangle =$$

WITH $S+L=$ even (PAULI EXCLUSION PRINCIPLE)

$$= \delta(S,S')\delta(M_S,M_S')\langle (\overset{i}{l}\,\overset{j}{l}) L M_L | (C_i^{(k)}\cdot C_j^{(k)}) | (\overset{i}{l}\,\overset{j}{l}) L' M_L'\rangle$$

FOR $K=0$

$$\langle \gamma \overset{l}{j_1} \overset{l}{j_2} \overset{L}{J} \overset{M_L}{M_J} | (T^{(k)}\cdot U^{(k)}) | \gamma' \overset{l}{j_1'} \overset{l}{j_2'} \overset{L}{J'} \overset{M_L}{M_J'}\rangle =$$

$$(-1)^{j_1'+j_2+J}\,\delta(J,J')\,\delta(M_J,M_J')\begin{Bmatrix} j_1' & j_2' & J\\ j_2 & j_1 & k\end{Bmatrix}\times$$

$$\sum_{\gamma''}\langle\gamma j_1\|T^{(k)}\|\gamma'' j_1'\rangle\langle\gamma'' j_2\|U^{(k)}\|\gamma' j_2'\rangle$$

$$= \delta(SS')\delta(M_SM_S')(-1)^{l+l+L}\,\delta(L,L')\,\delta(M_L,M_L')\begin{Bmatrix} l&l&k\\ l&l&L\end{Bmatrix}\langle l\|C^{(k)}\|l\rangle\langle l\|C^{(k)}\|l\rangle$$

THE MATRIX OF COULOMB INTERACTION IS DIAGONAL

$$(-1)^l\,[l,l]^{\frac{1}{2}}\begin{pmatrix} l&k&l\\0&0&0\end{pmatrix}$$

$l=3$: f^2

$$=\delta(SS')\delta(M_SM_S')\delta(LL')\delta(M_LM_L')\,49\,(-1)^L\begin{Bmatrix}3&3&k\\3&3&L\end{Bmatrix}\begin{pmatrix}3&k&3\\0&0&0\end{pmatrix}\begin{pmatrix}3&k&3\\0&0&0\end{pmatrix}$$

$k=$ even

$0\le k\le 6$

$k=0,2,4,6$

THE WHOLE MATRIX ELEMENT (INCLUDING THE RADIAL PART):

$$\langle f^2 S M_S L M_L | V_C | f^2 S' M_S' L' M_L' \rangle =$$

$$\sum_k 49 \, F^{(k)} (-1)^L \begin{pmatrix} 3 & k & 3 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} 3 & 3 & k \\ 3 & 3 & L \end{Bmatrix}$$

SLATER INTEGRALS

$$F^{(k)} = \int_0^\infty \int_0^\infty \frac{r_<^k}{r_>^{k+1}} \left[R_{nl}(r_i) R_{nl}(r_j) \right]^2 dr_i \, dr_j \equiv$$

$$\langle nl\, nl | \frac{r_<^k}{r_>^{k+1}} | nl\, nl \rangle \equiv F^{(k)}(nl\, nl\, nl\, nl)$$

$$R_{nl} = \frac{1}{r} P_{nl}(r)$$

PROBLEM: EVALUATE $E(^{2S+1}L)$ FOR THE FOLLOWING SPECTROSCOPIC TERMS OF f^2:

^1S, ^3P, ^1D, ^3F, ^1G, ^3H, ^1I