ANGULAR MOMENTUM COUPLING COEFFICIENTS

FORMALISM

TOOLS:

ANGULAR MOMENTUM OPERATORS ≡
GENERATORS OF ALL ROTATIONS IN SO(3)

$$[L_x, L_y] = i\hbar L_z\,,\quad [L_y, L_z] = i\hbar L_x,\quad [L_z, L_x] = i\hbar L_y$$

$$[L_z, L^2] = 0$$

PROPERTIES:

$$L_z Y_L^{M_L} = M_L \hbar\, Y_L^{M_L}$$

$$L_\pm Y_L^{M_L} = \sqrt{L(L+1) - M_L(M_L \pm 1)}\,\hbar\, Y_L^{M_L \pm 1}$$

$Y_L^{M_L}$ SPHERICAL HARMONICS

$$L_z |L M_L\rangle = M_L \hbar |L M_L\rangle$$

$$L_\pm |L M_L\rangle = \sqrt{L(L+1) - M_L(M_L \pm 1)}\,\hbar\, |L M_L \pm 1\rangle$$

$$l_z |l m_l\rangle = m_l \hbar |l m_l\rangle$$

$$l_\pm |l m_l\rangle = \sqrt{l(l+1) - m_l(m_l \pm 1)}\,\hbar\, |l m_l \pm 1\rangle$$

TRANSFORMATION PROPERTIES UNDER ROTATIONS IN SO(3)

$$J_z |J M_J\rangle = M_J \hbar |J M_J\rangle$$

$$J_\pm |J M_J\rangle = \sqrt{J(J+1) - M_J(M_J \pm 1)}\,\hbar\, |J M_J \pm 1\rangle$$

$$s_z |s m_s\rangle = m_s \hbar |s m_s\rangle$$

$$s_\pm |s m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}\,\hbar\, |s m_s \pm 1\rangle$$

$\Downarrow$

SYMMETRY OPERATIONS

...

$$[H, L^2] = 0 \qquad [H, L_z] = 0$$

QUANTUM NUMBERS

COUPLING OF ANGULAR MOMENTA:

$$\left.\begin{matrix} l_1 m_1 \\ l_2 m_2 \end{matrix}\right\} \quad |l_1 - l_2| \le L \le l_1 + l_2$$

$$M_L = m_1 + m_2$$

$$M_L = -L, \dots L \quad (2L+1 \text{ VALUES})$$

$$H_1 = \sum_{i<j}^{N} \frac{e^2}{r_{ij}}$$ COULOMB INTERACTION

$$H_2 = \sum_{i=1}^{N} \xi(r_i)\, \vec{s}_i \cdot \vec{l}_i$$ SPIN-ORBIT INTERACTION

$$\xi(r) = \frac{\hbar^2}{2m^2c^2 r} \frac{dU}{dr}$$

THE MOST IMPORTANT CORRECTIONS TO THE ENERGY!

HOW TO EVALUATE THEM? USING THE PROPERTIES OF ANGULAR MOMENTUM OPERATORS

H_1: $S^2 S_z L^2 L_z$ → **SPECTROSCOPIC TERMS**

LINEAR COMBINATIONS OF DETERMINANTAL FUNCTIONS $|\gamma S M_S L M_L\rangle$

H_2: $J^2 J_z$

LINEAR COMBINATIONS OF ATOMIC TERMS $|\gamma S L J M_J\rangle$

DIMENSIONS OF „SECULAR DET." ARE REDUCED!...

BUT OPERATORS ACT ON THE COORDINATES OF DISTINCT PARTICLE

$$\begin{tikzcd}
nls \arrow[r, shift left] \arrow[d, shift left] & nlm_s m_l \arrow[l, shift left] \arrow[dr, shift left] & \\
& \text{SHIFT OPERATORS?} & SM_S LM_L \arrow[ul, shift left] \arrow[dl, shift left] \\
jjJM \arrow[r, shift left] \arrow[u, shift left] & SLJM \arrow[l, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

SYSTEM OF TWO PARTS:

$$|j_1 m_1\rangle \qquad |j_2 m_2\rangle$$

$$\boxed{\begin{aligned} j^2(k)\,|j_k m_k\rangle &= j_k(j_k+1)\hbar^2\,|j_k m_k\rangle \\ j_z(k)\,|j_k m_k\rangle &= m_k\hbar\,|j_k m_k\rangle \end{aligned}} \qquad k=1,2$$

$$\vec{J} = \vec{j}(1)+\vec{j}(2) \quad : \quad [j_i(1), j_k(2)]=0$$

$\vec{J}^2, J_z$: EIGENFUNCTIONS $|(j_1 j_2) J M\rangle$

COUPLED TO J

$$\boxed{\begin{aligned} J^2\,|(j_1 j_2)JM\rangle &= J(J+1)\hbar^2\,|(j_1 j_2)JM\rangle \\ J_z\,|(j_1 j_2)JM\rangle &= M\hbar\,|(j_1 j_2)JM\rangle \end{aligned}}$$

PRODUCT FUNCTIONS

$$|j_1 m_1\rangle|j_2 m_2\rangle \equiv |j_1 m_1 j_2 m_2\rangle$$

EIGENFUNCTION OF $J_z = j_z(1)+j_z(2)$

$$J_z\,|j_1 m_1\rangle|j_2 m_2\rangle = \big(j_z(1)+j_z(2)\big)\,|j_1 m_1 j_2 m_2\rangle =$$

$$= (m_1+m_2)\hbar\,|j_1 m_1 j_2 m_2\rangle =$$

$$= M\hbar\,|j_1 m_1 j_2 m_2\rangle$$

LINEAR COMBINATION OF $|j_1 m_1 j_2 m_2\rangle$ IS THE EIGENFUNCTION OF $\hat{J}^2$;

HOW TO CONSTRUCT IT?

UNCOUPLED MOMENTA

$$|(j_1 j_2) JM\rangle = \sum_{\substack{m_1 m_2 \\ (m_1+m_2=M)}} |j_1 m_1 j_2 m_2\rangle \underbrace{\langle j_1 m_1 j_2 m_2 | (j_1 j_2) JM\rangle}$$

COUPLED MOMENTA

CLEBSH-GORDAN COEFFICIENTS
VECTOR-COUPLING COEFF.
ANGULAR MOMENTUM COUPLING COEFF.
RE-COUPLING COEFF.
(COEFFICIENT OF LINEAR COMBINATION)

$$\mathbb{1} = \sum_{m_1} |j_1 m_1\rangle\langle j_1 m_1|$$

$$\mathbb{1} = \sum_{m_2} |j_2 m_2\rangle\langle j_2 m_2|$$

$$\mathbb{1} = \sum_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2|$$

PHASE CONVENTION OF CONDON & SHORTLEY $\left\{ \right.$ $\langle j_1 m_1 j_2 m_2 | JM\rangle$ for $M=J$ AND $m_1 = j_1$ (MAX) ARE POSITIVE OR ZERO AND REAL

COUPLED MOMENTA

$$|j_1 m_1 j_2 m_2\rangle = \sum_{JM} |(j_1 j_2) JM\rangle \langle (j_1 j_2) JM | j_1 m_1 j_2 m_2\rangle$$

UNCOUPLED MOMENTA

CLEBSH-GORDAN COEFFICIENT

C-G COEFFICIENTS ARE REAL $\Rightarrow$ THE SAME SET OF COEFFICIENTS FOR BOTH TRANSFORMATION

$$\langle (j_1 j_2) JM | j_1 m_1 j_2 m_2\rangle = \langle j_1 m_1 j_2 m_2 | (j_1 j_2) JM\rangle$$

GENERAL PROCEDURE:

(JUDD p. 10/11)

USING

$$J_+ |(j_1 j_2) J M = J\rangle = 0$$

APPLYING

J_- $(J-M)$ TIMES ON $|(j_1 j_2) J J\rangle$

$\Downarrow$ ALGEBRAIC EXPRESSION FOR THE C-G COEFFICIENT

3-j SYMBOL ≡ 3-j COEFFICIENT ≡ WIGNER COEFFICIENT

$$\boxed{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-1/2}\langle j_1 m_1 j_2 m_2 | j_3 -m_3\rangle}$$

$$\langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3 -m_3\rangle$$

$$m_1+m_2+m_3=0$$

$$|j_1-j_2| \le j_3 \le j_1+j_2$$

TRIANGULAR CONDITION

OTHERWISE = 0 !

① $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$ (EVEN PERMUTATION)

② $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$ (ODD PERMUTATION

③ $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$

ORTHOGONALITY OF ANGULAR MOMENTUM STATES } ⇒ "ORTHOGONALITY" OF 3-j SYMBOLS

$$\sum_{j_3 m_3} |(j_1 j_2) j_3 m_3\rangle\langle (j_1 j_2) j_3 m_3| = 1$$

$$\langle j_1 m_1 j_2 m_2 | j_1 m_1' j_2 m_2'\rangle = \delta(m_1, m_1')\,\delta(m_2, m_2')$$

$$\sum_{j_3 m_3} \langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3 m_3\rangle\langle (j_1 j_2) j_3 m_3 | j_1 m_1' j_2 m_2'\rangle = \delta(m_1 m_1')\,\delta(m_2 m_2')$$

⇓ 3-j ⇓ 3-j

① $$\boxed{\sum_{j_3 m_3} (2j_3+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta(m_1 m_1')\,\delta(m_2 m_2')}$$

$$\sum_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2| = \mathbb{1}$$

$$\langle (j_1 j_2) j_3 m_3 | (j_1 j_2) j_3' m_3' \rangle = \delta(j_3 j_3')\,\delta(m_3 m_3')$$

$$\sum_{m_1 m_2} \langle (j_1 j_2) j_3 m_3 | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3' m_3' \rangle = \delta(j_3 j_3')\,\delta(m_3 m_3')$$

$$\Downarrow \qquad\qquad \Downarrow$$
$$3\text{-}j \qquad\qquad 3\text{-}j$$

2
$$\boxed{\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = (2j_3+1)^{-1}\,\delta(j_3 j_3')\,\delta(m_3 m_3')}$$

PROPERTY:

$$\sum_{m_3} (2) = 2j_3+1$$ (NUMBER OF VALUES OF m_3 FOR GIVEN j_3)

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$$

EXAMPLE: PERFORM TRANSFORMATION

$$|\gamma S L J M_J\rangle \longrightarrow |S M_S L M_L\rangle$$

IN THE CASE OF THE GROUND STATE $^3H_4$ OF f^2

A. TRADITIONAL WAY (USING SHIFT OPERATORS)

$S=1$, $L=5$, $J=4$, $M_J = -4, -3, -2, -1, 0, 1, 2, 3, 4_{MAX}$

$M_S = -1\ 0\ 1$

$M_L = -5\ -4\ -3\ -2\ -1\ 0\ 1\ 2\ 3\ 4\ 5$

$$M_J = M_L + M_S = 4 \text{ (MAX)}$$

$$|^3H_4 M_J=4\rangle = a\,|^3H\ M_S=1\ M_L=3\rangle + b\,|^3H\ 0\ 4\rangle + c\,|^3H\ -1\ 5\rangle$$

a, b, c – UNKNOWN COEFFICIENTS OF LINEAR COMBINATION

$$J_+ |^3H_4\, 4\rangle = 0\ , \qquad J_+ = S_+ + L_+$$

$$a(5\cdot6-3\cdot4)^{1/2}\,|{}^3H\,1\,4\rangle + b(1\cdot2-0\cdot1)^{1/2}\,|{}^3H\,1\,4\rangle +$$

$$b(5\cdot6-4\cdot5)^{1/2}\,|{}^3H\,0\,5\rangle + c(1\cdot2-(-1)\cdot0)^{1/2}\,|{}^3H\,0\,5\rangle = 0$$

$|\gamma S L M_S M_L\rangle$ ARE ORTHOGONAL

$$\Downarrow$$

$$\begin{cases} \sqrt{18}\,a + \sqrt{2}\,b = 0 \\ \sqrt{10}\,b + \sqrt{2}\,c = 0 \end{cases}$$

$$\langle {}^3H_4\,4 | {}^3H_4\,4\rangle = 1 \Rightarrow aa^* + bb^* + cc^* = 1$$

$$\langle {}^3H\,M_S M_L | {}^3H\,M_S' M_L'\rangle = \delta(M_S M_S')\,\delta(M_L M_L')$$

$$\Downarrow \quad 55aa^* = 1$$

$$a = \frac{1}{\sqrt{55}} \qquad b = \frac{-3}{\sqrt{55}} \qquad c = \frac{3}{\sqrt{11}}$$

$$|{}^3H_4\,4\rangle = \underbrace{\frac{1}{\sqrt{55}}}_{=0.1348}|{}^3H\,1\,3\rangle - \underbrace{\frac{3}{\sqrt{55}}}_{=0.4045}|{}^3H\,0\,4\rangle + \underbrace{\frac{3}{\sqrt{11}}}_{=0.9045}|{}^3H\,{-1}\,5\rangle$$

USING $J_- = S_- + L_-$ => OTHER STATES $|{}^3H_4\,M_J\rangle$

B. MODERN WAY

$$|\gamma S L J M_J\rangle = \sum_{M_S M_L} |S M_S L M_L\rangle\langle S M_S L M_L | \gamma S L J M_J\rangle$$

$$\Downarrow \text{C-G}$$

$$|\gamma\,1\,5\,4\,4\rangle = \sum_{M_S M_L} |1\,M_S\,5\,M_L\rangle\langle 1\,M_S\,5\,M_L | \gamma\,1\,5\,4\,4\rangle$$

$$\Downarrow$$

$$(-1)^{1-5+4}\underbrace{(2\cdot4+1)^{1/2}}_{\sqrt{9}=3}\begin{pmatrix} 1 & 5 & 4 \\ M_S & M_L & -4 \end{pmatrix}$$

$M_S + M_L - 4 = 0$

$M_S=1$ $M_L=3$	$M_S=0$ $M_L=4$	$M_S=-1$ $M_L=5$
$\begin{pmatrix} 1 & 5 & 4 \\ 1 & 3 & -4 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & 4 \\ 0 & 4 & -4 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & 4 \\ -1 & 5 & -4 \end{pmatrix}$
0.0449	−0.1348	0.3015
×3 = 0.1348	−0.4045	0.9045

Appendix C. Formulas of 3-*j* and 6-*j* Symbols

Reprint from: *Edmonds* [1957]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{1}{2}J}\left[\frac{(j_1+j_2-j_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(j_1+j_2+j_3+1)!}\right]^{\frac{1}{2}}$$

$$\frac{(\frac{1}{2}J)!}{(\frac{1}{2}J-j_1)!(\frac{1}{2}J-j_2)!(\frac{1}{2}J-j_3)!}$$

if J is even.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if} \quad J \text{ is odd} \quad \text{where} \quad J = j_1 + j_2 + j_3$$

$$\begin{pmatrix} J+\frac{1}{2} & J & \frac{1}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (-1)^{J-M-\frac{1}{2}}\left[\frac{J-M+\frac{1}{2}}{(2J+2)(2J+1)}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M-1}\left[\frac{(J-M)(J-M+1)}{(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M-1}\left[\frac{(J+M+1)(J-M+1)\cdot 2}{(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M}\left[\frac{(J-M)(J+M+1)\cdot 2}{(2J+2)(2J+1)(2J)}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M}\frac{M}{[(2J+1)(J+1)J]^{\frac{1}{2}}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b-1 \end{Bmatrix} = (-1)^{s}\left[\frac{s(s+1)(s-2a-1)(s-2a)}{(2b-1)2b(2b+1)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b \end{Bmatrix} = (-1)^{s}\left[\frac{2(s+1)(s-2a)(s-2b)(s-2c+1)}{2b(2b+1)(2b+2)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b+1 \end{Bmatrix} = (-1)^{s}\left[\frac{(s-2b-1)(s-2b)(s-2c+1)(s-2c+2)}{(2b+1)(2b+2)(2b+3)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} = (-1)^{s+1}\frac{2[b(b+1)+c(c+1)-a(a+1)]}{[2b(2b+1)(2b+2)2c(2c+1)(2c+2)]^{\frac{1}{2}}}$$

where $s = a + b + c$.

3-*j* Symbols

j_1	j_2	j_3	m_1	m_2	m_3		j_1	j_2	j_3	m_1	m_2	m_3	
1	1	0	0	0	0	$*0\underline{1}$	8	7	1	0	0	0	$3\underline{11}0,00\underline{1}$
2	1	1	0	0	0	$1\underline{11}$	8	7	3	0	0	0	$*22\underline{1}1,0\underline{111},$
2	2	0	0	0	0	$00\underline{1}$	8	7	5	0	0	0	$3210,\underline{1111},$
2	2	2	0	0	0	$*10\underline{11},$	8	7	7	0	0	0	$*1\underline{1}31,0\underline{111},\underline{1}$
3	2	1	0	0	0	$*01\underline{11},$	8	8	0	0	0	0	$0000,00\underline{1}$
3	3	0	0	0	0	$*000\underline{1},$	8	8	2	0	0	0	$*31\underline{1}0,00\underline{11},$
3	3	2	0	0	0	$2\underline{111},$	8	8	4	0	0	0	$2200,0\underline{111},$
4	2	2	0	0	0	$10\underline{11},$	8	8	6	0	0	0	$*3120,0\underline{111},\underline{1}$
4	3	1	0	0	0	$2\underline{2}0\underline{1},$	8	8	8	0	0	0	$1012,0\underline{111},\underline{1}$
4	3	3	0	0	0	$*100\underline{1},\underline{1}$	9	5	4	0	0	0	$*1202,\underline{1111},$
4	4	0	0	0	0	$0\underline{2}$	9	6	3	0	0	0	$*2101,0\underline{111},$
4	4	2	0	0	0	$*2\underline{2}1\underline{1},\underline{1}$	9	6	5	0	0	0	$2111,\underline{1111},$
4	4	4	0	0	0	$120\underline{1},\underline{11}$	9	7	2	0	0	0	$*22\underline{1}0,00\underline{11},$
5	3	2	0	0	0	$*1\underline{1}1\underline{1},\underline{1}$	9	7	4	0	0	0	$3010,0\underline{111},$
5	4	1	0	0	0	$*0\underline{2}10,\underline{1}$	9	7	6	0	0	0	$*1211,0\underline{111},\underline{1}$
5	4	3	0	0	0	$201\underline{1},\underline{11}$	9	8	1	0	0	0	$*0200,00\underline{11},$
5	5	0	0	0	0	$*0000,\underline{1}$	9	8	3	0	0	0	$310\underline{1},00\underline{11},$
5	5	2	0	0	0	$1\underline{1}10,\underline{11}$	9	8	5	0	0	0	$*2110,1\underline{111},\underline{1}$
5	5	4	0	0	0	$*1000,\underline{11}$	9	8	7	0	0	0	$3201,0\underline{111},\underline{1}$
6	3	3	0	0	0	$2\underline{1}2\underline{1},\underline{11}$	9	9	0	0	0	0	$*0000,000\underline{1},$
6	4	2	0	0	0	$0010,\underline{11}$	9	9	2	0	0	0	$111\underline{1},00\underline{11},$
6	4	4	0	0	0	$*2\underline{2}10,\underline{11}$	9	9	4	0	0	0	$*220\underline{1},10\underline{11},\underline{1}$
6	5	1	0	0	0	$1100,\underline{11}$	9	9	6	0	0	0	$4100,1\underline{111},\underline{1}$
6	5	3	0	0	0	$*0\underline{1}01,\underline{11}$	9	9	8	0	0	0	$*1\underline{1}02,00\underline{11},\underline{1}$
6	5	5	0	0	0	$4\underline{1}10,\underline{111}$	10	5	5	0	0	0	$2301,\underline{1111},$
6	6	0	0	0	0	$0000,0\underline{1}$	10	6	4	0	0	0	$1011,0\underline{111},$
6	6	2	0	0	0	$*10\underline{1}1,\underline{11}$	10	6	6	0	0	0	$*2301,0\underline{111},\underline{1}$
6	6	4	0	0	0	$2001,\underline{111}$	10	7	3	0	0	0	$301\underline{1},00\underline{11},$
6	6	6	0	0	0	$*4020,\underline{1111},$	10	7	5	0	0	0	$*1011,1\underline{111},\underline{1}$
7	4	3	0	0	0	$*0\underline{2}11,\underline{11}$	10	7	7	0	0	0	$35\underline{2}1,0\underline{111},\underline{1}$
7	5	2	0	0	0	$*01\underline{1}1,\underline{11}$	10	8	2	0	0	0	$021\underline{1},00\underline{11},$
7	5	4	0	0	0	$3\underline{2}11,\underline{111}$	10	8	4	0	0	0	$*301\underline{1},10\underline{11},\underline{1}$
7	6	1	0	0	0	$*0\underline{11}1,0\underline{1}$	10	8	6	0	0	0	$22\underline{1}1,1\underline{111},\underline{1}$
7	6	3	0	0	0	$31\underline{1}1,\underline{111}$	10	8	8	0	0	0	$*31\underline{1}1,00\underline{11},\underline{1}$
7	6	5	0	0	0	$*2111,\underline{1111},$	10	9	1	0	0	0	$1\underline{1}1\underline{1},000\underline{1},$
7	7	0	0	0	0	$*0\underline{11}$	10	9	3	0	0	0	$*021\underline{1},10\underline{11},\underline{1}$
7	7	2	0	0	0	$3\underline{11}1,0\underline{11}$	10	9	5	0	0	0	$42\underline{11},10\underline{11},\underline{1}$
7	7	4	0	0	0	$*24\underline{1}1,\underline{1111},$	10	9	7	0	0	0	$*2\underline{2}01,10\underline{11},\underline{1}$
7	7	6	0	0	0	$3\underline{1}30,\underline{1111},$	10	9	9	0	0	0	$21\underline{1}3,00\underline{11},\underline{11}$
8	4	4	0	0	0	$1\underline{2}12,\underline{111}$	10	10	0	0	0	0	$0\underline{1}0\underline{1},$
8	5	3	0	0	0	$3001,\underline{111}$	10	10	2	0	0	0	$*1\underline{1}1\underline{1},100\underline{1},\underline{1}$
8	5	5	0	0	0	$*1012,\underline{1111},$	10	10	4	0	0	0	$14\underline{11},10\underline{11},\underline{1}$
8	6	2	0	0	0	$20\underline{1}1,0\underline{11}$	10	10	6	0	0	0	$*4\underline{2}0\underline{1},1\underline{111},\underline{1}$
8	6	4	0	0	0	$*3201,\underline{1111},$	10	10	8	0	0	0	$1\underline{2}03,10\underline{11},\underline{11}$
8	6	6	0	0	0	$1021,\underline{1111},$	10	10	10	0	0	0	$*24\underline{1}3,00\underline{11},\underline{111}$

EXAMPLE: SHOW THAT FOR l^2 S+L = EVEN, WHERE S AND L ARE TOTAL ANGULAR AND SPIN QUANTUM NUMBERS DEFINED WITHIN THE L-S COUPLING SCHEME

$$|l^2 SLJM_J\rangle = \sum_{M_S M_L} |l^2 SM_S LM_L\rangle \underbrace{\langle l^2 SM_S LM_L | l^2 SLJM_J\rangle}_{\text{C-G COEFFICIENT}}$$

$$\langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3 m_3\rangle = (-1)^{-j_1 - j_2 + m_3} [j_3]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

$$|l^2 SLJM_J\rangle = \sum_{M_S M_L} (-1)^{-S-L+M_J} [J]^{1/2} \begin{pmatrix} S & L & J \\ M_S & M_L & -M_J \end{pmatrix} |l^2 SM_S LM_L\rangle$$

S AND L COUPLED TO J

UNCOUPLED MOMENTA

$\Downarrow$

$|SM_S\rangle |LM_L\rangle$

1° 2°

1° $$|SM_S\rangle = \sum_{m_{s_1} m_{s_2}} |s m_{s_1} s m_{s_2}\rangle \underbrace{\langle s m_{s_1} s m_{s_2} | (ss) SM_S\rangle}_{(-1)^{-s-s+M_S} [S]^{\frac{1}{2}} \begin{pmatrix} s & s & S \\ m_{s_1} & m_{s_2} & -M_S \end{pmatrix}}$$

$$|SM_S\rangle = \sum_{m_{s_1} m_{s_2}} (-1)^{-2s+M_S} [S]^{\frac{1}{2}} \begin{pmatrix} s & s & S \\ m_{s_1} & m_{s_2} & -M_S \end{pmatrix} |s m_{s_1} s m_{s_2}\rangle$$

$|s m_{s_1}\rangle |s m_{s_2}\rangle$

1. 2.

ONE-PARTIC[LE] FUNCTIONS α OR β

2° $$|LM_L\rangle = \sum_{m_{l_1} m_{l_2}} |l m_{l_1} l m_{l_2}\rangle \underbrace{\langle l m_{l_1} l m_{l_2} | (l^2) LM_L\rangle}_{(-1)^{2l+M_L} [L]^{\frac{1}{2}} \begin{pmatrix} l & l & L \\ m_{l_1} & m_{l_2} & -M_L \end{pmatrix}} =$$

$$|LM_L\rangle = \sum_{m_{l_1} m_{l_2}} (-1)^{2l+M_L} [L]^{\frac{1}{2}} \begin{pmatrix} l & l & L \\ m_{l_1} & m_{l_2} & -M_L \end{pmatrix} |l m_{l_1} l m_{l_2}\rangle$$

UNCOUPLED MOMENTA

$\Downarrow$

$|l m_{l_1}\rangle |l m_{l_2}\rangle$

1. 2.

ONE-PARTICLE FUNCTIONS: ORBITALS

$\bar{e}(1) \rightleftharpoons \bar{e}(2)$

INTERCHANGE

1° $(1 \longleftrightarrow 2)$

$$|(s^2)SM_S\rangle_{21} = \sum_{m_{s_1} m_{s_2}} (-1)^{2s+M_S} [S]^{\frac{1}{2}} \overset{(-1)^{2s+S}}{\begin{pmatrix} s & s & S \\ m_{s_2} & m_{s_1} & -M_S \end{pmatrix}} |s m_{s_1}\rangle |s m_{s_2}\rangle =$$

$$= \sum_{m_{s_1} m_{s_2}} (-1)^{2s+M_S} [S]^{1/2} (-1)^{2s+S} \begin{pmatrix} s & s & S \\ m_{s_1} & m_{s_2} & -M_S \end{pmatrix} |s m_{s_1}\rangle |s m_{s_2}\rangle$$

$$= (-1)^{2s+S} \; |(s^2)SM_S\rangle_{12}$$

2° $(1 \longleftrightarrow 2)$

$$|(l^2)LM_L\rangle_{21} = \sum_{m_{l_1} m_{l_2}} (-1)^{2l+M_L} [L]^{\frac{1}{2}} \overset{(-1)^{2l+L}}{\begin{pmatrix} l & l & L \\ m_{l_2} & m_{l_1} & -M_L \end{pmatrix}} |l m_{l_1}\rangle |l m_{l_2}\rangle$$

$$= (-1)^{2l+L} \; |(l^2)LM_L\rangle_{12}$$

$$|l^2 SM_S LM_L\rangle_{12} = (-1)^{\cancel{2l}+L+2s+S} \; |l^2 SM_S LM_L\rangle_{21}$$

$$= (-1)^{L+S+1} \; |l^2 SM_S LM_L\rangle_{21}$$

THE FUNCTION IS ANTISYMMETRIC $\Rightarrow$

$L+S = \text{even}$ FOR l^2

$L + S = J$

IF $S=0$ (SINGLET) $J=L$ $\Rightarrow$ $J = \text{even}$ $\quad |l^2 S=0 \; L=J \; JM\rangle$

TRIPLET

IF $S=1$: $J = L \pm 1$ $\Rightarrow$ $J = \text{even}$ $\quad |l^2 S=1 \; L=J\pm 1 \; JM\rangle$

$L+S = \text{even}$, $L = \text{odd}$

ODD $\quad$ ODD

THREE POSSIBLE STATES

IF $S=1$: $J=L$ $\Rightarrow$ $J = \text{odd}$ $\quad |l^2 S=1 \; L=J \; JM\rangle$

ARE ANY RESTRICTIONS ON THE SYMMETRY OF ANTISYMMETRIC FUNCTION OF TWO FERMIONS IN j-j COUPLING?

$$[H_{SO}, l_1^2] = 0 \quad [H_{SO}, l_2^2] = 0 \quad [H_{SO}, j_1^2] = 0 \quad [H_{SO}, j_2^2] = 0$$

$$[H_{SO}, J^2] = 0$$

IN j-j COUPLING H_{SO} INTERACTION SPLITS $l_1 l_2$ CONFIGURATION INTO FOUR CONFIGURATIONS:

$$(l_1 j_1\ l_2 j_2) \text{ WHERE } \quad j_1 = l_1 \pm \tfrac{1}{2} \quad j_2 = l_2 \pm \tfrac{1}{2}$$

$$\equiv ((s l_1) j_1 (s l_2) j_2)$$

$$|(s l_1) j_1 (s l_2) j_2 JM\rangle = \sum_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle \underbrace{\langle j_1 m_1 j_2 m_2 | (j_1 j_2) JM\rangle}_{(-1)^{-j_1-j_2+M} [J]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}}$$

COUPLED

$$|(s l_1) j_1 (s l_2) j_2 JM\rangle_{12} = \sum_{m_1 m_2} (-1)^{-j_1-j_2+M} [J]^{1/2} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} |j_1 m_1\rangle |j_2 m_2\rangle$$

$$|(s l_2) j_2 (s l_1) j_1 JM\rangle_{21} = \sum_{m_1 m_2} (-1)^{-j_1-j_2+M} [J]^{1/2} \begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & -M \end{pmatrix} |j_1 m_1\rangle |j_2 m_2\rangle$$

$$(-1)^{j_1+j_2+J}$$

IF $j_1 = j_2$ (EQUIVALENT)

$$|(lj)^2 JM\rangle_{12} = (-1)^{2j+J} |(lj)^2 JM\rangle_{21}$$

THE FUNCTION IS ANTISYMMETRIC!

j - HALF INTEGER

$2 \cdot j$ = odd $\quad \Rightarrow \quad J$ = even

THESE ARE THE ONLY ALLOWED STATES FOR IDENTICAL FERMIONS

HOW TO OBTAIN ONE RESTRICTION (LS-COUPLING) FROM THE OTHER (jj-COUPLING)?

9-j SYMBOL!

$$|(sl) j^2; JM\rangle = \sum_{SL} |(ss)S\,(ll)L; JM\rangle\langle (ss)S\,(ll)L; JM | (sl)j\,(sl)j\,; JM\rangle$$