COUPLING OF MORE THAN TWO ANGULAR MOMENTA

- RESULTANT ANGULAR MOMENTUM AND MAGNETIC QUANTUM NUMBER DO NOT CHARACTERIZE THE WAVE FUNCTION COMPLETELY

  + INTERMEDIATE ANGULAR MOMENTA

- WAVE FUNCTIONS OF THE COUPLED ANGULAR MOMENTA DEPEND UPON THE SCHEME OF COUPLING

2 A.M. $(1+2) \Rightarrow$ C-G $(= 3\text{-}j)$

3 A.M.

$(1+2)+3$, $1+(2+3)$

TRANSFORMATION MATRIX

$\langle (1+2)+3 \mid 1+(2+3) \rangle$ **6-j**

$\langle (j_1 j_2) j_{12}, j_3; j \mid j_1, (j_2 j_3) j_{23}; j \rangle$

INTERMEDIATE ANGULAR MOMENTA

4 A.M.

$(1+2)+(3+4)$, $1\ 2\ 3\ 4$

TRANSFORMATION MATRIX

$\langle (1+2)+(3+4) \mid (1+3)+(2+4) \rangle$ **9-j**

$\langle (j_1 j_2) j_{12}, (j_3 j_4) j_{34}; j \mid (j_1 j_3) j_{13}, (j_2 j_4) j_{24}; j \rangle$

$(((1+2)+3)+4)$, $(1+((2+3)+4))$

3 A.M. $\hat{J}_1, \hat{J}_2, \hat{J}_3$

$\hat{J}_1^2\ \hat{J}_2^2\ \hat{J}_3^2\ \hat{J}_{12}\ \hat{J}_{2z}\ \hat{J}_{3z}$ COMMUTING OPERATORS

EIGENFUNCTION (COMMON):

$$\varphi(j_1 j_2 j_3 m_1 m_2 m_3) = \psi(j_1 m_1)\psi(j_2 m_2)\psi(j_3 m_3) \equiv |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle$$

$\hat{J}_1^2\ \hat{J}_2^2\ \hat{J}_3^2\ \hat{J}_{12}^2\ \hat{J}^2\ \hat{J}_z$ COMMUTING OPERATORS

$$\hat{J}_{12} = \hat{J}_1 + \hat{J}_2$$
$$\hat{J} = \hat{J}_1 + \hat{J}_2 + \hat{J}_3$$

EIGENFUNCTION (COMMON):

$$\psi(j_1 j_2 j_3\ J_{12}\ J M) = \sum_{\substack{m_1 m_2 \\ m_3}} |j_1 m_1 j_2 m_2 j_3 m_3\rangle$$

$|\varphi\rangle$ $\langle\varphi|$

$$\langle j_1 m_1 j_2 m_2 j_3 m_3 | j_1 j_2 j_3\ J_{12}\ J M\rangle$$

GENERALIZED C-G COEFFICIENT

$\hat{J}_1^2\ \hat{J}_2^2\ \hat{J}_3^2\ \hat{J}_{23}^2\ \hat{J}^2\ \hat{J}_z$ COMMUTING OPERATORS

$$\hat{J}_{23} = \hat{J}_2 + \hat{J}_3$$
$$\hat{J} = \hat{J}_1 + \hat{J}_2 + \hat{J}_3$$

EIGENFUNCTION (COMMON)

$$\psi(j_1 j_2 j_3\ J_{23}\ J M) = \sum_{\substack{m_1 m_2 \\ m_3}} |j_1 m_1 j_2 m_2 j_3 m_3\rangle$$

$$\langle j_1 m_1 j_2 m_2 j_3 m_3 | j_1 j_2 j_3\ J_{23}\ J M\rangle$$

$|j_1 j_2 j_3 j m\rangle \longleftrightarrow |j_1 m_1 j_2 m_2 j_3 m_3\rangle$

COUPLING OF THREE ANGULAR MOMENTA

1° $|(j_1 j_2) j_{12}, j_3; j m\rangle \quad (\Rightarrow |(j_1 j_2) j_{12} m_{12}\rangle |j_3 m_3\rangle$

$|(j_1 j_2) j_{12} m_{12} j_3 m_3\rangle$

UNCOUPLED MOMENTA j_{12} & j_3

$\mathbb{1} = \sum_{m_{12}} |j_{12} m_{12}\rangle\langle j_{12} m_{12}|$

↖ EIGENFUNCTION OF $\hat{J}^2_{12}$, $J_{12,z}$

$\mathbb{1} = \sum_{m_3} |j_3 m_3\rangle\langle j_3 m_3|$

$\mathbb{1} = \sum_{m_{12} m_3} |(j_1 j_2) j_{12} m_{12} j_3 m_3\rangle\langle (j_1 j_2) j_{12} m_{12} j_3 m_3|$

$|(j_1 j_2) j_{12}, j_3; j m\rangle =$

TWO ANGULAR MOMENTA

$\sum_{m_{12} m_3} |(j_1 j_2) j_{12} m_{12} j_3 m_3\rangle\langle (j_1 j_2) j_{12} m_{12} j_3 m_3 | (j_1 j_2) j_{12}, j_3; j m\rangle$

C-G (3j)

j_{12} & j_3 - uncoupled

$(j_1 j_2)$ coupled to j_{12}

TWO ANGULAR MOMENTA

$|(j_1 j_2) j_{12} m_{12}\rangle = \sum_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_{12} m_{12}\rangle$

C-G (3j)

$|(j_1 j_2) j_{12}, j_3; j m\rangle =$

$\sum_{m_{12} m_3} \sum_{m_1 m_2} |j_1 m_1 j_2 m_2 j_3 m_3\rangle\langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_{12} m_{12}\rangle$

$\langle (j_1 j_2) j_{12} m_{12} j_3 m_3 | (j_1 j_2) j_{12}, j_3; j m\rangle$

LINEAR COMBINATION OF $|j_1 m_1 j_2 m_2 j_3 m_3\rangle$

FUNCTIONS OF UNCOUPLED MOMENTA

2° $|j_1,(j_2j_3)j_{23},jm\rangle =$

$$\sum_{m_1 m_{23}} |j_1 m_1,(j_2j_3)j_{23}m_{23}\rangle\langle j_1 m_1 (j_2j_3)j_{23}m_{23}|j_1,(j_2j_3)j_{23},jm\rangle$$

(1)

$$= \sum_{m_1 m_{23}} \sum_{m_2 m_3} |j_1 m_1 j_2 m_2 j_3 m_3\rangle\langle j_2 m_2 j_3 m_3 | (j_2j_3)j_{23}\, m_{23}\rangle$$

C-G(3-j)

$$\langle j_1 m_1 (j_2j_3)j_{23}\, m_{23}|j_1,(j_2j_3)j_{23},jm\rangle$$

C-G(3-j)

2°

$|j_1,(j_2j_3)j_{23},jm\rangle =$

$$\sum_{j_{12}} |(j_1j_2)j_{12},j_3,jm\rangle\langle(j_1j_2)j_{12},j_3,jm|j_1,(j_2j_3)j_{23},jm\rangle$$

1° TRANSITION MATRIX BETWEEN TWO DIFFERENT COUPLING SCHEMES OF THREE ANGULAR MOMENTA

COMPARING TERMS AT $|j_1m_1j_2m_2j_3m_3\rangle$ ON BOTH SIDES $\Rightarrow$ TRANSFORMATION COEFFICIENTS ARE INDEPENDENT OF m (OPERATE WITH j_+)

6-j SYMBOL

$$\begin{pmatrix} j_1 & j_2 & j_{12} \\ \cdot & \cdot & \cdot \end{pmatrix}_{1.\,\Delta} \quad \begin{pmatrix} j_{12} & j_3 & j \\ \cdot & \cdot & \cdot \end{pmatrix}_{2.\,\Delta} \quad \begin{pmatrix} j_2 & j_3 & j_{23} \\ \cdot & \cdot & \cdot \end{pmatrix}_{3.\,\Delta} \quad \begin{pmatrix} j_1 & j & j_{23} \\ \cdot & \cdot & \cdot \end{pmatrix}_{4.\,\Delta}$$

$$\langle (j_1j_2)j_{12},j_3,j|j_1,(j_2j_3)j_{23},j\rangle =$$

$$(-1)^{j_1+j_2+j_3+j}\left[(2j_{12}+1)(2j_{23}+1)\right]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

SYMMETRY PROPERTIES:

$\{3.\}$ $\{1.\}$ $\{2.\}$ $\{4.\}$

TRIANGULAR CONDITIONS: $\Delta(j_2j_3j_{23}), \Delta(j_1j_2j_{12}), \Delta(j_{12}j_3j), \Delta(j_{23}j_1j)$

SYMMETRY PROPERTIES (cont.)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \end{Bmatrix}$$

INVARIANT UNDER ANY PERMUTATION OF THE COLUMNS

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} l_1 & j_2 & l_3 \\ j_1 & l_2 & j_3 \end{Bmatrix}$$

INVARIANT UNDER EXCHANGE OF THE UPPER AND LOWER ARGUMENTS IN EACH OF ANY TWO COLUMNS

„ORTHOGONALITY" PROPERTIES

$$\langle j_1,(j_2j_3)j_{23},jm\,|\,j_1,(j_2j_3)j'_{23},jm\rangle = \delta(j_{23},j'_{23})$$

$|(j_2j_3)j_{23}m_{23}\rangle$: $\hat{j}^2_{23}$, $\hat{j}_{23,z}$ ORTHOGONAL

$$\sum_{j_{12}} |(j_1j_2)j_{12},j_3,j\rangle\langle(j_1j_2)j_{12},j_3,j| = 1$$

$$\sum_{j_{12}} \langle j_1,(j_2j_3)j_{23},jm\,|\,(j_1j_2)j_{12},j_3,j\rangle \times$$

$$\langle(j_1j_2)j_{12},j_3,j\,|\,j_1,(j_2j_3)j'_{23},j\rangle = \delta(j_{23},j'_{23})$$

$$\Downarrow$$

$$\sum_{j_{12}} [j_{12},j_{23}] \begin{Bmatrix} j_3 & j & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_3 & j & j_{12} \\ j_1 & j_2 & j'_{23} \end{Bmatrix} = \delta(j_{23},j'_{23})$$

$$[j_{12},j_{23}] = (2j_{12}+1)(2j_{23}+1)$$

PROPERTIES OF 6-j SYMBOLS

$1^0\ (j_1 j_2) j_{12} + j_3 \Rightarrow j$

$2^0\ j_1 + (j_2 j_3) j_{23} \Rightarrow j$

$3^0\ j_2 + (j_3 j_1) j_{31} \Rightarrow j$

$1^0 \overset{6\text{-}j}{\longleftrightarrow} 2^0$

$1^0 \overset{6\text{-}j}{\longleftrightarrow} 3^0$

$\langle (j_1 j_2) j_{12}, j_3; j \,|\, j_2, (j_3 j_1) j_{31}; j \rangle$

TRANSFORMATION MATRIX

$1^0 \longleftrightarrow 3^0$

$\sum |2^0\rangle\langle 2^0| = 1$

$\sum_{j_{23}} |j_1, (j_2 j_3) j_{23}; j\rangle\langle j_1, (j_2 j_3) j_{23}; j| = 1$

$\langle (j_1 j_2) j_{12}, j_3; j \,|\, j_2, (j_3 j_1) j_{31}; j \rangle =$

$= \sum_{j_{23}} \langle (j_1 j_2) j_{12}, j_3; j \,|\, j_1, (j_2 j_3) j_{23}; j \rangle \langle j_1, (j_2 j_3) j_{23}; j \,|\, j_2, (j_3 j_1) j_{31}; j \rangle$

$j_1\ j_2\ j_3$ $\qquad$ $j_2\ j_3\ j_1$

COUPLING OF TWO ANGULAR MOMENTA

PROPERTIES OF C-G COEFFICIENTS

$\langle j_1 m_1\ j_2 m_2 \,|\, j_1\ j_2\ j\, m \rangle = (-1)^{j_1 + j_2 - j} \langle j_2 m_2\ j_1 m_1 \,|\, j_2\ j_1\ j\, m \rangle$

$\begin{pmatrix} j_1 & j_2 & j \\ \cdot & \cdot & \cdot \end{pmatrix} = (-1)^{j_1 + j_2 + j} \begin{pmatrix} j_2 & j_1 & j \\ \cdot & \cdot & \cdot \end{pmatrix}$

$\langle j_1, (j_2 j_3) j_{23}; j| = (-1)^{j_1 + j_{23} - j} \langle (j_2 j_3) j_{23}, j_1; j|$

$j_1 + j_{23} \Rightarrow j$ $\qquad$ $j_{23} + j_1 \Rightarrow j$

$= \sum_{j_{23}} (-1)^{j_1 + j_{23} - j} \langle (j_1 j_2) j_{12}, j_3; j \,|\, j_1, (j_2 j_3) j_{23}; j \rangle^*$

$\langle (j_2 j_3) j_{23}, j_1; j \,|\, j_2, (j_3 j_1) j_{31}; j \rangle \Rightarrow$

$$\langle (j_1 j_2) j_{12}, j_3; j \,|\, j_2\ (j_3 j_1) j_{31}; j \rangle$$

$(-1)^{j_1+j_2-j_{12}}$ $(j_1 \leftrightarrow j_2)$ $\qquad$ $(-1)^{j_3+j_1-j_{31}}$ $(j_3 \leftrightarrow j_1)$

$$= (-1)^{j_1+j_2-j_{12}+j_3+j_1-j_{31}} \langle (j_2 j_1) j_{12}, j_3; j \,|\, j_2, (j_1 j_3) j_{31}; j \rangle$$

$$\sum_{j_{23}} (-1)^{j_1+j_{23}-j} \langle (j_1 j_2) j_{12}, j_3; j \,|\, j_1, (j_2 j_3) j_{23}; j \rangle \times$$

$$(-1)^{j_1+j_2+j_3+j} [j_{12}, j_{23}]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

$$\langle (j_2 j_3) j_{23}, j_1; j \,|\, j_2, (j_3 j_1) j_{31}; j \rangle =$$

$$(-1)^{j_1+j_2+j_3+j} [j_{23}, j_{31}]^{\frac{1}{2}} \begin{Bmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{31} \end{Bmatrix}$$

$$(-1)^{j_1+j_2-j_{12}+j_3+j_1-j_{31}} \langle (j_2 j_1) j_{12}, j_3; j \,|\, j_2, (j_1 j_3) j_{31}; j \rangle$$

$$(-1)^{j_1+j_2+j_3+j} [j_{12}, j_{31}]^{\frac{1}{2}} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{31} \end{Bmatrix}$$

$$\langle (j_1 j_2) j_{12}, j_3; j \,|\, j_1, (j_2 j_3) j_{23}; j \rangle = [j_{12}, j_{23}]^{\frac{1}{2}} (-1)^{j_1+j_2+j_3+j} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

$$\sum_{j_{23}} (-1)^{j_{23}+j_1-j} [j_{23}] \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{31} \end{Bmatrix} =$$

$$= (-1)^{j_1+j_2-j_{12}+j_1+j_3-j_{31}-(j_1+j_2+j_3+j)} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{31} \end{Bmatrix}$$

$$\begin{Bmatrix} j_3 & j & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix} \qquad \begin{Bmatrix} j_1 & j & j_{23} \\ j_2 & j_3 & j_{31} \end{Bmatrix} \qquad \begin{Bmatrix} j_3 & j & j_{12} \\ j_2 & j_1 & j_{31} \end{Bmatrix}$$

(3-12)

JUDD: „OPERATOR TECHNIQUES...