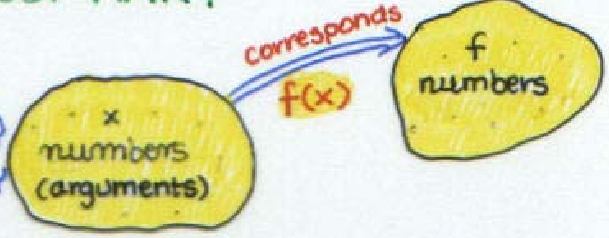
BASIC LANGUAGE OF QUANTUM MECHANICS: SUMMARY

1. FUNCTIONS

MUST BE WELL BEHAVED
THEY HAVE TO SATISFY
THE SO-CALLED



A. NATURAL CONDITIONS (CLASS Q):

* WAVEFUNCTIONS AND THEIR FIRST DERIVATIVES HAVE TO BE:
  - SINGLE-VALUED
  - CONTINOUS
  - FINITE

* $\int_{\alpha}^{\beta} |u_n(x)|^2 dx < \infty$ (PHYSICAL INTERPRETATION ≡ PROBABILITY)

B. HERMITEAN PRODUCT (INNER, SCALAR)

$$\int_{\alpha}^{\beta} f^*(x)\, g(x)\, dx \equiv (f,g) \quad \text{(NUMBER)} \left.\right\} \equiv \langle f | g \rangle$$

DIRAC'S NOTATION

$(f,g) = (g,f)^*$ $\qquad$ $(\alpha f, g) = \alpha^* (f,g)$

$(f, g+h) = (f,g) + (f,h)$ $\qquad$ $(f, \alpha g) = \alpha (f,g)$

$(f+g, h) = (f,h) + (g,h)$ $\qquad$ $(f,f) \geqslant 0$; $(f,f) = 0 \Rightarrow f \equiv 0$

$\wedge_{g(x)}$ $(g,f) = 0 \Rightarrow f(x) = 0$

$\wedge_{f(x)}$ $(f,g) = (f,h) \Rightarrow g(x) = h(x)$

C. LINEAR DEPENDENCE

$\{u_n\}_1^r$ IS LINEARLY DEPENDENT IF

$$\vee_{\lambda_k} : \sum_{k}^{r} \lambda_k u_k(x) = 0$$

ONLY LINEARLY **INDEPENDENT** FUNCTIONS ARE REGARDED (COUNTED) AS DIFFERENT!

D. ORTHOGONALITY AND NORMALIZATION ≡ ORTHONORMALITY

$$(u_m, u_n) = \delta_{nm} \quad \begin{cases} 0 & n \neq m \text{ orthogonal} \\ 1 & n = m \text{ normalized} \end{cases}$$

E. COMPLETNESS – PARSEVAL'S IDENTITY:

$$(f, g) = \sum_n (f, u_n)(u_n, g) \left(\equiv \sum_n \langle f | u_n \rangle \langle u_n | g \rangle \right)$$

ARBITRARILY CHOSEN f AND g

$$\sum_n |u_n\rangle\langle u_n| = \hat{I}$$

CLOSURE

PROJECTION OPERATOR ONTO THE WHOLE SPACE

IN PRACTICE USED AS DEFINITION OF COMPLETE SET $\{u_n\}$

(JARGON: „EXPANSION OF UNITY")

2. OPERATORS

SYMBOL OF MATHEMATICAL OPERATION (ACTION)

$f(x) = \hat{O} g(x)$

OPERATOR DOES NOT EXIST BY ITSELF – ALWAYS IN ACTION ON FUNCTION

A. LINEAR OPERATORS

$$\hat{A}(c f(x)) = c \hat{A} f(x) \quad \text{(c-NUMBER)}$$

$$\hat{A}(f_1(x) + f_2(x)) = \hat{A} f_1 + \hat{A} f_2$$

B. COMMUTATOR

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \Rightarrow \text{„SYMMETRY IN PHYSICS"}$$

DOES NOT EXIST BY ITSELF – WHY?

$$\left[\frac{\partial}{\partial x}, f(xyz)\right] g(x) = \frac{\partial}{\partial x} f(xyz) g(x) - f(xyz) \frac{\partial}{\partial x} g(x) =$$

$$\frac{\partial f}{\partial x} g(x) + \cancel{f(x) \frac{\partial g}{\partial x}} - \cancel{f(x) \frac{\partial g}{\partial x}} = \frac{\partial f}{\partial x} g(x)$$

FOR ARBITRARY FUNCTION $g(x)$

THEREFORE:

$$\left[\frac{\partial}{\partial x}, f(xyz)\right] = \frac{\partial f}{\partial x}$$

C. HERMITICITY = SELFADJOINT OPERATOR

$$A = A^{+}$$

WHERE A^{+} IS DEFINED BY THE RELATION

$$(f, Ag) = (g, A^{+}f)^{*}$$

$$\int f^{*} A g \, d\tau = \left(\int g^{*} A^{+} f \, d\tau \right)^{*}$$

EXAMPLE: SHOW THAT $\frac{d}{dx}$ IS NOT A HERMITEAN OPERATOR

FROM DEF.:

$$\int_{-\infty}^{\infty} f^{*} \frac{d}{dx} g(x) dx \overset{?}{=} \left(\int_{-\infty}^{\infty} g^{*}(x) \frac{d}{dx} f(x) dx \right)^{*}$$

$$= \left(\int_{-\infty}^{\infty} g^{*}(x) \left(\frac{d}{dx}\right)^{+} f(x) dx \right)^{*}$$

$$f^{*}(x) g(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x) \frac{d}{dx} f^{*}(x) dx = \left(\int_{-\infty}^{\infty} g^{*}(x) \left(-\frac{d}{dx}\right) f(x) dx \right)^{*}$$

$f^{*}(x) g(x) \Big|_{-\infty}^{\infty} \to 0$

FUNCTIONS OF CLASS Q

$$\left(\frac{d}{dx}\right)^{+} = -\frac{d}{dx} \neq \frac{d}{dx}$$

D. UNITARY OPERATORS

$$S^{+} = S^{-1} \quad : \quad S^{+}S = SS^{+} = 1$$

PARTICULAR CASE OF NORMAL OPERATORS

$$[N, N^{+}] = 0$$

EIGENVALUE PROBLEM

THE ONLY POSSIBLE VALUES OF AN OBSERVABLE A ARE THE EIGENVALUES α

$$\hat{A}\, a(x) = \alpha\, a(x)$$

eigenfunctions (limited to Q class)

eigenvalues (numbers)
spectrum of operator

IF $\alpha: \begin{cases} a_1(x) \\ a_2(x) \\ \vdots \\ a_r(x) \end{cases}$ (ALL FUNCTIONS ARE LINEARLY INDEPENDENT)

α IS r- FOLD DEGENERATE

IF $[A, B] = 0$ BOTH OPERATORS HAVE COMMON EIGENFUNCTIONS → "SYMMETRY IN PHYSICS"

$$Aa = \alpha a$$

B

$$BAa = B\alpha a \qquad A\,Ba = \alpha \underbrace{Ba}$$

NEW EIGENFUNCTION IT HAS TO BE LINEARLY DEPENDENT WITH a

$$Ba = \beta a$$

EIGENVALUE PROBLEM OF B

a IS ALSO EIGENFUNCTION FOR B

IF $A = A^{+} \Rightarrow \alpha = \alpha^{*}$ EIGENVALUE OF HERMITEAN OPERATOR IS REAL

DEF. $(f, Ag) = (g, A^{+}f)^{*}$

$$Aa = \alpha a$$

a^{*}

$\int d\tau$

$$\left.\begin{aligned} &(a, Aa) = \alpha\,(a, a) \\ &\quad\ \| \\ &\quad (a, A^{+}a)^{*} = (a, Aa)^{*} = \alpha^{*}(a, a) \end{aligned}\right\}\ \alpha = \alpha^{*}$$

(EIGENVALUES) (EIGENFUNCTIONS)

IF $\alpha^* = \alpha \;\wedge\; \{a_n\}$ - COMPLETE

$\Downarrow$ $A = A^+$ (HERMITEAN)

$1 = \sum_n |a_n)(a_n|$

$$(f, Ag) = \sum_n \sum_m (f, a_n)(a_n, A \overset{\text{eigenfunctions}}{a_m})(a_m, g) =$$

$1 = \sum_m |a_m\rangle\langle a_m|$ $\qquad A a_m = \alpha_m a_m \;,\; (a_n, a_m) = \delta_{nm}$

$$= \sum_n (f, a_n)(a_n, g) \alpha_n \qquad \alpha = \alpha^*$$

$$(g, Af)^* = \left\{ \sum_n (g, a_n)(a_n, f) \alpha_n \right\}^* = \sum_n (a_n, g)(f, a_n) \alpha_n^*$$

$$(f, Ag) = (g, Af)^* \quad \textbf{A - HERMITEAN!}$$

EXPECTATION VALUES

IF A SYSTEM IS IN A STATE f, THE AVERAGE OF SEQUENCE OF MEASUREMENTS (QUANTUM AVERAGE):

$$\langle A \rangle_f = \langle f, Af \rangle$$

$\langle A+B \rangle_f = \langle A \rangle_f + \langle B \rangle_f$

$\langle AB \rangle_f = \langle A \rangle_f \langle B \rangle_f$ ONLY IF $\hat{A}$ AND $\hat{B}$ OPERATE IN TWO DISTINCT SPACES !!!

✱ IF $\langle A \rangle_f = \langle B \rangle_f \Rightarrow A \equiv B$

✱ IF $\langle A \rangle_f = \langle B \rangle_f^* \Rightarrow B = A^+$

$(f, Af) = (f, A^+ f)^*$

$\langle A \rangle_f = \langle A^+ \rangle_f^*$ ✱ IF $A = A^+$

$\langle A \rangle_f = \langle A \rangle_f^*$ REAL AVERAGE VALUES OF OBSERVABLE REPRESENTED BY A

MATRIX REPRESENTATION

1. FUNCTION $f(x)$ FOR A COMPLETE SET $\{u_n(x)\}$

COLUMN OF FOURIER COEFFICIENTS

$$\check{f} = \begin{bmatrix} (u_1, f) \\ (u_2, f) \\ \vdots \end{bmatrix} \qquad \check{f}^{+} = \left[(f, u_1)\,(f, u_2) \dots \right]$$

$$(f, g) = \left[(f, u_1)(f, u_2) \dots \right] \begin{bmatrix} (u_1, g) \\ (u_2, g) \\ \vdots \end{bmatrix} = \sum_n (f, u_n)(u_n, g)$$

PRODUCT OF TWO MATRICES

2. OPERATOR

$$\check{A} = \{A_{nm}\} \text{ matrix built of}$$

MATRIX ELEMENTS $A_{nm} = (u_n, Au_m)$

FOR GIVEN COMPLETE SET OF FUNCTIONS

$$(f, Ag) = \sum_n \sum_m (f, u_n) \underbrace{(u_m, Au_m)}_{A_{nm} \text{ MATRIX ELEMENTS}} (u_m, g)$$

IF IN ADDITION $Au_m = d_m u_m$

$$(u_n, Au_m) = d_m \delta_{nm}$$

$$(f, Ag) = \sum_m (f, u_n)(u_n, g)\, d_m$$

IF $f = g$:

$$\langle A_f \rangle = \sum_m |(f, u_n)|^2 d_n$$

SPECTRAL EXPANSION

GENERAL CONCLUSION: MATRIX REPRESENTATIONS OBEY THE SAME RULES AS OPERATORS; EQUATION SATISFIED BY OPERATORS AND FUNCTIONS IS ALSO SATISFIED BY THEIR MATRIX REPRESENTATIONS (AND VICE VERSA).

$\Downarrow$

EQUIVALENCE BETWEEN MATRIX QUANTUM MECHANICS OF HEISENBERG AND WAVE QUANTUM MECHANICS OF SCHRÖDINGER

PARTICULAR CHOICE OF A COMPETE SET OF FUNCTIONS:
EIGENFUNCTIONS OF POSITION OPERATORS $\bar{r}$

$\Psi_\alpha(\bar{r})$: MATRIX WITH ONE COLUMN, ROWS ARE LABELED BY THE COORDINATE $\bar{r}$

GEOMETRICAL PICTURE OF STATE FUNCTION: STATE VECTOR

IN AN INFINITE-DIMENSIONAL HILBERT SPACE

EACH DIMENSION CORRESPONDS TO ONE OF THE ROWS OF ONE-COLUMN MATRIX; COMPONENT OF THE STATE VECTOR ALONG THAT AXIS OF THE HILBERT SPACE = CORRESPONDING MATRIX ELEMENT.
DIFFERENT CHOICES FOR THE ORIENTATION OF THE AXES IN THE HILBERT SPACE $\Rightarrow$ DIFFERENT REPRESENTATIONS

DIRAC'S BRA AND KET NOTATION

$$\left.\begin{array}{l}\text{STATE FUNCTION}\\ \text{STATE VECTOR}\end{array}\right\} \Psi_\alpha : |\alpha\rangle \quad \text{KET VECTOR}$$

$$\Psi_\alpha^+ : \langle\alpha| \quad \text{BRA VECTOR}$$

INNER PRODUCT

$$\langle\Psi_\alpha|\Psi_\beta\rangle \ \ \underset{\text{EXPRESSION}}{\text{BRACKET}} = \int \Psi_\alpha^+ \Psi_\beta \, d\tau = (\Psi_\alpha, \Psi_\beta) \ \text{NUMBER}$$

$$\mathbb{1} = \int |\bar{r}\rangle\langle\bar{r}| \, d\bar{r}$$

$$\int \underset{\text{matrix}}{\langle\Psi_\alpha|\bar{r}\rangle} \underset{\text{matrix}}{\langle\bar{r}|\Psi_\beta\rangle} d\bar{r} = \int \Psi_\alpha^+(\bar{r}) \Psi_\beta(\bar{r}) \, d\bar{r}$$

IN THE CASE OF OPERATORS:

$$\Omega \Psi_\alpha(\bar{r}) = \omega \Psi_\alpha(\bar{r})$$

$$\int d\bar{r} \langle\bar{r}|\Omega|\bar{r}'\rangle\langle\bar{r}'|\alpha\rangle = \omega \langle\bar{r}|\alpha\rangle$$

$$\mathbb{1} \qquad \Downarrow \ \Omega|\alpha\rangle = \omega|\alpha\rangle$$